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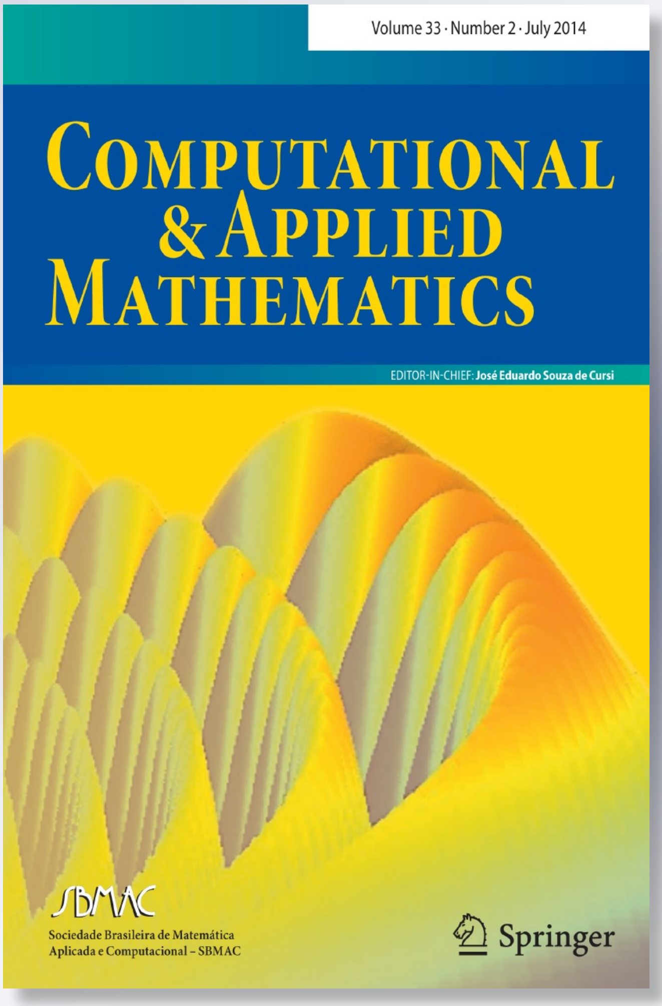

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A stopping rule for an iterative algorithm in systems of integral equations

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Abstract In this paper, an important inequality for T, a contraction integral operator, is obtained. From a practical programming point of view, this inequality allows us to express our iterative algorithm with a "for loop" rather than a "while loop". The main tool used in our research is the fixed point theorem in the Banach space of continuous functions, $X := C([a, b], \mathbb{R}^k)$.

Keywords Integral operator · Successive approximation method · Approximation error

Mathematics Subject Classification (2000) 45D05, 47J25

1 Introduction

The solutions of integral equations play an important role in the various fields of sciences and engineering [15,24]. Most of physical phenomena can be modeled by differential equations, integral equations, integro-differential equations or a system of these equations [6,10]. Since only few of these equations have explicit solution, we often have to use numerical methods [3,19]. There are several numerical methods for solving linear system of Volterra integral equations of the second kind, such as Galerkin method [11], Collocation method [7], Taylor series [20], Legendre wavelets [21,32], Jacobi polynomials [17] and recently Chebyshev polynomials [9], homotopy perturbation method [5,14,25], Block-Pulse functions [23] and expansion methods [30,31]. On the other hand, investigations on existence theorems for

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diverse functional-integral equations have been presented in other references such as [1,4,8,12,13,16,18,22,26–28]. Nevertheless, it seems that no one has studied the systems of integral equations by the analogue method mentioned in this paper yet.

The paper is organized as follows. In Sect. 2, by the successive approximation method a contraction mapping for $\mathcal{K}$ is obtained. Thereafter in Sect. 3, a simple technique the stopping rule for our iterative algorithm has been introduced. Finally, in Sect. 4, we give numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme.

Consider the system of linear Volterra integral equations of second kind of the form:

$$\mathbf{U}(x) = \mathbf{F}(x) + \int_a^x \mathbf{K}(x,t)\mathbf{U}(t)\mathrm{d}t \equiv \mathcal{K}\mathbf{U}, \quad (a \le x \le b,\ \mathbf{U} \in X), \tag{1}$$

where

$$\begin{aligned}
\mathbf{U}(x) &= [u_1(x), \quad u_2(x), \ldots, u_l(x)]^T,\\
\mathbf{F}(x) &= [f_1(x), \quad f_2(x), \ldots, f_l(x)]^T,\\
\mathbf{K}(x,t) &= [k_{ij}(x,t)], \quad i, j = 1, 2, \ldots, l.
\end{aligned}$$

In (1), the vector function $\mathbf{F}$ and the matrix function $\mathbf{K}$ are given, and $\mathbf{U}$ is the vector function of the solution that will be determined. We assume that $\mathbf{F}$ and $\mathbf{K}$ are continuous on the interval $[a, b]$ and the triangular region $D := \{(x,t) : x \in [a,b], t \in [a,x]\}$, respectively.

2 A contraction mapping for the Volterra equation

In this section, first we prove that $\mathcal{K}^n$ in (1) is contraction when n is enough large.

Theorem 2.1 *The mapping $\mathcal{K}^n$ is contraction when n is sufficiently large.*

Proof We write

$$\begin{aligned}
\mathcal{K}\mathbf{U} &= \mathbf{F}(x) + \int_a^x \mathbf{K}(x,\zeta)\mathbf{U}(\zeta)\mathrm{d}\zeta,\\
\mathcal{K}^2\mathbf{U} &= \mathbf{F}(x) + \int_a^x \mathbf{K}(x,\zeta)[\mathbf{F}(\zeta) + \int_a^\zeta \mathbf{K}(\zeta,t)\mathbf{U}(t)\mathrm{d}t]\mathrm{d}\zeta\\
&= \mathbf{F}(x) + \int_a^x \mathbf{K}(x,\zeta)\mathbf{F}(\zeta)\mathrm{d}\zeta + \int_a^x\int_a^\zeta \mathbf{K}(x,\zeta)\mathbf{K}(\zeta,t)\mathbf{U}(t)\mathrm{d}t\mathrm{d}\zeta\\
&= \mathbf{F}(x) + \int_a^x \mathbf{K}(x,\zeta)\mathbf{F}(\zeta)\mathrm{d}\zeta + \int_a^x \mathbf{K}_2(x,\zeta)\mathbf{U}(\zeta)\mathrm{d}\zeta,
\end{aligned}$$

where $\mathbf{K}_2(x,\zeta) = \int_\zeta^x \mathbf{K}(x,t)\mathbf{K}(t,\zeta)\mathrm{d}t$.
We repeat this successive process to get

$$\begin{aligned}\mathcal{K}^n\mathbf{U} = \mathbf{F}(x) + \int_a^x \mathbf{K}_1(x,\zeta)\mathbf{F}(\zeta)\mathrm{d}\zeta + \int_a^x \mathbf{K}_2(x,\zeta)\mathbf{F}(\zeta)\mathrm{d}\zeta \\ + \cdots + \int_a^x \mathbf{K}_{n-1}(x,\zeta)\mathbf{F}(\zeta)\mathrm{d}\zeta + \int_a^x \mathbf{K}_n(x,\zeta)\mathbf{U}(\zeta)\mathrm{d}\zeta,\end{aligned}$$

where $\mathbf{K}_{n+1}(x,\zeta) = \int_\zeta^x \mathbf{K}(x,t)\mathbf{K}_n(t,\zeta)\mathrm{d}t$ and $\mathbf{K}_1(x,\zeta) = \mathbf{K}(x,\zeta)$.

Since $\mathbf{K}(x,\zeta)$ is assumed to be continuous on domain D, then there exists a positive number M such that $\|\mathbf{K}(x,\zeta)\|_\infty \le M$, where

$$\|\mathbf{K}(x,\zeta)\|_\infty = \max_{1\le i\le l}\sum_{j=1}^{l}|M_{ij}|,\ \ M_{ij} = \sup_D |k_{ij}(x,\zeta)|.$$

On the other hand, the following bound can be obtained for $\mathbf{K}_n(x,\zeta)$:

$$\|\mathbf{K}_n(x,\zeta)\|_\infty \le \frac{M^n}{(n-1)!}(x-\zeta)^{n-1}, \qquad a \le \zeta \le x, \tag{2}$$

since,

$$\begin{aligned}\|\mathbf{K}_{n+1}(x,\zeta)\|_\infty &= \|\int_\zeta^x \mathbf{K}(x,t_1)\mathbf{K}_n(t_1,\zeta)\mathrm{d}t_1\|_\infty \\ &\le \int_\zeta^x \|\mathbf{K}(x,t_1)\|_\infty\|\mathbf{K}_n(t_1,\zeta)\|_\infty\,\mathrm{d}t_1 \\ &\le \int_\zeta^x \|\mathbf{K}(x,t_1)\|_\infty\|\int_\zeta^{t_1}\mathbf{K}(t_1,t_2)\mathbf{K}_{n-1}(t_2,\zeta)\mathrm{d}t_2\|_\infty\,\mathrm{d}t_1 \\ &\le \int_\zeta^x \|\mathbf{K}(x,t_1)\|_\infty\int_\zeta^{t_1}\|\mathbf{K}(t_1,t_2)\mathbf{K}_{n-1}(t_2,\zeta)\|_\infty\mathrm{d}t_2\,\mathrm{d}t_1 \\ &\le \int_\zeta^x\int_\zeta^{t_1}\|\mathbf{K}(x,t_1)\|_\infty\|\mathbf{K}(t_1,t_2)\|_\infty\|\mathbf{K}_{n-1}(t_2,\zeta)\|_\infty\mathrm{d}t_2\,\mathrm{d}t_1 \\ &\ \ \vdots \\ &\le \int_\zeta^x\int_\zeta^{t_1}\cdots\int_\zeta^{t_{n-1}}\|\mathbf{K}(x,t_1)\|_\infty\|\mathbf{K}(t_1,t_2)\|_\infty\cdots\|\mathbf{K}(t_n,\zeta)\|_\infty\,\mathrm{d}t_n\cdots\mathrm{d}t_1\end{aligned}$$

$$
\begin{aligned}
&\le M^{n+1} \int_{\zeta}^{x} \int_{\zeta}^{t_1} \cdots \int_{\zeta}^{t_{n-1}} \mathrm{d}t_n \cdots \mathrm{d}t_1 \\
&\le M^{n+1} \frac{(x-\zeta)^n}{n!}.
\end{aligned}
$$

By (2), it can be shown that

$$
\begin{aligned}
\mathrm{d}(\mathcal{K}^n \mathbf{U}, \mathcal{K}^n \mathbf{V}) &= \|\mathcal{K}^n \mathbf{U} - \mathcal{K}^n \mathbf{V}\|_\infty \\
&= \| \int_a^x \mathbf{K}_n(x,\zeta)[\mathbf{U}(\zeta) - \mathbf{V}(\zeta)]\mathrm{d}\zeta \, \|_\infty \\
&\le \int_a^x \|\mathbf{K}_n(x,\zeta)\|_\infty \|\mathbf{U}(\zeta) - \mathbf{V}(\zeta)\|_\infty \mathrm{d}\zeta \\
&\le \int_a^x \frac{M^n}{(n-1)!}(x-\zeta)^{n-1} \|\mathbf{U}(\zeta) - \mathbf{V}(\zeta)\|_\infty \mathrm{d}\zeta \\
&\le M^n \|\mathbf{U}(\zeta) - \mathbf{V}(\zeta)\|_\infty \int_a^x \frac{(x-\zeta)^{n-1}}{(n-1)!}\mathrm{d}\zeta \\
&\le M^n \frac{(b-a)^n}{n!} d(\mathbf{U}, \mathbf{V}) = \alpha_n \mathrm{d}(\mathbf{U}, \mathbf{V}),
\end{aligned}
$$

where $\alpha_n = \frac{M^n (b-a)^n}{n!}$. Since $\lim_{n \to \infty} \alpha_n = 0$, there exists $N_1 \in \mathbb{N}$ such that $\alpha_n < 1$, for any $n \ge N_1$ and $n \in \mathbb{N}$. So, the proof is complete.

3 Main results

Suppose N_1 is the smallest number in $\mathbb{N}$, such that $\alpha_{N_1} < 1$. Therefore $\mathcal{K}^{N_1}$ is a contraction. From now, Let $T := \mathcal{K}^{N_1}$.

Since $T : X \to X$ is a contraction mapping, then

$$
\mathrm{d}(T^m \mathbf{U}_1, T^m \mathbf{U}_2) \le K^m \mathrm{d}(\mathbf{U}_1, \mathbf{U}_2), \;\; m \ge 1,
$$

where $K := \frac{M^{N_1}(b-a)^{N_1}}{N_1!}$.
By the triangle inequality, we have

$$
\begin{aligned}
\mathrm{d}(\mathbf{U}_1, \mathbf{U}_2) &\le \mathrm{d}(\mathbf{U}_1, T\mathbf{U}_1) + d(T\mathbf{U}_1, T\mathbf{U}_2) + \mathrm{d}(\mathbf{U}_2, T\mathbf{U}_2) \\
&\le \mathrm{d}(\mathbf{U}_1, T\mathbf{U}_1) + K\mathrm{d}(\mathbf{U}_1, \mathbf{U}_2) + \mathrm{d}(\mathbf{U}_2, T\mathbf{U}_2),
\end{aligned}
$$

Thus

$$
\mathrm{d}(\mathbf{U}_1, \mathbf{U}_2) \le \frac{1}{1-K}(\mathrm{d}(\mathbf{U}_1, T\mathbf{U}_1) + \mathrm{d}(\mathbf{U}_2, T\mathbf{U}_2)). \tag{3}
$$

In particular, if $\mathbf{U}_1$ and $\mathbf{U}_2$ be the fixed points of T, we get $\mathrm{d}(\mathbf{U}_1, \mathbf{U}_2) = 0$, hence the contraction mapping T can have at most one fixed point. For any $\mathbf{U} \in X$, by substituting $\mathbf{U}_1 = T^n \mathbf{U}$ and $\mathbf{U}_2 = T^m \mathbf{U}$ in (3), we have

$$\begin{aligned} \mathrm{d}(T^n\mathbf{U}, T^m\mathbf{U}) &\le \frac{1}{1-K}\left(\mathrm{d}(T^n\mathbf{U}, T^n(T\mathbf{U})) + \mathrm{d}(T^m\mathbf{U}, T^m(T\mathbf{U}))\right) \\ &\le \frac{K^n + K^m}{1-K}\mathrm{d}(\mathbf{U}, T\mathbf{U}), \end{aligned}$$

and since $K < 1$, $K^n \to 0$, so $d(T^n\mathbf{U}, T^m\mathbf{U}) \to 0$ as n and m tend to infinity. Because X is a complete metric space, this Cauchy sequence converges to a point $\mathbf{U}^*$ of X, and this $\mathbf{U}^*$ is clearly a fixed point of T.

Stopping rule Now if we let m tends to infinity in the latter inequality, an important inequality is obtained as follows

$$\mathrm{d}(T^n\mathbf{U}, \mathbf{U}^*) \le \frac{K^n}{1-K}\mathrm{d}(\mathbf{U}, T\mathbf{U}). \tag{4}$$

To show the importance of the inequality (4), suppose we are going to reach an error of ϵ, i.e., instead of the actual fixed point $\mathbf{U}^*$ of T we will be satisfied with a point $\mathbf{U}_n$ satisfying $\mathrm{d}(\mathbf{U}_n, \mathbf{U}^*) < \epsilon$, and also suppose that we start our iteration with some point $\mathbf{U}_0$ in X. Since we want $\mathrm{d}(\mathbf{U}_n, \mathbf{U}^*) < \epsilon$, we just have to pick N_2 so large that $\frac{K^{N_2}}{1-K}\mathrm{d}(\mathbf{U}_0, \mathbf{U}_1) < \epsilon$. Now the quantity $\mathrm{d}(\mathbf{U}_0, \mathbf{U}_1)$ is something that we can compute after the first iteration and then by taking the log of the above inequality and solving for N_2 (remember that $\log(K)$ is negative), we can compute how large N_2 must be. The result is as follows:
If $\beta := \mathrm{d}(\mathbf{U}_0, \mathbf{U}_1)$ and

$$N_2 > \frac{\log(\epsilon) + \log(1-K) - \log(\beta)}{\log(K)}$$

then $\mathrm{d}(\mathbf{U}_{N_2}, \mathbf{U}^*) < \epsilon$. From a practical programming point of view, this inequality allows us to express our iterative algorithm with a "for loop" rather than a "while loop". Also it has another interesting interpretation. Suppose we take $\epsilon = 10^{-m}$ in our stopping rule inequality. What we see is that the growth of N_2 with m is a constant plus $\frac{m}{|\log(K)|}$, or in other words, to get one more decimal digit of precision we have to do (approximately) $\frac{1}{|\log(K)|}$ more iteration steps. From a different angle of view, if we need N_2 iteration steps to get m decimal digits of precision, then we need another N_2 iterations to double the precision to $2m$ digits.

Note Clearly, there is a reverse relation between N_1 and N_2. Thus, by increasing N_1, the parameter $N = N_1 \times N_2$ will be decreased.

4 Numerical examples

In this section, we present some examples of classical integral and functional equations which are particular cases of Eq. (1) and subsequently, for some initial guesses, the value of parameters have been calculated.

Example 4.1 (see [29]) For the first example, consider the following linear Volterra integral equation

$$u(x) = f(x) - \lambda \int_0^x \sin[A(x-t)]u(t)\mathrm{d}t \equiv \mathcal{K}(u). \quad (x \in [0, 1]) \tag{5}$$

Table 1 Numerical results for Example 1

u_0	β	K	N_1	N_2	N	$\|u^* - u_N\|_\infty$
$\cos(x)$	0.6379	0.0083	5	1	5	1.1498×10^{-7}
1	1.1678	0.0417	4	2	8	6.4433×10^{-11}
x	1	0.0083	5	1	5	1.3868×10^{-5}

Table 2 Numerical results for Example 2

$\mathbf{U}_0$	β	K	N_1	N_2	N	$\|\mathbf{U}^* - \mathbf{U}_N\|_\infty$
$\mathbf{F}(x)$	0.0813	0.0208	3	1	3	5.1175×10^{-7}
$\mathbf{x}$	0.2169	0.0208	3	1	3	5.42319×10^{-5}
$\mathbf{0}$	0.5156	0.1250	2	2	4	5.1175×10^{-7}

For $A(A+\lambda) > 0$, the exact solution is

$$u(x) = f(x) - \frac{A\lambda}{k} \int_0^x \sin[k(x-t)] f(t) \mathrm{d}t, \quad k = \sqrt{A(A+\lambda)}.$$

In particular, for $f(x) = \cos(x)$, $\lambda = 1$ and $A = 2$, this solution becomes $u^*(x) = 0.6\cos(x) + 0.4\cos(\sqrt{6}\,x)$. On the other hand, the operator $\mathcal{K}^{N_1}$ is a contraction mapping with contraction coefficient K. So, let $T := \mathcal{K}^{N_1}$. Now by taking $\epsilon = 10^{-2}$, we guess that after N iterative steps, $m = 2$ decimal digits of precision must be obtained. In Table 1, for some initial guesses u_0, the value of parameters are calculated.

Example 4.2 (see [2]) For the second example, consider the following system of linear Volterra integral equations in interval $x \in [0, \frac{1}{2}]$.

$$\begin{cases} u_1(x) = f_1(x) + \int_0^x (x^2 - t)(u_1(t) + u_2(t)) \mathrm{d}t, \\ u_2(x) = f_2(x) + \int_0^x x(u_1(t) + u_2(t)) \mathrm{d}t, \end{cases} \tag{6}$$

where $f_1(x) = -\frac{x^5}{3} - \frac{x^4}{4} + \frac{x^3}{3} + x$ and $f_2(x) = -\frac{x^4}{3} - \frac{x^3}{2} + x^2$. The exact solution is $\mathbf{U}^*(x) = \begin{pmatrix} x \\ x^2 \end{pmatrix}$. In Table 2, for $\epsilon = 10^{-2}$ and some initial guesses, $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ x \end{pmatrix}$ and $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the value of parameters are calculated.

5 Conclusions

In this paper, an iterative method for solving functional integral equations has been discussed. The proof of existence and uniqueness of the solution for systems of linear Volterra integral equations has been presented. From a practical programming point of view, an important inequality is proposed that allows us to express our iterative algorithm with a "for loop" rather than a "while loop". Moreover, in this paper, we have shown that to get one more decimal digit of precision we have to do (approximately) $\frac{1}{|\log(K)|}$ more iteration steps.

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