A GENERALIZATION ON THE CONJUGATE GRAPH

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ABSTRACT. Let $G$ be a finite group. In this paper we introduce
the generalized conjugate graph $\Gamma^m_{G,n}$ which is a graph whose
vertices are, all the non-central subsets of $G$ with $n$ elements and
two distinct vertices $X$ and $Y$ join by an edge if $X = Y^g$ for
some $g \in G$. We present a condition under which two generalized
conjugate graph are isomorphic. Moreover, this graph is a key to
define the probability that two subsets of the group $G$ with the
same cardinality be conjugate.

1. INTRODUCTION

Recently, joining branches of group theory and graph theory together
is one the most interesting topics. Erfanian et al. introduced conjugate
graph $\Gamma^m_{G}$ associated to a non-abelian group $G$ with vertex set
$G \setminus Z(G)$ such that two distinct vertices are adjacent if they are con-
jugate. The graph theoretical properties such as planarity, regularity

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and completeness of the conjugate graph are verified (see [2] for details). The idea of defining such a graph was obtained from the work of Blackburn et al., whom presented $P^*(G)$ as the probability that two elements of the group are conjugate (see [1]). By these facts in mind, we define a graph associated to a finite group $G$, with vertex set \( \{X \subseteq G : |X| = n, X \nsubseteq Z(G)\} \) such that two distinct vertices $X$ and $Y$ join by an edge if there exists an element $g \in G$ with $X = Y^g$. We denote it by $\Gamma^c_{(G,n)}$ and call it, the generalized conjugate graph. If we put $n = 1$, then $\Gamma^c_{(G,1)}$ is known as the conjugate graph.

We discuss about some preliminary results of generalized conjugate graph. We try to combine graph theory with probability. The notion $P^c(G,n)$, which is the probability that two subsets of the group $G$ with the same cardinality $n$ are conjugate, is defined. By use of this probability we present a formula for the number of edges of the generalized conjugate graph. Upper and lower bounds are obtained for $P^c(G,n)$. Furthermore, we found an upper bound for $P^c(G,n)$ by use of the energy of $\Gamma^c_{(G,n)}$.

2. Main Results

Let us start with the following definition.

**Definition 2.1.** Let $G$ be a finite group. We define the generalized conjugate graph with vertex set $V(\Gamma^c_{(G,n)}) = \{X \subseteq G : |X| = n, X \nsubseteq Z(G)\}$ such that two distinct vertices $X$ and $Y$ join by an edge if there exists an element $g \in G$ such that $X = Y^g$.

It is clear that when $G$ is abelian then $\Gamma^c_{(G,n)}$ is a null graph for all $n \geq 1$, so we may always assume that $G$ is non-abelian. If $n = 1$, then $\Gamma^c_{(G,1)}$ is coincide to the known conjugate graph as denoted by $\Gamma_G$ in [2].

Now, assume that $K_G$ is the set of all subsets of $G$ with $n$ elements. Define the action of $G$ on $K_G$ by $(A, g) \mapsto A^g := g^{-1}Ag$, for all $A$ in $K_G$ and $g \in G$. If $A_i^G$ is the orbit of $A_i$ in $K_G$ and $K(G)$ is the number of orbits then one can easily see that

$$K(G) = \left(\frac{|Z(G)|}{n}\right) + r,$$

where $r$ is the number of orbits which have more than one element.

It is obvious

$$|E(\Gamma^c_{(G,n)})| = \sum_{i=1}^{r} \left(\frac{|A_i^G|}{2}\right).$$

Since $A_i \nsubseteq Z(G)$ so $|G : G_A| \geq 2$ for all $1 \leq i \leq r$. Thus the following lower bound can be deduced.
It is clear that, if $\Gamma^c_{(G,n)}$ has $t$ components, then its complement $\overline{\Gamma^c_{(G,n)}}$ is complete $t$-partite.

**Proposition 2.2.** Let $G$ be a group. Then

(i) $\text{diam}(\overline{\Gamma^c_{(G,n)}}) = 2$.
(ii) $\text{girth}(\overline{\Gamma^c_{(G,n)}}) = 3$ or 4, where $t > 2$ or $t = 2$ respectively.
(iii) $\chi(\overline{\Gamma^c_{(G,n)}}) = \omega(\overline{\Gamma^c_{(G,n)}}) = t$.
(iv) Let $\overline{\Gamma^c_{(G,n)}} = K_{r_1, \ldots, r_t}$, where $r_1 \leq r_2 \leq \cdots \leq r_t$. If $r_1 + \cdots + r_{t-1} \geq r_t$, then $\overline{\Gamma^c_{(G,n)}}$ is Hamiltonian.

**Theorem 2.3.** If $\Gamma^c_{(G,n)}$ has $t$ complete components, $K_{r_1}, \ldots, K_{r_t}$, then $E(\Gamma^c_{(G,n)}) = 2(r_1 + r_2 + \cdots + r_t) - 2t$.

The energy of the graph $\Gamma$ is the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph, which is denoted by $E(\Gamma)$. The graph $\Gamma$ of order $n$ whose energy satisfies $E(\Gamma) > 2(n-1)$ is called hyper-energetic and otherwise nonhyper-energetic. Clearly $\Gamma^c_{(G,n)}$ is a nonhyper-energetic. Recall that $\Gamma$ is an integral graph whenever all eigenvalues of its adjacency matrix are integer. Obviously $\Gamma^c_{(G,n)}$ is an integral graph.

**Theorem 2.4.** Let $\Gamma^c_{(G,n)}$ be generalized conjugate graph with complete components $K_{r_i}, 1 \leq i \leq t$. Then the number of spanning forests of $\Gamma^c_{(G,n)}$ is $\prod_{i=1}^{t} r_i^{r_i-2}$.

3. GENERALIZED CONJUGATE GRAPH AND PROBABILITY

Blackburn in [1] introduced the probability that a pair of elements of a finite group are conjugate. We generalized it as follows.

**Definition 3.1.** Let $G$ be a finite group. We define the probability that two sets with the same cardinality are conjugate by the following ratio,

$$P^c(G, n) = \frac{|\{(X, Y) \in K_G \times K_G : X = Y^g\}|}{|K_G|^2},$$

where $K_G$ is the set of all $n$-element subsets of $G$.

It is clear that if $K_G$ is the set of all singletons of the group $G$ then $P^c(G, n)$ corresponds to the probability which was defined by Blackburn. Consider the set $A = \{(X, Y) \in K_G \times K_G : X = Y^g\}$. Now by use of $P^c(G, n)$ we can obtain the number of edges of $\Gamma^c_{(G,n)}$. We
have
\[ |K_G|^2 P^c(G, n) = |A| = |K_G| + |\{(X, Y) \in K_G \times K_G : X = Y^g, X \neq Y\}| \]
\[ = |K_G| + 2|E(\Gamma^c_{G, n})|, \]
where \(|E(\Gamma^c_{G, n})|\) denote the number of edges of the graph and \(|K_G| = \binom{|G|}{n}\). Therefore we have
\[ |E(\Gamma^c_{G, n})| = \frac{|K_G|(|K_G|P^c(G, n) - 1)}{2}. \tag{3.1} \]

**Theorem 3.2.** Let \(\Gamma^c_{G, n}\) be the generalized conjugate graph associated to the group \(G\). Then
\[ P^c(G, n) \leq \frac{\frac{1}{2}E(\Gamma^c_{G, n}) + |K_G|}{|K_G|^2}. \]

**Proposition 3.3.** Let \(G\) be a finite group. Then
\[ P^c(G, n) \leq \frac{|Z(G)|!(|G| - n)!}{|G|!(|Z(G)| - n)!}. \]

By mimicking the proof of Theorem 1.2 in [1] we conclude the following theorem.

**Theorem 3.4.** If \(G\) and \(H\) are two isoclinic groups, then \(|K_G|P^c(G, n) = |L_H|P^c(H, n)|\), where \(K_G\) and \(L_H\) is the set of all \(n\) element sets of \(G\) and \(H\) respectively.

**References**
