EXACT SOLUTION OF AN AXISYMMETRIC STAGNATION-POINT FLOW AND HEAT TRANSFER OF VISCOUS, COMPRESSIBLE FLUID ON AN MOVING CYLINDER

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ABSTRACT

The steady-state viscous, compressible flow and heat transfer in the vicinity of an axisymmetric stagnation-point of a cylinder moving axially with a constant velocity are investigated. The impinging free stream is steady and with a constant strain-rate $k$. An exact solution of the Navier-Stokes equations and energy equation is derived in this problem. A reduction of these equations is obtained by use of appropriate transformations. The general self-similar solution is obtained when the wall temperature of the cylinder or its wall heat flux is constant. All the solutions above are presented for the case of low Mach number and for Reynolds numbers, $Re = \frac{ka^2}{2\nu}$, ranging from 0.1 to 1000, selected values of compressibility factor, and different values of Prandtl numbers where $a$ is the cylinder radius and $\nu$ is kinematic viscosity of the fluid. Shear-stress is presented as well. Axial movement of the cylinder does not have any effect on radial component of the velocity and also heat transfer but its increase increases the axial component of fluid velocity field and shear-stress.

INTRODUCTION

Axial movement of a cylinder in the case of stagnation-point flow and heat transfer has many applications in manufacturing processes. For example, cooling and also cleaning processes of punching instruments and drilling tools are some industrial applications. Existing solutions of the problem of axisymmetric stagnation-point flow and heat transfer on either a cylinder or a flat plate are for viscous, incompressible fluid. These studies started by Hiemenz [1] who obtained an exact solution of the Navier-Stokes equations governing the two-dimensional stagnation-point flow on a flat plate and went on by Homann [2] which was an analogous axisymmetric study and by Howarth [3], Davey [4] where results for stagnation flow against a flat plate for axisymmetric cases were presented. Wang [5 - 6] was first to find exact solution for the problem of axisymmetric stagnation-flow on an infinite stationary circular cylinder and continued by Gorla [7 - 11] which are a series of steady and unsteady flows and heat transfer over a circular cylinder in the vicinity of the stagnation-point for the cases of constant axial movement and the special case of axial harmonic motion of a non-
rotating cylinder. Cunning et al. [12] have considered the stagnation flow problem on a rotating circular cylinder with constant angular velocity and Grosch et al. [13], and Takhar et al. [14] who studied special cases of unsteady viscous flow on an infinite circular cylinder. The more recent works of the same types are the ones by Saleh and Rahimi [15], Rahimi and Saleh [16-17] which are exact solution studies of a stagnation-point flow and heat transfer on a circular cylinder with time-dependent axial and rotational movements and studies by Shokrgozar and Rahimi [18-21] which are exact solutions of stagnation-point flow and heat transfer but on a flat plate. Existing compressible flow studies are all general studies in the stagnation region of a body and by using boundary layer equations. The only study that deals with stagnation-point flow and heat transfer of a viscous, compressible fluid on a cylinder is by Mohammadian and Rahimi [22]. They obtain an exact solution of the Navier-Stokes equations for the case of a stationary cylinder.

The problem of stagnation-point flow and heat transfer for the case of a compressible fluid and when the cylinder is moving axially has not been considered so far. In this research work, solution of the problem of axisymmetric stagnation-point flow and heat transfer is presented for the case of compressible, viscous fluid on a cylinder when it is moving axially with a constant velocity. An exact solution of the Navier-Stokes and energy equations in cylindrical polar coordinates governing the axisymmetric compressible flow and heat transfer are:

**PROBLEM FORMULATION**

Flow is considered in cylindrical coordinates \((r, \varphi, z)\) with corresponding velocity components \((u, v, w)\), see Fig. 1. We consider the laminar flow of a steady, viscous, compressible fluid and heat transfer in the neighborhood of an axisymmetric stagnation-point of an infinite circular cylinder moving with a constant axial velocity. An external axisymmetric radial stagnation flow of strain rate \(k\) impinges on the cylinder of radius \(a\), centered at \(r = 0\). The steady Navier-Stokes and energy equations in cylindrical polar coordinates are:

**Mass:**

\[
\frac{\partial (\rho u)}{\partial r} + \frac{\rho u}{r} + \frac{\partial (\rho w)}{\partial z} = 0
\]  

**r-Momentum:**

\[
u \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial (\rho u)}{\partial r} \right] - \frac{\partial (\rho u)}{\partial r} + \frac{\partial^2 (\rho u)}{\partial z^2} \right\} = -\frac{\partial P}{\partial r} +
\]

\[
u \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial (\rho u)}{\partial r} \right] - \frac{\partial (\rho u)}{\partial r} + \frac{\partial^2 (\rho u)}{\partial z^2} \right\} = -\frac{\partial P}{\partial r} +
\]

\[
\frac{\partial (\rho u)}{\partial r} + \frac{\rho u}{r} + \frac{\partial (\rho w)}{\partial z} = 0
\]  

\[
u \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial (\rho u)}{\partial r} \right] - \frac{\partial (\rho u)}{\partial r} + \frac{\partial^2 (\rho u)}{\partial z^2} \right\} = -\frac{\partial P}{\partial r} +
\]

**Fig. 1. Schematic diagram of an axially moving cylinder**
z-Momentum:
\[
\frac{\partial (\rho w)}{\partial r} + w \frac{\partial (\rho w)}{\partial z} = - \frac{\partial P}{\partial z} + \nu \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial (\rho w)}{\partial r} \right] + \frac{\partial^2 (\rho w)}{\partial z^2} \right\}
\]

\( (3) \)

Energy,
\[
\rho \frac{\partial T}{\partial r} + \rho w \frac{\partial T}{\partial z} = \frac{\mu}{Pr} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right)
\]

\( (4) \)

Where \( p, \rho, \nu, \) and \( T \) are the fluid pressure, density, kinematic viscosity, and temperature.

The boundary conditions for velocity field are:
\[ r = a : \quad u = 0, \quad w = V \]

\( (5) \)

\[ r \rightarrow \infty : \quad u = -\kappa (r - a^2 / r), \quad w = 2\kappa z \]

\( (6) \)

in which, (5) are no-slip conditions on the cylinder wall and \( V \) is the axial velocity of the cylinder. Relations (6) show that the viscous flow solution approaches, in a manner analogous to the Hiemenz flow, the potential flow solution as \( r \rightarrow \infty \), Ref. [12].

For the temperature field we have:
\[
\begin{align*}
T & = T_w = \text{const}\tan t \\
\frac{\partial T}{\partial r} & = - \frac{q_w}{k}
\end{align*}
\]

\( r = a \)

\[
\begin{align*}
T & \rightarrow T_w \\
r \rightarrow \infty & : \quad T \rightarrow T_\infty
\end{align*}
\]

\( (7) \)

Where \( k \) is the thermal conductivity of the fluid and \( T_w \) and \( q_w \) are temperature and heat flux at the cylinder wall, respectively, and \( T_\infty \) is the free stream temperature.

A reduction of the Navier-Stokes equations is obtained by the following coordinate separation of the velocity field:

\[
\begin{align*}
u & = -\frac{ka^2}{r} \frac{\rho_\infty}{\rho(\eta)} f(\eta), \\
w & = \frac{\rho_\infty}{\rho(\eta)} [2kcf'(\eta)z + H(\eta)], \\
p & = \rho_\infty \kappa^2 a^2 P
\end{align*}
\]

Where
\[
\eta = \frac{2}{a^2} \int_a^r \frac{\rho r}{\rho_\infty} dr
\]

is dimensionless radial variable, Ref. [22], and prime denotes differentiation with respect to \( \eta \) and \( \rho_\infty \) is free stream density. For the case of incompressible flow \( (\rho(\eta) = \text{constant}) \), this variable is similar to the one in Wang (Ref. [5]) except that it changes from zero to infinity instead of one to infinity. Transformations (8) satisfy (1) automatically and their insertion into (2) – (3) yields a coupled system of differential equations in terms of \( f(\eta), H(\eta) \) and an expression for the pressure:

\[
\begin{align*}
\Gamma [c^3 f'' + 3c^2c' f'' + c^2c' f' + (c')^2 c f''] & + c^2 f'' + cc'f' + \\
+ \text{Re} [1 + c' f' + cf'' - c(f')^2] &= 0
\end{align*}
\]

\( (10) \)

\[
\begin{align*}
\Gamma (c^2 H'' + c c'H') & + c H' + \\
+ \text{Re} (f H' - f' H) &= 0
\end{align*}
\]

\( (11) \)

\[
p - p_0 = \int_0^\eta \left[ \frac{1}{2} \left( \frac{f'}{c} \right)^2 - \frac{f''}{c} - \frac{1}{\text{Re}} (c f'')' \right] d\eta - 2 \left( \frac{\zeta}{a} \right)^2
\]

\( (12) \)

In these equations

\[
c(\eta) = \frac{\rho(\eta)}{\rho_\infty}, \quad \text{Re} = \frac{ka^2}{2\nu},
\]
\[
\Gamma(\eta) = 1 + \int_0^\eta \frac{d\eta}{c(\eta)}, \quad \text{and prime indicates differentiation with respect to } \eta. \text{ Details of the derivation of the above equations have been presented in the appendix. From conditions (5) and (6), the boundary conditions for (10) and (11) are as follows:} \\
\eta = 0: \quad f = 0, \quad f' = 0, \quad H = V c(0) \\
\eta \to \infty: \quad f' = 1, \quad H = 0
\] (13)

To model the variation of density with respect to temperature, the following Boussinesq approximation is used assuming low Mach number flow:
\[
\rho \approx \rho_\infty [1 - \beta(T - T_\infty)] \quad \Rightarrow \quad \rho / \rho_\infty = c(\eta) = 1 - \beta(T - T_\infty)
\] (14)

In the above relation, \( \rho_\infty \) and \( \beta \) are free stream density and compressibility factor, respectively. To transform the energy equation into a non-dimensional form, we introduce:
\[
\theta(\eta) = \left\{ \begin{array}{ll}
\frac{T(\eta) - T_\infty}{T_w - T_\infty}, & \eta = 0 \\
\frac{T(\eta) - T_\infty}{T(\eta) - T_\infty} - \frac{a_q w}{2k} & \eta \to \infty \end{array} \right.
\] (15)

Where \( \gamma = \frac{a_q w}{2k} \) is used in figures and presented results.

Making use of (8) and (15), as shown in the appendix, the energy equation may be written as:
\[
\frac{1}{\text{Re.Pr}}[\Gamma(c^2 \theta'' + cc' \theta') + c \theta'] + \frac{f\theta'}{0} = 0
\] (16)

With boundary conditions as:
\[
\eta = 0: \quad \theta = 1, \\
\eta \to \infty: \quad \theta = 0,
\] (17)

The local heat transfer coefficient is given by:
\[
h = \frac{\frac{q_w}{T_w - T_\infty}}{\frac{2k}{a}} \theta'(0) c(0),
\] (18)

Because of \( c(\eta) \), the Eqs. (10) - (12) and (16) are coupled. Note that, for the case of incompressible fluid, \( \rho(\eta) = \rho_\infty \), Eq. (10) is exactly reduced to the equation obtained in Wang (Ref. [5]) for radial component of velocity and also Eq. (16) reduces to the equation obtained in Gorla, Ref. [7], with consideration of starting value for the variable \( \eta \).

**SHEAR-STRESS:**

The shear-stress on the surface of the cylinder is obtained from:
\[
\sigma = \mu \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right]_{z=a} \quad \text{or} \\
\sigma = \mu \left[ \frac{\partial w}{\partial r} \right]_{z=a}
\]

But
\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial r} = \frac{2k}{C} \left( \frac{f'' z + H'}{C} - \frac{HC''}{a^2} \right) \frac{2r}{C(\eta)}
\] (19)
Since \( \eta = 0 \) at \( r = a \), then we have

\[
\sigma = \mu \left[ 2k f'(0) z + \frac{H'(0) - H(0) C'(0)}{C(0)} \frac{2}{a(0)} C(0) \right] \tag{20}
\]

Results of \( \left[ \frac{\sigma a}{2 \mu} \right]_{z=0} \) for different values of \( Re \) with \( Pr \) held constant and for different values of \( Pr \) with \( Re \) held constant are presented in later sections.

**NUMERICAL PROCEDURES**

Equations (10)-(12) and (16) along with boundary conditions (13) and (17) are solved by using the fourth-order Runge-Kutta method of integration along with a shooting method, press et al. [23]. Using this method, the initial values are guessed and the integration is repeated until convergence is obtained. In these computations the step size in \( \eta \)-direction is optimized and \( \Delta \eta = 0.001 \) is used throughout the computations. The truncation error was set to be 1E-9.

**RESULTS AND DISCUSSIONS**

In this section the solution of the self-similar Eq. (11) along with surface shear-stresses for prescribed values of surface temperature and heat flux for selected values of Reynolds and Prandtl numbers are presented. Equations (10), (12) and (16) are the same as in Ref. [22] which is for a stationary cylinder. Therefore axial movement of the cylinder does not have any effect on radial component of the velocity and also heat transfer.

Figures 2-5 show that axial component of the velocity field increases by increase of Reynolds number and also by increase of axial speed of the cylinder in both cases of constant wall temperature and constant heat flux.

**CONCLUSIONS**

An exact solution of the Navier-Stokes equations and energy equation have been derived for the steady-state viscous, compressible flow and heat transfer in the vicinity of an axisymmetric stagnation-point of a cylinder moving axially with a constant velocity. A reduction of these equations has been obtained by use of appropriate transformations. The general self-similar solution has been obtained when the wall temperature of the cylinder or its wall heat flux was constant. All the solutions above have been presented for low Mach number and Reynolds numbers ranging from 0.1 to 1000, selected values of compressibility factor, and different values of Prandtl numbers. Shear-stress has been presented as well. Axial movement of the cylinder does not have any effect on radial component of the velocity and also heat transfer but the axial component of fluid velocity field increases as the speed of this movement increases. The axial component of the fluid velocity field for a specified axial velocity of the cylinder decreases as Reynolds number or compressibility factor or surface temperature or surface heat flux increases and this decrease in all cases is larger when cylinder moves faster. It is worth mentioning that in each case incompressible fluid produces the largest amount of change in axial component of the fluid velocity field. In the contrary, the axial component of the fluid velocity field increases as Prandtl number increases and this increase is larger when cylinder moves faster. Also cylinder axial speed increases the absolute value of the shear-stress and the amount of this shear-stress is the least for the case of an incompressible fluid.
REFERENCES

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Fig. 2. Variation of axial velocity filed versus \( \eta \) at \( Re=10\), \( T_w=500k\), \( \beta=0.0033\), \( a=0.1m\), \( Pr=1 \) for selected values of cylinder axial speed

Fig. 3. Variation of axial velocity filed versus \( \eta \) at \( Re=100\), \( T_w=500k\), \( \beta=0.0033\), \( a=0.1m\), \( Pr=1 \) for selected values of cylinder axial speed
Fig. 4. Variation of axial velocity filed versus $\eta$ at $Re = 10$, $\gamma = 10$, $\beta = 0.0033$, $a = 0.1m$, $Pr = 1$ for selected values of cylinder axial speed.

Fig. 5. Variation of axial velocity file versus $\eta$ at $Re = 100$, $\gamma = 10$, $\beta = 0.0033$, axial speed $a = 0.1m$, $Pr = 1$ for selected values of cylinder a.