An analytical model for vibration and control of a PR-PRP parallel robot with flexible platform and prismatic joint

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Abstract
An analytical solution for vibration of a parallel robot where its end-effector is flexible and has a passive prismatic joint(s) has not been presented before. In this research vibration analysis of a PR-PRP parallel robot using an analytical method is investigated. The PR-PRP parallel robot has two planar degrees of freedom and moves by means of two active prismatic joints. The robot moving platform is a flexible link with one passive revolute and one passive prismatic joint. First, the motion equation for a flexible link with the passive prismatic joint is developed. The motion equation is solved by employing an approximate analytical method called “constrained assumed modes method”. A time-variant constraint is written for the passive prismatic joint. The developed model allows for inclusion of the effect of physical length for the passive prismatic joint in contact with the moving platform on the vibration response of the system. For verification of the presented model and solution, three case studies are presented and results of the analytical solution are compared with results of a commercial finite element method software. For each case study, two different lengths for the passive prismatic joint are considered. Finally, active vibration control is performed for an applicable motion of the robot using the proportional-integral-derivative controller.

Keywords
Control, flexible, parallel manipulator, prismatic joint, time-variant constraint, vibration

1. Introduction
High accuracy, high speed and high payload capacity are some of the key characteristics which differentiate parallel robots from their serial counterparts. The stiffness per weight ratio of parallel robots is normally higher than serial robots. However, even with high stiffness, the need for high speed and high payload can result in undesirable vibration and consequently, a decrease in robot accuracy. Therefore, vibration analysis of parallel robots is an important problem. However, the vibration study of parallel robots is challenging and there are few studies presenting its analytical model or solution. This may be due to the existence of many links, joints and their complicated kinematics. Furthermore, when a prismatic joint is added to flexible parallel or series robots, dynamic modeling and the solution to the robot system become more difficult.

In this research, the authors focus on a parallel robot in which the flexible end-effector has a passive prismatic joint(s). The end-effector is assumed to be a flexible beam. Then, the motion of a flexible beam with a prismatic joint is the basic problem for this class of parallel robots. In this problem, the prismatic joint moves along the flexible beam. Therefore, the lengths of both sides of the flexible beam are time-variant. Consequently, boundary conditions of the beam are time-variant.

Many researchers have studied the motion of a flexible beam with a prismatic joint. For example, Tabarrok et al. (1974), Banerjee and Kane (1987),

A well organized and comprehensive literature review for flexible multi-body dynamics is presented by Shabana (1997). The most recent literature review for dynamic modeling of flexible manipulators is presented by Dwivedy and Eberhard (2006). Different four-bar mechanisms (Winfrey, 1971; Erdman et al., 1972; Cleghorn et al., 1981; Turcic and Midha, 1984; Yang and Sadler, 1990) and slider-crank mechanisms (Kitis, 1990; Wu et al., 1995; Shabana, 1996) were used by many researchers in order to investigate flexible multi-body formulations. However, not many flexible parallel robots were considered in the flexible multi-body formulations. Five-bar mechanisms are also simple parallel robots which have been studied for the control of undesired vibrations (Madani and Moallem, 2011; Junfeng et al., 2012). Ubertini (2000) presents an inventive iterative numerical method for inverse kinematics of flexible robots. Fattah et al. (1994a) present the direct kinematics of a spatial flexible parallel robot (3-RRS). In another reference, Fattah et al. (1994b) present the modeling of a planar parallel robot with a flexible link and use the potential energy of an Euler-Bernoulli beam element for “planar beam-shaped flexible links”. Using this method, the effect of link flexibility is included in the motion equations. Fattah et al. (1995) presented a dynamics model of the 3-RRS with a rigid moving platform. In their FEM model, the potential energy of Euler-Bernoulli beam elements for flexible links, as previously employed, are used. Additionally, a natural orthogonal complement method is used to eliminate the constraint forces and to derive the minimum number of equations of motion, which are then solved using FEM. Zeng et al. (2010) also perform the dynamic modeling of a 3-RRS parallel robot with flexible links. Ibrahimbegovic and Mamouri (2000) present a nonlinear dynamic model for flexible systems based on FEM and apply it to mechanisms with only one flexible link. This method is rather complicated and may be difficult to apply to parallel robots. This research also presents a symbolic model for a prismatic joint but does not present a numerical example for the prismatic joint. Bauchau (2000) also uses the FEM to model prismatic joints used in flexible multi-body systems.

Most researchers have used the FEM for dynamic modeling of flexible parallel robots. In general, when the FEM is used to obtain flexibility, the elastic potential energy of each element is first calculated and then summed together. Zhou et al. (2006) present the vibrational modeling of a 3-PRS manipulator with flexible links using the FEM. In Wang and Mills (2006), the dynamic modeling of a 3-PRR robot is performed employing a subtracting approach based on the FEM. Zhaocai and Yueqing (2008) present an approximation method for inverse dynamics of flexible parallel robots using FEM. This research employs Kineto-Elastodynamics theory and Timoshenko beam theory to derive the essential equations. It uses rigid body driving forces to obtain flexible case driving forces; therefore, its inverse method is not precise and contains some errors. Abedi et al. (2008) perform dynamic modeling of a flexible two-link series robot using the FEM.

Some researchers have investigated the vibration control of flexible parallel robots. Kang and Mills (2002) present the dynamic model of a three degree-of-freedom flexible planar 3-PRR parallel robot with a rigid platform. Lagrange multipliers are used in dynamic equations, and active vibration control is performed using piezoelectric (PZT) actuators. The modeling presented is rather simple as the two ends of the flexible links are each attached to two revolute joints. Consequently, there is not any prismatic joint on the flexible links. In Piras et al. (2005), the authors perform the dynamic analysis of the same robot as used in Kang and Mills (2002) by using the FEM and investigate the effect of high-speed motion on the deflection of the robot. Wang and Mills (2005) performed FEM dynamic modeling and active vibration control of the 3-PRR robot. Using the 3-PRR flexible parallel robot and a similar method as used by Kang and Mills (2002), researchers Zhang et al. (2008) present a reduced order and an analytical linearized model that enables real-time control. They also perform active control based on strain rate feedback and show that the deflections caused by structural free vibration are mostly eliminated, but that the deflections caused by the inertial forces are not reduced. Zhang et al. (2010) also perform multi-mode vibration control and position error analysis for the 3-PRP flexible parallel robot. Bottega et al. (2009) perform trajectory control for series robots with flexible non-prismatic links using PZT sensors and actuators; they also derive the motion equations using Lagrange’s method and solve the equations via the FEM. Recently Zhang et al. (2013) presented trajectory tracking and vibration suppression of a 3-PRR parallel with a triangular base.

Most researchers have not considered the effect of mass or moment of inertia of the prismatic joint in dynamic modeling. Stoenescu and Marghitu (2004) investigate the moment of inertia effect for the
prismatic joint in rigid planar kinematic chains using Lagrange's equations. They show that the moment of inertia effect for a prismatic joint can be significant at high speeds.

In the authors' opinion, it seems that no significant progress or variation has been presented as the solution method for the vibration analysis of flexible parallel robots. However, more recently, Faris et al. (2009) presented the mathematical model of a two-link series robot using the extended Hamilton's principle. Then, the authors designed an optimal trajectory for the robot by employing a genetic algorithm. Shanzen et al. (2010) derive dynamic equations of the 3-RRS robot as used by Fattah et al. (1994) and solve the equations by employing the Newmark numerical method. In an effort to present a new computationally efficient analytical method, Celentano and Coppola (2011) offer a method based on the assumed modes method (AMM) and obtain a simpler analytical model for flexible series robots. They consider an analytic expression of the kinetic energy for the effector and show that the computational efficiency is significantly increased. The authors of this paper recently considered the vibration of a two-link robot with two prismatic joints. The first vertical rigid link is attached to an actuated prismatic joint and a second horizontal flexible link, moving in axial direction, is attached to a passive prismatic joint. They employed the use of a body coordinate system which resulted in a rather simpler form of the dynamic model. The derived analytical solution simultaneously solves direct/inverse dynamics problem and obtains the vibration response of the system, (Sharifinia and Akbarzadeh, 2014). Some researchers have considered the joint flexibility in addition to link flexibility (Al-Bedoor and Almusallam, 2000).

Many researchers, such as Hwang and Haug (1990), Sugiyama et al. (2003) and Lee et al. (2008), have presented several formulations for joint constraints. Although Hwang and Haug (1990) used modal coordinates for dynamic modeling, in more recent studies, joint constraint equations are mostly used in FEMs such as the floating frame of reference formulation (Shabana, 2005) and absolute nodal coordinate formulation (Sugiyama et al., 2003). Generalized coordinates used in dynamic modeling can create complexities in the joint constraint equations and nonlinearity of the motion equations (Sugiyama et al., 2003). When a body coordinate system and an elastic coordinate system are used for the dynamic modeling of a prismatic joint, geometric nonlinearities appear because of interference of these two coordinate systems. Additionally, modal coordinates are mostly used for problems with constant boundary conditions. Therefore, it seems that there is a challenge for employing the prismatic joint constraints in an assumed modes method and other mesh-free methods.

Considering the body of the available research on the flexibility of parallel manipulators, it may be concluded that in the past works:

1. Derived dynamic equations of flexible robots are complicated and therefore, most researchers used the FEM or numerical methods to obtain the solution. Therefore, any simplification of the dynamic equations which does not reduce the generality of the problem can potentially lead into obtaining better analytical solutions.

2. Inverse dynamics of a rigid robot is first obtained and then used for its direct dynamics.

3. There are very few existing studies on flexible robots with prismatic joints. There is no existing study where the end-effector is flexible and has a passive prismatic joint(s).

4. When analytical methods are presented, the motion equation of the flexible link on the two sides of the prismatic joint is solved separately.

5. The effect of actual length of the prismatic joint on the vibration response is not considered.

6. It seems that the prismatic joint constraints are not used in mesh-free methods such as the assumed modes method.

7. Vibration control is often considered for the flexible links not sliding through a prismatic joint.

The present study aims to offer certain advantages for each of the above seven shortcomings. To do this, the authors' previous study on vibration of a flexible link with a prismatic joint (Sharifinia and Akbarzadeh, 2014) is extended by considering the additional effect of rotational motion for the flexible link as well as the dynamic effect of the prismatic joint length. Next, vibration analysis of a PR-PRP parallel robot is investigated using an analytical method. The goal is to present a method which can be extended to study the analytical solution of more complicated flexible parallel robots having prismatic joint(s).

The rest of this paper is organized as follows. In Section 2, a dynamic model of the PR-PRP parallel robot is presented and a new motion equation for the flexible link with a prismatic joint is developed. In Section 3, the solution of the motion equation and vibration analysis is presented using an analytical method. In Section 4, numerical results for three case studies are presented. While including the length of the prismatic joint, the vibration response of the flexible link, right and left sides of the prismatic joint, is simultaneously calculated. Finally, active vibration control is performed for an applicable motion of the robot using the proportional-integral-derivative (PID) controller.
2. Dynamical model

2.1. Robot structure and assumptions

The PR-PRP parallel robot is a two degrees-of-freedom robot which moves by means of two prismatic actuators. The robot’s end-effector is a flexible link to which a passive revolute joint and a passive prismatic joint are attached. The PR-PRP parallel robot is shown in Figure 1. The robot has two actuated prismatic joints that are attached to two vertical rigid links and move in Y-direction. Motions of the two rigid links cause motion of the third flexible link, called moving platform.

Consider Figure 2(a). A passive prismatic joint located at point B moves along the flexible link. The passive prismatic joint is joined to the first rigid link by a revolute joint. A second rigid link is joined to one end of the flexible beam, point A, by a revolute joint. To focus on the vibration behavior of the flexible link, the mass of the two rigid links is assumed to be zero. Robot inputs, vertical displacements $z_1(t)$ and $z_2(t)$, are applied to the two rigid links which cause a change in the position of working point of the robot, point C. The flexible end-effector, ABC, is modeled as a beam with three boundary conditions at points A (revolute joint), B (prismatic joint) and C (free end). Simple-free boundary conditions may be used for the two ends, A and C. However, this treatment does not allow us to determine constraint force at point A. Therefore, instead of using the simple boundary condition at point A, a free boundary condition plus a constraint of zero vertical displacement are used. This treatment allows us to calculate the constraint force of this point using a Lagrange multiplier. To model the passive prismatic joint, a constraint is considered at point B that can move along the beam.

The robot operates on vertical $XZ$ plane. It is assumed that the motion of link ABC starts at zero
initial conditions. The origin of a coordinate system, \( x_w \), called "rigid body coordinate system" is attached to the end-effector at the point A. The direction of its x-axis coincides with the direction of line AB. Therefore, \( x_w \) has the same motion as rigid motion of the end-effector. As a result, deflection or vibration of the flexible link is measured in the \( x_w \) coordinate system. The flexible link is assumed to be an Euler-Bernoulli beam. Therefore, shear deformation effects, rotational inertia and axial deformation are neglected.

2.2. Acceleration analysis of the rigid robot

Rigid acceleration components of a beam element are shown in Figure 3.

Each element of the beam with distance \( x \) from the origin of \( x_w \) coordinate system, has a rigid body motion with the acceleration components of \( a_x(x, t) \) and \( a_w(x, t) \) in \( x \) and \( w \) directions, respectively. Considering the gravity effects, components of the gravity acceleration, \( g \), are also added to \( a_x(x, t) \) and \( a_w(x, t) \), therefore, the following can be written

\[ a_x(x, t) = a_A \sin(\beta) - x\dot{\beta}^2 + g \sin(\beta) = \dot{z}_2(t) \sin(\beta) - x\dot{\beta}^2 + g \sin(\beta) \]  

\[ a_w(x, t) = a_A \cos(\beta) + x\dot{\beta} + g \cos(\beta) = \ddot{z}_2(t) \cos(\beta) + x\dot{\beta} + g \cos(\beta) \]  

As can be seen from equations (1) and (2), the acceleration components of the rigid body motion for each beam element are functions of time and the distance \( x \). Consider Figure 2(a). For angle \( \beta \), the following can be written

\[ \beta = \tan^{-1}\left(\frac{z_1(t) - z_2(t)}{d}\right) \]  

2.3. Motion equation in differential form

Unlike the previous studies in which a fixed coordinate system is placed on the fixed prismatic joint, the present study employs the rigid body coordinate system, \( x_w \), which has the same motion as the rigid motion of the end-effector. The new treatment for the \( x_w \) causes the time derivative of element distance to be zero (\( \frac{dx}{dt} = 0 \)) and total acceleration of the beam element in \( w \)-direction can be written as

\[ a_w(x, t) + \frac{\ddot{d}w}{dt^2} - \dot{\beta}^2 w = a_w(x, t) + \left(\frac{\ddot{d}w}{dt^2} + 2\frac{\dot{d}x}{dt} \frac{\dot{d}w}{dt} + \frac{\partial}{\partial x} \left(\frac{\ddot{d}w}{dt^2} + \frac{\partial^2 d}{\partial x^2} \frac{\partial w}{\partial x}\right)\right) \]

\[ - \dot{\beta}^2 w = a_w(x, t) + \left(\frac{\ddot{d}w}{dt^2} - \dot{\beta}^2 w\right) \]  

in which \( -\dot{\beta}^2 w \) is acceleration of the beam element in \( w \)-direction due to the elastic motion of the beam and rotational motion of the rigid body coordinate system, \( \dot{\beta} \), and \( \ddot{d}w/dt^2 \) represents acceleration of the beam element in \( w \)-direction due to the elastic motion of the beam in the rigid body coordinate system, \( x_w \), and \( a_w(x, t) \) is...
acceleration of the beam element in w-direction due to the rigid motion of the beam. The treatment for the rigid body coordinate system will result in a simpler differential equation. Consider Figure 2(b). Parameters \( T, V, M, \theta \) and \( L \) represent axial force, shear force, bending moment, slope angle of \( w \) curve and the beam length, respectively. Assume small values for angle \( \theta \). In \( w \)-direction using Newton’s law for a free body diagram of the beam element there follows

\[
\rho \Delta x \left( \dot{a}_w(x, t) + \frac{\partial^2 w}{\partial t^2} - \ddot{\beta}w \right) = - \left( \frac{\partial V}{\partial x} \right) \Delta x + \frac{\partial \theta}{\partial x} \Delta x \tag{5}
\]

Additionally, in the \( x \)-direction there can be written

\[
\rho \Delta x \left( \dot{a}_x(x, t) - 2\dot{\beta} \frac{\partial w}{\partial t} - \ddot{\beta}w \right) - \frac{\partial T}{\partial x} \Delta x \tag{6}
\]

in which \(-2\dot{\beta}(\partial w/\partial t) - \ddot{\beta}w\) is the acceleration of the beam element in \( x \)-direction due to the elastic motion of the beam and rotational motion of the rigid body coordinate system, \( \beta \), and \( \dot{a}_x(x, t) \) is the acceleration of the beam element in \( x \)-direction due to rigid motion of the beam. Parameter \( \rho \) is the mass per length of the beam. Beam deformation and rotational inertia effects are assumed to be negligible. Therefore, \( V = \partial M/\partial x \) can be written. Next, consider Figure 2(c). In \( x \)-direction, Newton’s law for part of the beam that is to the right side of the beam element can be written as

\[
\rho(L-x) \left( a_x(x, t) \right) - \left( \frac{L-x}{2} \right) \ddot{\beta}w - 2\dot{\beta} \frac{\partial w}{\partial t} \left( \frac{x+L}{2}, t \right) \tag{7}
\]

and defined as

\[
a_c^0(x, t) = a_x(x, t) - \left( \frac{L-x}{2} \right) \ddot{\beta}w \tag{8}
\]

in which \( a_c^0(x, t) \) represents axial acceleration for point \( D_x \) center of mass of the part of the beam that is to the right side of the beam element, due to the rigid motion, Figure 2(c). Also for the Euler-Bernoulli beam, there follows

\[
M = EI \frac{\partial^2 w}{\partial x^2}, \quad V = \frac{\partial M}{\partial x}, \quad \theta = \frac{\partial w}{\partial x} \tag{9}
\]

Using equations (4) through (9), there can be written

\[
\rho \left( \dot{a}_w(x, t) + \frac{\partial^2 w}{\partial t^2} - \ddot{\beta}w \right) + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right)
\]

\[
+ \frac{\partial}{\partial x} \left( \rho(L-x) \left( a_c^0(x, t) - 2\dot{\beta} \frac{\partial w}{\partial t} \left( \frac{x+L}{2}, t \right) \right) \right)
\]

\[
- \ddot{\beta}w \left( \frac{x+L}{2}, t \right) \frac{\partial w}{\partial x} \right) = 0 \tag{10}
\]

If the beam has no rotational and vertical motion then \( \dot{\beta} = 0, a_n(x, t) = 0 \) and \( a_c^0(x, t) = a_n(0, t) \) therefore by expanding equation (10), there is

\[
\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) - \rho a_n(0, t) \frac{\partial w}{\partial x} + \rho(L-x)a_n(0, t) \frac{\partial^2 w}{\partial x^2} = 0 \tag{10-1}
\]

This differential equation (10–1) is similar to the differential equations that other researchers used for the axially moving beam, (Tabarrok et al., 1974; Banerjee and Kane, 1987; Wang and Wei, 1987). However, beam length, \( L \), is constant in the above equation, which simplifies the differential equation.

2.4. Motion equation in virtual form

In what follows, the proposed equation (10) is further developed to include the effect of rigid motion and the reaction forces of the prismatic joint. Consider Figure 3. The prismatic joint at \( x = x_p(t) \) lies on the beam. The entire length of the flexible beam is considered as the domain of the differential equation of motion. A constraint force, \( F_p \), instead of the prismatic joint, is applied to the beam. Using D’Alembert’s principle, virtual work of vertical inertia force, \(-\rho a_n(x,t)\delta w(x)\), can be added to the right side of the differential equation (10). Additionally, virtual work of the constraint forces can be written as \( F_p^v \delta w(0) \) and \( F_p^v \delta w(x_p) \). These terms as external forces effects can also be added to the right side of the motion equation. Therefore, equation (10) can now be re-written as

\[
\rho \left( \frac{\partial^2 w}{\partial t^2} - \ddot{\beta}w \right) + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) + \frac{\partial}{\partial x} \left( \rho(L-x) \left( a_c^0(x, t) \right) \right)
\]

\[
- 2\dot{\beta} \frac{\partial w}{\partial t} \left( \frac{x+L}{2}, t \right) \frac{\partial w}{\partial x} \right) \delta w(x)
\]

\[
= F_p^v \delta w(0) + F_p^v \delta w(x_p) \delta w(x_p)
\]

\[
- \rho a_n(x, t) \delta w(x) \tag{11}
\]

in which \( \delta w \) returns the Dirac delta function. The proposed motion equation (11) has several advantages when compared with previous studies. First, equation (11) simultaneously considers the length of a flexible beam on the left and right sides of its prismatic joint. Secondly, equation (11) is rather simpler than the motion equation of previous studies. Note that in
equation (11), the parameter $L$ is no longer time-variant. Additionally, some extra terms shown in equation (4) do not exist in equation (11).

By substituting equations (1), (2) and (3) into equation (11), acceleration components caused by the rigid body motion of the beam element enter the motion equation. These acceleration components are reasons for the vibrations of the flexible end-effector.

Note that, if desired, the axial constraint force of the revolute joint, $F_A^x$, can be calculated using the axial acceleration of mass center of the beam, $a_c(L/2, t)$ shown in equation (8).

$$F_A^x = \rho L a_c(L/2, t) = \rho L (\ddot{z}_2(t) \sin(\beta) - \dot{\beta}^2 L/2 + g \sin(\beta))$$

(12)

### 2.5. Boundary conditions and selected mode shapes

All boundary conditions of the problem at the points A, B and C are as follows

$$w(0, t) = 0, \frac{\partial w}{\partial x}(0, t) = 0 \quad \text{for point A, revolute joint}$$

(13a)

$$w(x_p(t), t) = 0 \quad \text{for point B, prismatic joint}$$

(13b)

$$\frac{\partial^2 w}{\partial x^2}(L, t) = 0, \frac{\partial^3 w}{\partial x^3}(L, t) = 0 \quad \text{for point C, free end}$$

(13c)

Some of the boundary conditions in the equation (13) are automatically satisfied due to the type of the selected assumed modes. For satisfying the remaining boundary condition equations, they can be imposed as geometric constraints. Additionally, geometric constraints allow constraint forces to be calculated as the Lagrange multipliers. For example, if simple-free modes are used, boundary conditions of equations (13a) and (13c) are automatically satisfied. Because $w(0, t) = 0$ and then $\delta w(0) = 0$, the constraint force $F_A^x$ is eliminated in the motion equation (11). However, the boundary condition of equation (13b) is not automatically satisfied and is therefore imposed as a geometric constraint. Because $w(x_p(t), t) \neq 0$ and then $\delta w(x_p) \neq 0$, the constraint force $F_p$ is not eliminated in the motion equation and can consequently be calculated. However, it is desirable to calculate all joint forces and therefore displacement boundary conditions at joints A and B must be imposed as geometric constraints. If free-free modes are used, except $w(0, t) = 0$ and $w(x_p(t), t) = 0$, the other boundary conditions of equation (13) are automatically satisfied. On the other hand, if free-free mode shapes are used, the virtual displacements of $\delta w(0)$ and $\delta w(x_p)$ will not be zero in equation (11) and therefore their coefficients, $F_A^x$ and $F_p$, will not be eliminated from the motion equation and can also be calculated. Consequently, free-free mode shapes are used instead of the simple-free mode shapes.

The procedure mentioned above is a common method in multi-body dynamics for calculation of constraint forces. However, in flexible multi-body dynamics, it is mostly used for FEMs (Sugiyama et al., 2003; Lee et al., 2008). In the authors’ opinion, this procedure can be further extended to the assumed modes method and consequently suitable for use in more complicated problems.

Using the free-free mode shapes and imposing the following geometric constraints, there can be written

$$\begin{align*}
  w(0, t) &= 0 \\
  w(x_p(t), t) &= 0
\end{align*}$$

(14)

To include the effect of the prismatic joint length, the number of constraint equations at neighborhood $x = x_p(t)$ must be increased. Assume the length of the prismatic joint is $L_p$, then it will be necessary to add the following constraints to the constraint equation (14).

$$\begin{align*}
  w \left(x_p(t) + \frac{L_p}{2}, t\right) &= -\frac{L_p}{2} \frac{\partial w}{\partial x}(x_p(t), t) \\
  w \left(x_p(t) - \frac{L_p}{2}, t\right) &= -\frac{L_p}{2} \frac{\partial w}{\partial x}(x_p(t), t)
\end{align*}$$

(15)

The additional geometrical constraints, equation (15), require the addition of constraint forces $F_p^R$ and $F_p^L$ to the motion equation. $F_p^R$ and $F_p^L$ are forces applied to the beam by the prismatic joint at right and left edges of the prismatic joint. Their corresponding virtual works as $F_p^R \text{dirac}(x - x_p - L_p/2) \times \delta w(x_p + L_p/2)$ and $F_p^L \text{dirac}(x - x_p + L_p/2) \times \delta w(x_p - L_p/2)$ will need to be added to the right side of the motion equation (11).

### 3. The solution method and vibration analysis

In order to solve the motion equation (11), a “constrained assumed modes method” is utilized (Sharifinia and Akbarzadeh, 2014). In the assumed modes method, each of the assumed mode shapes must satisfy all the geometrical boundary conditions. However, using “constrained assumed modes method”, the assumed mode shapes do not each satisfy the geometrical boundary conditions of the point where
the prismatic joint is located, point B. Instead, by writing additional constraint equations at point B, the combination of the assumed modes will satisfy the geometrical boundary condition(s) at location of the prismatic joint. Consider Figure 2(a). A beam with free-free boundary conditions at its two ends is constrained by a revolute joint at \( x = 0 \) and a prismatic joint at \( x = x_p(t) \). Therefore, mode-shapes of the free-free beam are used as assumed modes. The characteristic equation or frequency equation of the free-free beam is as follows

\[
\begin{align*}
\cos(k\pi)\cosh(k\pi) &= 1 \\
k_1 &= k_2 = 0, \quad k_3 = 1.5056, \quad k_4 = 2.4998, \\
k_5 &= 3.5000, \ldots, \quad k_n = n - 1.5
\end{align*}
\]

(16)

The first two rigid modes and other flexible modes are

\[
\begin{align*}
\varphi_1(x) &= 1/\sqrt{L} \\
\varphi_2(x) &= \sqrt{2x}/L \\
\varphi_n(x) &= A_n (\sin(\beta_n x) + \sinh(\beta_n x)) \\
&\quad + B_n (\cos(\beta_n x) + \cosh(\beta_n x)) \quad n > 2
\end{align*}
\]

(17)

in which \( \beta_n = k_n \pi/L \) and parameters \( A_n \) and \( B_n \) are obtained by using the mode normalization. Using assumed modes, the vibration response of the beam can be written as

\[
w(x, t) = \sum_{n=1}^{N} \alpha_n(t) \varphi_n(x)
\]

(18)

Therefore

\[
\delta w(x, t) = \sum_{n=1}^{N} \psi_n(x) \delta \alpha_n(t)
\]

(19)

\[
\delta w(x_p, t) = \sum_{n=1}^{N} \psi_n(x_p) \delta \alpha_n(t)
\]

(20)

Using eigen value characteristic for the assumed modes, there is

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{EI}{\rho} \frac{d^4 \varphi_n}{dx^4} = \lambda_n \varphi_n \\
\lambda_n = \omega^2_n = \frac{EI}{\rho} \beta_n^4
\end{array} \right.
\]

(21)

Substituting equations (18) through (21) in equation (11):

\[
\begin{align*}
&\left( \alpha_n(x, t) + \sum_{n=1}^{N} \varphi_n(x) (\ddot{\alpha}_n(t) - \beta^2 \alpha_n(t)) \right) \left( \sum_{m=1}^{N} \psi_m(x) \delta \alpha_m(t) \right) \\
&+ \left( \sum_{n=1}^{N} \lambda_n \varphi_n(x) \alpha_n(t) \right) \\
&+ \frac{\partial}{\partial x} \left( (L-x) \alpha_n^2(x, t) \sum_{n=1}^{N} \varphi_n^2(x) \alpha_n(t) \right) \psi_m(x) \\
&- \frac{\partial}{\partial x} \left( (L-x) \sum_{j=1}^{N} \sum_{n=1}^{N} \varphi_n \left( \frac{x + L}{2} \right) \psi_j^2(x) \right)
\end{align*}
\]

(22)

According to equation (18), using time-variant coefficients \( \alpha_n(t) \), the beam vibration response is written as a linear composition of assumed modes \( \varphi_n(x) \). However, the four constraints, equations (14) and (15), cause four of the \( N \) coefficients, \( \alpha_n(t) \), to be dependent on the remaining coefficients. Equation (22) can be classified in terms of the coefficients \( \alpha_n(t) \). The constraint forces \( F_p, F_p^R, F_p^L \) and \( F_p^A \) in equation (22) vary until the multiplier expression for each of the four dependent \( \alpha_n(t) \) become zero. Therefore, from equation (22), there are \( N \) independent equations

\[
\begin{align*}
&\left( \alpha_n(x, t) + \sum_{n=1}^{N} \varphi_n(x) (\ddot{\alpha}_n(t) - \beta^2 \alpha_n(t)) \right) \psi_m(x) \\
&+ \left( \sum_{n=1}^{N} \lambda_n \varphi_n(x) \alpha_n(t) \right) \\
&+ \frac{\partial}{\partial x} \left( (L-x) \alpha_n^2(x, t) \sum_{n=1}^{N} \varphi_n^2(x) \alpha_n(t) \right) \psi_m(x) \\
&- \frac{\partial}{\partial x} \left( (L-x) \sum_{j=1}^{N} \sum_{n=1}^{N} \varphi_n \left( \frac{x + L}{2} \right) \psi_j^2(x) \right)
\end{align*}
\]
\[ \times \left( 2 \ddot{\alpha}_m(t) + \beta \dot{\alpha}_m(t) \right) \phi_m(x) - \frac{F_w}{\rho} \text{dirac}(x - 0) \phi_m(0) \]
\[ - \frac{F_p}{\rho} \text{dirac}(x - x_p) \phi_m(x_p) - \frac{F_R}{\rho} \text{dirac} \left( x - x_p - \frac{L_p}{2} \right) \phi_m \]
\[ \times \left( x_p + \frac{L_p}{2} \right) - \frac{F_l}{\rho} \text{dirac} \left( x - x_p + \frac{L_p}{2} \right) \phi_m \left( x_p - \frac{L_p}{2} \right) = 0 \quad m = 1, 2, 3, \ldots, N \]

The above \( N \) equations must be integrated along the beam length and consequently, transform to equations that are only a function of time, as follows

\[ \ddot{\alpha}_m(t) + \left( \lambda_m - \beta^2 \right) \alpha_m(t) + \int_0^L \left( \alpha_m(x, t) \phi_m(x) \right) dx \]
\[ + \sum_{n=1}^N \alpha_n(t) \int_0^L \frac{\partial}{\partial x} \left( (L - x) \alpha_n(x, t) \phi_n(x) \right) \phi_n(x) dx \]
\[ - \sum_{n=1}^N \sum_{j=1}^N \left( 2 \beta \dot{\alpha}_n(t) + \beta \ddot{\alpha}_n(t) \right) \alpha_j(t) \]
\[ \times \int_0^L \frac{\partial}{\partial x} \left( (L - x) \phi_n(x) \left( \frac{L}{2} \right) \phi_j(x) \right) \phi_n(x) dx \]
\[ - \frac{F_w}{\rho} \phi_m(0) - \frac{F_p}{\rho} \phi_m(x_p) - \frac{F_R}{\rho} \phi_m \left( x_p + \frac{L_p}{2} \right) \]
\[ - \frac{F_l}{\rho} \phi_m \left( x_p - \frac{L_p}{2} \right) = 0 \quad m = 1, 2, 3, \ldots, N \]

(24)

Substituting equation (18) into equations (14) and (15), results in the following constraint equations

\[ \sum_{n=1}^N \alpha_n(t) \phi_n(0) = 0 \]

(25)

\[ \sum_{n=1}^N \alpha_n(t) \phi_n(x_p(t)) = 0 \]

(26)

\[ \sum_{n=1}^N \alpha_n(t) \phi_n \left( x_p(t) + \frac{L_p}{2} \right) = \frac{L_p}{2} \alpha_n(t) \frac{\partial \phi_n}{\partial x} (x_p(t)) \]

(27)

\[ \sum_{n=1}^N \alpha_n(t) \phi_n \left( x_p(t) - \frac{L_p}{2} \right) = -\frac{L_p}{2} \alpha_n(t) \frac{\partial \phi_n}{\partial x} (x_p(t)) \]

(28)

According to the authors’ analytical modeling, there are \( N + 4 \) equations, which consist of \( N \) motion equations in the form of ordinary differential equations in equation (24) plus four algebraic constraint equations, equations (25) through (28). Additionally, there are known parameters \( F_w^m, F_p^m, F_l^m, F_R^m \) and \( \alpha_m(t) \) in which \( m = 1, 2, 3, \ldots, N \). By finding unknown parameters \( \alpha_m(t) \), the lateral vibration response, \( w \), can be obtained from equation (18).

4. Numerical results

4.1. Verification with a commercial finite element software

In this section, three numerical case studies are presented for the PR-PRP parallel robot. The analytical results are verified using results of a commercial FEM software. Consider Figure 1(b). The numerical parameters are: \( E = 200 \times 10^3 \text{N/m}^2 \), \( I = 1/12 \times 0.03 \times (0.0012)^3 \text{m}^4 \), \( \rho = 7800 \times (0.03 \times 0.0012) \text{kg/m} \), \( L = 0.5 \text{m} \) and \( d = 0.18 \text{m} \). Three case studies, each with a unique end-effector trajectory, are considered for the robot. The three case studies result in three different types of motion, translational, pure rotational and general plane motion for the end-effector. Specifically, the tip of the end-effector will follow a straight line, a circular arc and an elliptical arc. Considering the three actuators input and Figure 4, it can be seen that the tip of the end-effector robot, point C, starts at position \( X_C(0) = L, Z_C(0) = 0 \) with initial zero velocity. Additionally, to investigate the effect of the prismatic joint length on the vibration response, two different lengths for the passive prismatic joint are selected, \( L_p = 0.5 \text{cm} \) and \( L_p = 3 \text{cm} \). The end-effector motion is generated by motion of the two actuated prismatic joints as shown in Table 1.

![Figure 4. Path of the end-effector working point in XZ plane.](image-url)
The path of the end-effector working point, point C, in the XZ plane for the three case studies is shown in Figure 4.

The vibration responses of the robot’s working point for the three case studies are shown in Figure 5. For each of the case studies, the vibration response of the developed analytical model is compared with the response of commercial FEM software. As shown in Figure 5, the curves closely follow each other, which demonstrate the correctness of the analytical formulation. For the FEM solution, the number of Euler-Bernoulli beam elements used is 100. For the analytical solution, in all three case studies, the number of assumed modes is \( N = 50 \). Note that when the shorter prismatic joint length, \( L_p = 0.5 \) cm, is used, the analytical solution treats it as a point contact and consequently, the constraint equations (25) and (26) are used. However, with the longer prismatic joint length, \( L_p = 3.0 \) cm, all four constraint equations (25) through (28) are used. Figure 5 also shows the effect of the two prismatic joint lengths on the vibration response. The effect of joint length is clear for the first

<table>
<thead>
<tr>
<th>Case study</th>
<th>Input: actuator 1, ( z_1(t)(m) )</th>
<th>Input: actuator 2, ( z_2(t)(m) )</th>
<th>Prismatic joint length ( L_p ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case study 1</td>
<td>0.3t^2</td>
<td>0.3t^2</td>
<td>(a) ( L_p = 0.5 )</td>
</tr>
<tr>
<td>Case study 2</td>
<td>0.3t^2</td>
<td>0</td>
<td>(b) ( L_p = 3.0 )</td>
</tr>
<tr>
<td>Case study 3</td>
<td>( \frac{1}{20\pi} (1 - \cos(2\pi t)) )</td>
<td>( \frac{1}{20\pi} (1 - \cos(2\pi t)) )</td>
<td>(c) ( L_p )</td>
</tr>
</tbody>
</table>

**Figure 5.** Vibration response of the end-effector end point for \( L_p = 0.5 \) cm and \( L_p = 3.0 \) cm; (a) Case study 1; (b) Case study 2; (c) Case study 3.
two studies. As shown in Figure 5, with the increase in joint length, vibration amplitude is decreased and its frequency is increased. This is because an increase in the number of the constraint results in an increase of system stiffness. In most physical systems, the length of the prismatic joint is commonly significant and its effect must therefore be considered. For the three case studies, the vertical constraint force of the revolute joint, $F^w_A$, and the constraint force of the prismatic joint, $F_p$, are shown in Figures 6 and 7, respectively and compared with response of the FEM software. Note that for the cases where $L_p = 3\text{ cm}$, to account for increased prismatic length, instead of $F_p$, the combined sum of $F_p + F_p^R + F_p^L$ is shown in Figure 7. As shown, the length of the prismatic joint mostly affects the frequency of the constraint force. As demonstrated, the presented analytical method in this study allows modeling the dynamic effect of the prismatic joint length on the vibration response of the system.

As shown in this section, the numerical results obtained by the analytical model and commercial FEM software closely follow each other. All the material properties, geometric conditions, input trajectories and loads have been same in the FEM software and the analytical model. Also both the FEM software and the analytical model use the Euler-Bernoulli theory. Additionally, variations of the dynamical loads are so that the assumption of small deflections of the robot in analytical model is valid. Therefore, it is naturally expected for FEM and AMM solutions to closely follow each other. The minor difference between the FEM and AMM solutions is because of number of the finite elements and of the assumed modes. Note that in the FEM, the number of the finite elements for obtaining the best solution has an optimum value. On the other hand in AMM, an additional number of assumed modes results in improved solution accuracy. Therefore, the very small difference between the FEM and AMM results demonstrates the correctness of the analytical model.

4.2. Active vibration control

Structural flexibility causes undesirable vibration and inaccuracy in positioning for mechanisms and robots. Vibrations can be attenuated using the smart materials. In this section, a circular trajectory with high acceleration is considered for the PR-PRP robot and active
vibration control is implemented using PZT patches. Using this motion (case study 4), the robot’s working point, point C, reciprocates on a circular path as follows

\[
X_c(t) = -0.1 \cos(\pi \cos(\pi t)) + 0.34 \\
Z_c(t) = +0.1 \sin(\pi \cos(\pi t))
\] (29)

This circular path and start configuration of the robot are shown in Figure 8. Using the trajectory, equation (29), the robot completes a full circle during 1 second. It then repeats the same circular path in a reverse direction during another 1 second. From equation (29), it is obvious that velocity of the point C at the start and end point of the circular path is zero. Simulation time is assumed to be 4 seconds. Using the inverse kinematics for this robot, any desired path within the robot workspace is produced when the driving actuators move as follows

\[
z_2(t) = Z_c(t) + \sqrt{L^2 - X_c(t)^2} \\
z_1(t) = \frac{(Z_c(t) - z_2(t))d}{X_c(t)} + z_2(t)
\] (30)

Three PZT actuators and two accelerometers are used for active vibration control of the robot. Determining the number and location of PZT actuators and accelerometers as well as control gains are complicated problems when multiple vibration modes need to be controlled. Optimization methods are usually used...
for solving such problems. However, using the advanced control or optimization methods is not the purpose of the authors of this paper. Therefore, no rigorous attempts are made to optimize the stated parameters for this part of the study. As shown in Figure 9, accelerometer 2 measures the acceleration of point 2, the center of the PZT actuator 2, and provides feedback for the PZT actuator 2. Accelerometer 1 measures the acceleration of point 1, working point of the robot, and provides feedback for the PZT actuator 1 and 3. Therefore, for simplicity in indexing for formulation, the authors call point 3 to coincide with the point 1. Two independent PID controllers are used to control the output of the three PZT actuators 1, 2 and 3, respectively. Lateral elastic displacements of the measured points can be calculated using their measured accelerations.

The virtual work of the PZT actuators are added to right side of the motion equation (11) as

\[
\delta w_{PZT} = k_e V_{e}(t) \sum_{n=1}^{N} (\psi_n(x^n_R) - \psi_n(x^n_I)) \delta x_n(t) \quad \text{for } i = 1, 2, 3
\]

in which index \(i\) denotes the number of the PZT actuator. Parameters \(x^n_R\) and \(x^n_I\) represent \(x\)-coordinate of right and left end of PZT actuator \(i\), respectively. Parameter \(V_e(t)\) is the control voltage applied to the PZT actuator \(i\). Parameter \(k_e\) is a positive constant which depends on properties of the PZT actuators and the beam as follows

\[
k_e = -d_{31} \frac{E h b^2 b_p}{E h b + E_p h_p b_p} \quad \text{for } i = 1, 2, 3
\]

in which \(d_{31} = -270 \times 10^{-12} \text{C/N}\) is the PZT constant, \(E = 200 \times 10^9 \text{N/m}^2\) and \(E_p = 63 \times 10^9 \text{N/m}^2\) are the Young’s modulus, \(h = 0.03 \text{m}\) and \(b_p = 0.025 \text{m}\) the widths, \(h_p = 0.025 \text{m}\) the thicknesses of the beam and PZT actuators, respectively.

Therefore, using equation (32), \(k_1 = k_2 = k_3 = 0.00048384\). The vertical elastic displacement of measured point \(i\) is shown with \(w_i = w(x_i, t)\), where \(x_i\) is the \(x\)-coordinate of the measured, sensed, point \(i\). The control voltage applied to the PZT actuators is as follows

\[
V_i(t) = k_P w_i + k_I \int_0^t w_i \, dt + k_D \frac{d}{dt}(w_i)
\]

\[
= k_P \left( \sum_{n=1}^{N} \alpha_n(t) \psi_n(x_i) \right) + k_I \int_0^t \left( \sum_{n=1}^{N} \alpha_n(t) \psi_n(x_i) \right) \, dt + k_D \frac{d}{dt} \left( \sum_{n=1}^{N} \alpha_n(t) \psi_n(x_i) \right)
\]

\[
= k_P \left( \sum_{n=1}^{N} \alpha_n(t) \psi_n(x_i) \right) + k_I \int_0^t \alpha_n(t) \, dt + k_D \frac{d}{dt} \left( \sum_{n=1}^{N} \alpha_n(t) \psi_n(x_i) \right)
\]

for \(i = 1, 2, 3\) (33)

Substituting \(V_i(t)\) from equation (33) in equation (31), there is

\[
\delta w_{PZT} = k_i \left[ \sum_{n=1}^{N} \psi_n(x_i) \left( k_P \alpha_n + k_I \int_0^t \alpha_n(t) \, dt + k_D \frac{d}{dt} \alpha_n(t) \right) \right] \times \left( \sum_{n=1}^{N} \left( \psi_n(x_i^R) - \psi_n(x_i^I) \right) \right) \delta x_n(t) \quad \text{(34)}
\]

For the active vibration control in case study 4, the authors use \(L_p = 3\text{ cm}\) and set the control gains \(k_1k_1 = -20 \text{ N}, k_2k_2 = 200 \text{ N}, k_3k_3 = -20 \text{ N}\) and \(k_1k_1 = -3000 \text{ N/s}, k_2k_2 = 10000 \text{ N/s}, k_3k_3 = -3000 \text{ N/s}\) and \(k_1k_1 = -0.6 \text{ N/s}, k_2k_2 = 8 \text{ N/s}, k_3k_3 = -0.6 \text{ N/s}\). Vibration responses of robot’s working point with and without control are shown in Figure 10. As shown in this figure, structural free-vibration, which has higher frequency, is completely attenuated. However, forced vibration, which has lower frequency,
Figure 10. Vibration responses of robot’s working point with and without control.

Figure 11. Amount of voltages applied to the piezoelectric (PZT) actuators.

Figure 12. Vertical constraint force of the revolute joint, $F_{A}^w$.

Figure 13. Constraint force of the prismatic joint, $F_p + F_{p}^R + F_{p}^L$. 
is not completely attenuated at some times during the motion. More advanced control algorithms such as adaptive or robust control methods can be used for further attenuation of the forced vibration.

The voltages applied to the PZT actuators are shown in Figure 11. These voltages are important in order to reduce consuming control power and to reduce maximum voltage applied to the PZT actuators. To decrease value of the voltages used in Figure 11, a higher bending stiffness must be designed for the flexible link of the robot. However, additional material for the robot increases the required input power. Therefore, considering the limitation of the PZT actuators voltage, an optimum stiffness of the flexible link can create optimum values for the input and control powers. This is an optimizing problem which can be solved using optimizing procedures. The vertical constraint force of the revolute joint, \( F_R^w \), and constraint force of the prismatic joint, \( F_p^L + F_p^R + F_p^p \), are shown in Figures 12 and 13, respectively.

5. Conclusion

In this research, motion equation of a flexible link with a prismatic joint was developed. An analytical model was presented for vibration of the PR-PRP parallel robot. Motion equation of the analytical model was solved using an approximate analytical method called the “constrained assumed modes method”. Using this method combination of the assumed modes satisfied the geometrical boundary conditions at the prismatic joint.

Both the dynamic modeling, as well as the solution method, contributes by: (1) Considering the effect of the prismatic joint length. It was shown that if the prismatic joint length is considerable, stiffness increases and therefore vibration amplitude decreases while the frequency increases; (2) Simultaneously obtaining the vibration response on both sides of the flexible link; (3) Allowing the time-variant boundary conditions of the prismatic joint to be modeled as time-variant geometrical constraints; (4) Allowing the addition of an arbitrary number of kinematic constraints to the motion equations. Consequently, the approximate analytical solution can approximate the response of the actual physical systems with acceptable accuracy; (5) Allowing constraint forces at the time-variant and time-invariant boundaries to be obtained; (6) Allowing active vibration control for a flexible link sliding through a prismatic joint.

The presented analytical method may potentially be employed in modeling and obtaining an approximate solution for the vibration of more complex parallel robots with a flexible moving platform having prismatic joint(s). The authors are on the final stages of extending the present study to obtain the vibration response of a 3-PRP parallel robot.

The main contributions of this paper are: (1) A new motion equation is developed for a PR-PRP parallel robot having a flexible moving platform with a prismatic joint as well as presenting its approximate analytical solution; (2) For the first time, using an analytical model, the dynamic effect of the prismatic joint length on the vibration response of a flexible link is investigated. Additional contributions are: (3) Both the dynamic modeling as well as the solution method allows simultaneous consideration of the length of a flexible beam on the left and right side of its prismatic joint; (4) The prismatic joint constraints are used in a mesh-free method, assumed modes method, and constraint forces are obtained for the inverse dynamics; (5) Active vibration control is presented for a flexible link sliding through a prismatic joint.

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