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A GENERALIZATION ON THE CONJUGATE GRAPH

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ABSTRACT. Let G be a finite group. In this paper we introduce the generalized conjugate graph $\Gamma^c_{(G,n)}$ which is a graph whose vertices are, all the non-central subsets of G with n elements and two distinct vertices X and Y join by an edge if $X = Y^g$ for some $g \in G$. We present a condition under which two generalized conjugate graph are isomorphic. Moreover, this graph is a key to define the probability that two subsets of the group G with the same cardinality be conjugate.

1. INTRODUCTION

Recently, joining branches of group theory and graph theory together is one the most interesting topics. Erfanian et al. introduced conjugate graph Γ^c_G associated to a non-abelian group G with vertex set $G \setminus Z(G)$ such that two distinct vertices are adjacent if they are conjugate. The graph theoretical properties such as planarity, regularity

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and completeness of the conjugate graph are verified (see [2] for details). The idea of defining such a graph was obtained from the work of Blackburn et al., whom presented $P^c(G)$ as the probability that two elements of the group are conjugate (see [1]). By these facts in mind, we define a graph associated to a finite group G, with vertex set $\{X \subseteq G : |X| = n, X \nsubseteq Z(G)\}$ such that two distinct vertices X and Y join by an edge if there exists an element $g \in G$ with $X = Y^g$. We denote it by $\Gamma^c_{(G,n)}$ and call it, the generalized conjugate graph. If we put $n = 1$, then $\Gamma^c_{(G,1)}$ is known as the conjugate graph.
We discuss about some preliminary results of generalized conjugate graph. We try to combine graph theory with probability. The notion $P^c(G, n)$, which is the probability that two subsets of the group G with the same cardinality n are conjugate, is defined. By use of this probability we present a formula for the number of edges of the generalized conjugate graph. Upper and lower bounds are obtained for $P^c(G, n)$. Furthermore, we found an upper bound for $P^c(G, n)$ by use of the energy of $\Gamma^c_{(G,n)}$.

2. Main results

Let us start with the following definition.

Definition 2.1. Let G be a finite group. We define the generalized conjugate graph with vertex set $V(\Gamma^c_{(G,n)}) = \{X \subseteq G : |X| = n, X \nsubseteq Z(G)\}$ such that two distinct vertices X and Y join by an edge if there exists an element $g \in G$ such that $X = Y^g$.

It is clear that when G is abelian then $\Gamma^c_{(G,n)}$ is a null graph for all $n \geq 1$, so we may always assume that G is non-abelian. If $n = 1$, then $\Gamma^c_{(G,1)}$ is coincide to the known conjugate graph as denoted by Γ^c_G in [2]. Now, assume that K_G is the set of all subsets of G with n elements. Define the action of G on K_G by $(A, g) \mapsto A^g := g^{-1}Ag$, for all A in K_G and $g \in G$. If A_i^G is the orbit of A_i in K_G and $K(G)$ is the number of orbits then one can easily see that

$$K(G) = \binom{|Z(G)|}{n} + r,$$

where r is the number of orbits which have more than one element.
It is obvious

$$|E(\Gamma^c_{(G,n)})| = \sum_{i=1}^{r} \binom{|A_i^G|}{2}.$$

Since $A_i \nsubseteq Z(G)$ so $[G : G_{A_i}] \geq 2$ for all $1 \leq i \leq r$. Thus the following lower bound can be deduced.

It is clear that, if $\Gamma^c_{(G,n)}$ has t components, then its complement $\overline{\Gamma^c_{(G,n)}}$ is complete t-partite.

Proposition 2.2. *Let G be a group. Then*

(i) $\mathrm{diam}(\overline{\Gamma^c_{(G,n)}}) = 2$.
(ii) $\mathrm{girth}(\overline{\Gamma^c_{(G,n)}}) = 3$ *or* 4, *where* $t > 2$ *or* $t = 2$ *respectively.*
(iii) $\chi(\overline{\Gamma^c_{(G,n)}}) = \omega(\overline{\Gamma^c_{(G,n)}}) = t$.
(iv) *Let* $\overline{\Gamma^c_{(G,n)}} = K_{r_1,\cdots,r_t}$, *where* $r_1 \leq r_2 \leq \cdots \leq r_t$. *If* $r_1 + \cdots + r_{t-1} \geq r_t$, *then* $\overline{\Gamma^c_{(G,n)}}$ *is Hamiltonian.*

Theorem 2.3. *If* $\Gamma^c_{(G,n)}$ *has* t *complete components,* $K_{r_1}, \cdots, K_{r_t}$, *then* $\mathcal{E}(\Gamma^c_{(G,n)}) = 2(r_1 + r_2 + \cdots + r_t) - 2t$.

The energy of the graph Γ is the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph, which is denoted by $\mathcal{E}(\Gamma)$. The graph Γ of order n whose energy satisfies $\mathcal{E}(\Gamma) > 2(n-1)$ is called hyper-energetic and otherwise nonhyper-energetic. Clearly $\Gamma^c_{(G,n)}$ is a nonhyper-energetic. Recall that Γ is an integral graph whenever all eigenvalues of its adjacency matrix are integer. Obviously $\Gamma^c_{(G,n)}$ is an integral graph.

Theorem 2.4. *Let* $\Gamma^c_{(G,n)}$ *be generalized conjugate graph with complete components* K_{r_i}, $1 \leq i \leq t$. *Then the number of spanning forests of* $\Gamma^c_{(G,n)}$ *is* $\prod_{i=1}^{t} r_i^{r_i-2}$.

3. Generalized conjugate graph and probability

Blackburn in [1] introduced the probability that a pair of elements of a finite group are conjugate. We generalized it as follows.

Definition 3.1. Let G be a finite group. We define the probability that two sets with the same cardinality are conjugate by the following ratio,

$$P^c(G,n) = \frac{|\{(X,Y) \in K_G \times K_G : X = Y^g\}|}{|K_G|^2},$$

where K_G is the set of all n-element subsets of G.

It is clear that if K_G is the set of all singletons of the group G then $P^c(G,n)$ corresponds to the probability which was defined by Blackburn. Consider the set $A = \{(X,Y) \in K_G \times K_G : X = Y^g\}$. Now by use of $P^c(G,n)$ we can obtain the number of edges of $\Gamma^c_{(G,n)}$. We have

$$\begin{aligned} |K_G|^2 P^c(G,n) &= |A| = |K_G| + |\{(X,Y) \in K_G \times K_G : X = Y^g, X \neq Y\}| \\ &= |K_G| + 2|E(\Gamma^c_{(G,n)})|, \end{aligned}$$

where $|E(\Gamma^c_{(G,n)})|$ denote the number of edges of the graph and $|K_G| = \binom{|G|}{n}$. Therefore we have

$$|E(\Gamma^c_{(G,n)})| = \frac{|K_G|(|K_G|P^c(G,n) - 1)}{2}. \tag{3.1}$$

Theorem 3.2. *Let $\Gamma^c_{(G,n)}$ be the generalized conjugate graph associated to the group G. Then*

$$P^c(G,n) \leq \frac{\frac{1}{2}\mathcal{E}(\Gamma^c_{(G,n)}) + |K_G|}{|K_G|^2}$$

Proposition 3.3. *Let G be a finite group. Then*

$$P^c(G,n) < \frac{|Z(G)|!(|G|-n)!}{|G|!(|Z(G)|-n)!}$$

By mimicking the proof of Theorem 1.2 in [1] we conclude the following theorem.

Theorem 3.4. *If G and H are two isoclinic groups, then $|K_G|P^c(G,n) = |L_H|P^c(H,n)$, where K_G and L_H is the set of all n element sets of G and H respectively.*

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