Finite Element Method for solving Linear Volterra Integro-Differential Equations of the second kind
Mortaza Gachpazan, Asghar Kerayechian, Hamed Zeidabadi
Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
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Abstract. In this paper, we present a method for numerical solution of linear Volterra integro-differential equations with boundary conditions. First, we obtain variational form of the problem, and then, finite element method and basis functions will be used. Also, the error analysis of the method is considered. Furthermore, the efficiency of the proposed method will be considered through numerical examples.

Keywords: Linear Volterra Integro-Differential Equations, Finite element method, Error estimation.

1. Introduction

Many authors have studied finite element methods for integral equations. See, Atkinson[2] , Ikebe [7], Nedelec [12], Sloan [16], and Wendland [19]. Adaptive finite element methods for integral equations have been considered more recently. See,[13,19].

Integro-differential equations have been discussed in many applied fields, such as biological, physical and engineering problems. They are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. There are several methods for solving integro-differential equations, Yanik and Fairweather in [20], used finite element methods for solving integro-differential equation of parabolic type and obtained an $O(h^{r+1} + (\Delta t)^i)$ order estimate for $L^2$ norm of the error.


In this paper, we use Lagrange polynomials with Finite element method to obtain an approximate solution of the problem. To illustrate the basic approach, we consider the following volterra integro-differential equation

$$-u'' + b(x)u'(x) + c(x)u(x) = f(x) + \int_a^x K(x,t)u(t)dtu(a) = 0, \quad u(b) = 0, \quad \Omega = [a,b]$$

We assume that $K(x,t)$ and $f(x)$ are continuous functions respect to their arguments, and $b(x)$ and $c(x)$ are nonnegative functions and belong to $C^1(\Omega)$. First, for using finite element method, by suitable linear transform, we convert the essential boundary condition to homogeneous one, and then we define

$$V = H^1_0(\Omega) = \{v \in H^1(\Omega), \quad v(a) = v(b) = 0\}$$

where $V$ is a Sobolev space together with following norm:

$$\|u\|^2 = \|u\|^2_{L^2(\Omega)} + \|u'\|^2_{L^2(\Omega)}.$$

For obtaining varational form, we let $B:V \times V \to R$ and $L:V \to R$ be bilinear form and linear functional, respectively.

The varational form of the problem is given as follows.
\[ B(u,v) = L(v), \quad \forall v \in V, \] 

where

\[
B(u, v) = \int_{\Omega} u'(x)v'(x)\,dx + \int_{\Omega} b(x)u'(x)v(x)\,dx + \int_{\Omega} c(x)u(x)v(x)\,dx - \int_{\Omega} v(x)\left(\int_{a}^{b} K(x,t)u(t)\,dt\right)\,dx \quad \in L^2(\Omega).
\]

Lemma 1.1 Let \( B \) be bilinear form defined by (3). If \( M_1 \leq c(x) \leq M_2 \) and \( P_1 \leq b(x) \leq P_2 \), then \( B \) is continuous.

Proof. For \( B \), we can write,

\[
|B(u,v)| \leq \|u\|_{H^1} \|v\|_{H^1} + P_2 \|u\|_{H^1} \|v\|_{H^1} + M_2 \|u\|_{H^1} \|v\|_{H^1} + K\|u\|_{H^1} \|v\|_{H^1} = (1+P_2+M_2+KR)\|u\|_{H^1} \|v\|_{H^1} = C\|u\|_{H^1} \|v\|_{H^1}.
\]

Using the Cauchy-Schwarz inequality and Sobolev norm, we have

\[
|B(u,v)| \leq \|u\|_{H^1} \|v\|_{H^1} + P_2 \|u\|_{H^1} \|v\|_{H^1} + M_2 \|u\|_{H^1} \|v\|_{H^1} + K\|u\|_{H^1} \|v\|_{H^1} = (1+P_2+M_2+KR)\|u\|_{H^1} \|v\|_{H^1} = C\|u\|_{H^1} \|v\|_{H^1}.
\]

which \( K = \max_{x \in \Omega} |K(x,t)|, \quad R = \|1\|_{L^2(\Omega)}^2 \) and \( C = 1+P_2+M_2+KR \). So \( B \) is continuous.

In addition of the hypothesis of lemma 1.1, suppose \( 0 \leq b'(x) \leq T_2 \). Now we consider the \( V \)-ellipticity of \( B \). For this purpose we write

\[
\int_{\Omega} v'(x)v'(x)\,dx + \int_{\Omega} c(x)v(x)v(x)\,dx \geq \int_{\Omega} v'^2(x)\,dx \geq \frac{1}{1+c} \|v\|_{H^1}^2,
\]

and

\[
\int_{\Omega} b(x)v'(x)v(x)\,dx = -\frac{1}{2} \int_{a}^{b} b'(x)(v(x))^2\,dx \geq \frac{T_2}{2} \int_{a}^{b} (v(x))^2\,dx \geq \frac{T_2}{2} \|v\|_{H^1}^2,
\]

also

\[
-\int_{\Omega} v(x)\left(\int_{a}^{b} K(x,t)v(t)\,dt\right)\,dx \geq -\int_{\Omega} v(x)\left(\int_{a}^{b} K(x,t)v(t)\,dt\right)\,dx \geq -KR \|v\|_{L^2}^2 \geq -KR \|v\|_{H^1}^2.
\]

By ((4)), ((5)), ((6)), we have

\[
B(v,v) \geq \left(\frac{1}{1+c} - \frac{T_2}{2} - KR\right) \|v\|_{H^1}^2,
\]

or

\[
B(v,v) \geq \alpha \|v\|_{H^1}^2,
\]

where \( \alpha = \left(\frac{1}{1+c} - \frac{T_2}{2} - KR\right) \), \( c \) is poincare's constant. So, the following lemma can be expressed.

Lemma 1.2 If \( \alpha > 0 \), \( B \) is \( V \)-elliptic.

By using Lax-Milgram theorem and lemmas 1.1, 1.2, the problem ((1)) has a unique solution.

2. Finite element method
Now, we explain how to solve the problem with finite element method. Since V is an infinite dimensional space, we choose a subspace of V with finite dimension and call it $V_h$. So the problem is converted to find $u_h \in V_h$ such that $B(u_h, v_h) = L(v_h), \ \forall v_h \in V_h$. We consider a set of basis continuous piecewise polynomials functions of degree at most $m$ such as $\{\phi_i\}_{i=1}^n$, which $V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$. Let $\{\Omega_{e}\}_{e=1}^M$ be a regular partition of $\Omega = [a,b]$. We choose $m+1$ nodes in each subinterval and denote nodes by $\{x_1, x_2, \ldots, x_N\}$. Corresponding to each node, we construct a basis function, such that satisfies the following properties:

i) $\phi_j(x_j) = \delta_j, \quad i, j = 1, 2, \ldots, N$

ii) $\phi_j |_{\Omega_e} = \psi_i^{(e)}(x), \quad \psi_i^{(e)}(x_j) = \delta_j, \quad i, j = 1, 2, \ldots, N$

where $\psi_i^{(e)}$ are called local functions. We can write $u_h(x)$ and $v_h(x)$ as a linear combinations of the basis functions of $V_h$, so we have

$$u_h(x) = \sum_{i=1}^{n} a_i \phi_i(x) \quad v_h(x) = \sum_{j=1}^{n} b_j \phi_j(x)$$

(9)

Hence, by substituting (9) in variational formulation of the problem, we have

$$\sum_{j=1}^{n} b_j \left( \sum_{i=1}^{n} a_i \int_{\Omega} \phi_i(x) \phi_j(x) dx + \int_{\Omega} b(x) \phi_i(x) \phi_j(x) dx + \int_{\Omega} c(x) \phi_i(x) \phi_j(x) dx \right) - \int_{\Omega} \phi_j(x) \left( \int_{a}^{b} K(x,t) \phi_i(t) dt \right) dx = 0$$

(10)

Since, the $b_j$'s are arbitrary, we have

$$\sum_{i=1}^{n} a_i \int_{\Omega} \phi_i(x) \phi_j(x) dx + \int_{\Omega} b(x) \phi_i(x) \phi_j(x) dx + \int_{\Omega} c(x) \phi_i(x) \phi_j(x) dx \right) - \int_{\Omega} \phi_j(x) \left( \int_{a}^{b} K(x,t) \phi_i(t) dt \right) dx = 0$$

(11)

Now, we define

$$C_{i,j} = \int_{\Omega} \phi_i(x) \phi_j(x) dx + \int_{\Omega} b(x) \phi_i(x) \phi_j(x) dx + \int_{\Omega} c(x) \phi_i(x) \phi_j(x) dx \right) - \int_{\Omega} \phi_j(x) \left( \int_{a}^{b} K(x,t) \phi_i(t) dt \right) dx \quad i, j = 1, 2, \ldots, n.$$ 

(12)

and

$$F_j = \int_{\Omega} f(x) \phi_j(x) dx \quad j = 1, 2, \ldots, n.$$ 

(13)

In this case, the following system is obtained

$$\sum_{i=1}^{n} C_{i,j} a_i = F_j \quad j = 1, 2, \ldots, n.$$ 

(14)

We assume, $a = [a_1, a_2, \ldots, a_n]^T$, $F = [F_1, F_2, \ldots, F_n]^T$ and $C = (C_{i,j})$, then system (14) can be written as $C a = F$. Let $C_{ij}^{(e)}$ and $F_j^{(e)}$ are restriction of $C_{ij}$ and $F_j$ respectively. So, we have

$$C = \sum_{e=1}^{n} C^{(e)}, \quad C^{(e)} = (C_{ij}^{(e)}) \quad i, j = 1, 2, \ldots, n.$$ 

(15)
Now, by solving the system \( C^T a = F \), the coefficients \( a_j \) is obtained, and with these coefficients, we can obtain the approximate solution.

3. Error Analysis

Suppose \( u \) is the exact solution of the problem and \( u_h \) be its approximate solution, then we have

\[
B(u, v_h) = l(v_h) \quad \forall v_h \in V_h, 
\]

and also we have

\[
B(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h. 
\]

If \( e = u - u_h \), then

\[
B(e, v_h) = 0 \quad \forall v_h \in V_h. 
\]

**Definition 3.1** Let \( V \) be a Hilbert space, and suppose \( B \) be a symmetric and \( V \)-elliptic bilinear form. We define an inner product as follows

\[
(.,.) : V \times V \rightarrow \mathbb{R} 
\]

\[
(u, v)_B = B(u, v) 
\]

which is called the inner product energy. Also we define energy norm as follows

\[
\|u\|_E^2 = (u, u)_B 
\]

By Schwarz inequality, we have the following relation between energy norm and inner product,

\[
|B(v, w)| \leq \|v\|_E \|w\|_E \quad \forall v, w \in V. 
\]

So, from ((18)) we obtain

\[
(e,v_h)_B = B(e,v_h) = 0. 
\]

Therefore \( e \) is orthogonal to each \( v_h \).

\[
(u - u_h, v_h) = 0 
\]

According to [4], we have the following theorem and lemma.

**Theorem 3.2** \( \|u - u_h\|_E = \min \{\|u - v_h\|_E \mid v_h \in V_h\} \).

**Lemma 3.3** (Cea’s Lemma)

Suppose \( V \) is a Hilbert space, and \( B \) is a continuous bilinear form and \( V \)-elliptic, and \( l \) is a continuous linear functional on \( V \). Then, there is a constant \( c \) independent of \( h \) such that

\[
\|u - u_h\|_V \leq c \inf_{v_h \in V_h} \|u - v_h\|_V. 
\]

**Definition 3.4** (projection operator)

Projection operators are defined as follows:

\[
\Pi : V \rightarrow V_h 
\]

\[
\Pi u = \bar{u}_h = \sum_{i=1}^n \bar{a}_i \phi_i(x) 
\]

In other words, each member of \( V \), correspond to its interpolated function by projection operator. Since for each particular \( \bar{v}_h \) in \( V_h \), we have

\[
\inf \|u - v_h\|_V \leq \|u - \bar{v}_h\|_V 
\]

for finding an upper bound for \( u - u_h \), we can take \( \bar{v}_h \) equal to \( \bar{u}_h \). Then

\[
\|u - u_h\|_V \leq c \|u - \bar{u}_h\|_V. 
\]
Therefore it is sufficient to get an upper bound for the interpolation error. If the bilinear form is symmetric, then from theorem 3.2, we have
\[ \| u - u_h \|_E = \min_{v_h \in V_h} \| u - v_h \|_E \]
then
\[ \alpha \| u - u_h \|_E^2 \leq B(u - u_h, u - u_h) = \| u - u_h \|_E^2 \Rightarrow \| u - u_h \|_E \leq \frac{1}{\sqrt{\alpha}} \| u - u_h \|_E, \] (21)
by continuity of \( B \), we have
\[ \| u - v_h \|_E^2 = B(u - v_h, u - v_h) \leq C \| u - v_h \|_V \| u - v_h \|_V \Rightarrow \| u - v_h \|_E \leq \sqrt{C} \| u - v_h \|_V \] (22)
and from ((21)) and ((22))
\[ \| u - u_h \|_V \leq \frac{\sqrt{C}}{\sqrt{\alpha}} \min_{v_h \in V_h} \| u - v_h \|_V. \] (23)
The above inequality is a special case of lemma 3.3. [4]

Let the basis functions be piecewise quadratic polynomial, we define
\[ E(x) = u(x) - \bar{u}_h(x) \] (interpolation error)
First, we examine the error on \( \Omega_e \), we have
\[ E(\varphi_1^{(e)}) = E(\varphi_2^{(e)}) = E(\varphi_3^{(e)}) = 0 \]
so, according to Rolle's theorem there exists \( \xi \in (\varphi_1^{(e)}, \varphi_2^{(e)}) \) and \( \xi \in (\varphi_2^{(e)}, \varphi_3^{(e)}) \), such that \( E'(\xi) = 0 \) and \( E'(\xi) = 0 \), and so there exists \( \eta \in (\varphi_1^{(e)}, \varphi_3^{(e)}) \), such that \( E''(\eta) = 0 \) and note that \( E''(x) = u''(x) - u''_h(x) \), then
\[ E''(x) = \int_\eta E'''(t) dt. \]
Since the polynomial interpolation is piecewise quadratic polynomial,
\[ |E''(x)| = |\int_\eta u'''(t) dt| \leq \int_\eta |u'''(t)| |dt| \leq \int_{\varphi_1^{(e)}}^{\varphi_3^{(e)}} |u'''(t)| |dt| \]
and by using the Cauchy - Schwarz inequality, we have
\[ |E''(x)| \leq \|1\|_2 \|u'''\|_2 \]
or
\[ |E''(x)| \leq h^{\frac{1}{2}} |u|_{H^3(\Omega_e)} \]
then
\[ \int_{\Omega_e} |E''(x)|^2 dx \leq h |u|_{H^3(\Omega_e)}^2 \int_{\Omega_e} dx \Rightarrow \|E''\|_{L^2(\Omega_e)}^2 = \|E''\|_{L^2(\Omega_e)}^2 \leq h^2 |u|_{H^3(\Omega_e)}^2 \]
Now, we can obtain an upper bound on \( \Omega \), as follows:
\[ \|E''\|_{L^2(\Omega)}^2 = \int_{\Omega} |E''(x)|^2 dx = \sum_{e=1}^M \int_{\Omega_e} |E''(x)|^2 dx = \sum_{e=1}^M |E''|_{L^2(\Omega_e)}^2 \leq h^2 \sum_{e=1}^M |u|_{H^3(\Omega_e)}^2 \]
\[ h^2 \sum_{e=1}^M |u|_{H^3(\Omega_e)}^2 = h^2 \sum_{e=1}^M \int_{\Omega_e} (u''(x))^2 dx = h^2 \int_{\Omega} (u''(x))^2 dx = h^2 \|u''\|_{H^3(\Omega)}^2 \]
thus
\[ |E(x)|_{H^3(\Omega)} \leq h |u|_{H^3(\Omega)} \] (24)
Similarly, we can write
And so, we can obtain
\[ \| E \|_{L^2(\Omega)} \leq h^2 \| E \|_{H^1(\Omega)} \]  
\hfill (25)
From ((24)), ((25)) and (26), we have
\[ \| E \|_{L^2(\Omega)} \leq h^2 \| E \|_{H^1(\Omega)} \]  
\hfill (26)
By using Sobolev norm, we have
\[ \| E \|_{H^1(\Omega)}^2 = \| E \|_{L^2(\Omega)}^2 + \| E \|_{H^1(\Omega)}^2 \leq h^6 \| u \|_{H^3(\Omega)}^2 + \| E \|_{H^1(\Omega)}^2 \]  
\hfill (27)
\[ \leq h^6 \| u \|_{H^3(\Omega)}^2 + h^4 \| u \|_{H^3(\Omega)}^2 \leq 2 h^4 \| u \|_{H^3(\Omega)}^2 \rightarrow \| E \|_{H^1(\Omega)} \leq \sqrt{2} h^2 \| u \|_{H^3(\Omega)} \]  
\hfill (28)

Upper bound for the interpolation error
Since the variational form has a unique solution, therefore \( \| u \|_{H^2(\Omega)} \) is a constant number.
However, according to Lemma 3.3
\[ \| u - u_h \|_V \leq \frac{C}{\alpha} \| u - \bar{u}_h \|_V \]
where \( C \) is the continuity constant and \( \alpha \) is the \( V \)-ellipticity constant. Then
\[ \| u - u_h \|_{H^1(\Omega)} \leq C_i \sqrt{2} h^2 \| u \|_{H^3(\Omega)} \quad C_i := \frac{C}{\alpha} \]
By the above error bounded, one can see that the order of method is \( O(h^2) \). As, we observe, since \( \| u \|_{H^3(\Omega)} \) is constant, the norm of error tends to zero as \( h \rightarrow 0 \), and the convergence of the method is demonstrated.

4. Numerical Examples

Example 4.1 Consider the following Volterra Integro-Differential Equation:
\[ -u''(x) + 4u(x) = -f(x) - \int_0^x (xt)u(t)dt, \quad u(0) = u(1) = 0 \]
where
\[ f(x) = \frac{x^3}{2} - \cosh(1) - \frac{x^2}{2} \sinh(2x - 1) + \frac{x}{4} \cosh(2x - 1) - \frac{x}{4} \cosh(1) + 4 \cosh(1) \]
and \( t, x \in [0,1] \), with the exact solution \( u(x) = \cosh(2x - 1) - \cosh(1) \).

For \( M = 20 \) and using polynomials of degree 2, exact and approximate values at some points are given in Table 4.1, and approximation error is shown in Figure 4.1.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0.1 )</th>
<th>( 0.3 )</th>
<th>( 0.5 )</th>
<th>( 0.7 )</th>
<th>( 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_h(x) )</td>
<td>-0.20564574</td>
<td>-0.46200834</td>
<td>-0.54308068</td>
<td>-0.46200832</td>
<td>-0.20564572</td>
</tr>
<tr>
<td>( u(x) )</td>
<td>-0.20564568</td>
<td>-0.46200826</td>
<td>-0.54308063</td>
<td>-0.46200826</td>
<td>-0.20564568</td>
</tr>
</tbody>
</table>
Example 4.2 Consider the following Volterra Integro-Differential Equation:

\[-y''(x) - 3y(x) = f(x) - \int_0^x \sin(x + t)y(t)\,dt \quad y(0) = y(1) = 0,\]

where

\[f(x) = 2 - 3x + 3x^2 + (x^2 - x - 2)\cos(2x) - (2x - 1)\sin(2x) - \sin(x) + 2\cos(x)\]

with exact solution \(y(x) = x^2 - x\). For \(M = 20\) and \(m = 2\) graph of error is shown in Figure 4.2. As it can be seen the maximum error is less than \(\frac{3}{2} \times 10^{-6}\).

Example 4.3 Let us consider the following linear Volterra Integro-Differential Equation

\[-y''(x) + \frac{\pi^2}{\cos(\frac{x}{\pi})}y' - \frac{1}{\pi^2}y - \int_0^x (xt + 1)y(t)\,dt = \pi(1 + x^2)\cos\left(\frac{x}{\pi}\right) - x\pi^2 \sin\left(\frac{x}{\pi}\right)\]

with boundary condition \(y(0) = 0\), \(y(1) = \sin(\frac{1}{\pi})a\) and the exact solution \(y(x) = \sin(\frac{x}{\pi})\). At first, we convert the boundary condition at \(x = 1\) such that to be homogeneous. For \(M = 30\) and \(m = 2\) graph of error is shown in Figure 4.3.
As Figure 4.3 shows the maximum error is less than $1 \times 10^{-8}$.

5. References


JIC email for contribution: editor@jic.org.uk


