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A new approach for solving a class of fuzzy optimal control systems under generalized Hukuhara differentiability

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Abstract

This paper investigates linear time varying fuzzy controlled systems with fuzzy boundary conditions, and fuzzy optimal control systems where the boundary conditions are described by fuzzy numbers. We use the α-cut sets of fuzzy system and Heaviside functions as well to find the solution of so-called fuzzy systems, while we profit the generalized Hukuhara differentiability concept. Three theorems are proved for the richness of the theory. Finally three numerical examples are given to verify the reliability and efficiency of the proposed approach.

1. Introduction

Fuzzy differential equations (FDEs) form a suitable setting for mathematical modeling of many real-world problems. There are several approaches to study an FDE, the first and the most popular approach is using Hukuhara derivative [8]. But this approach has a drawback: the solution becomes fuzzier as time goes on (see [6,3]). Hence, the fuzzy solution behaves quite differently from the crisp solution. Hüllermeier [7] interpreted a fuzzy differential equation as a

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family of differential inclusions. The main shortcoming of this approach is in not using fuzzy derivative, and so reduces the fuzziness effective.

Another approach can be found in [4] where the authors have used the extension principle in order to extend crisp differential equation. Bede and Gal [3] overcame the above-mentioned shortcoming under strongly generalized differentiability. But the disadvantage of this approach is that derivative does not always exists. So a new derivative, named generalized Hukuhara derivative, introduced by Stefanini in [19], Bede and Stefanini [1].

Nieto et al. in [17] considered fuzzy systems under strongly generalized differentiability. In [9] sufficient conditions for the global existence of a unique (2)-solution to an initial value problem for fuzzy functional differential equations were provided. Nieto and Rodríguez-López in [16] solved explicitly an impulsive linear fuzzy differential equation subject to boundary value conditions. Recently a new technique based on the quasi-level-wise system for solving a class of fuzzy linear differential equation was presented in [13].

Mazandarani and Kamyad in [12] defined the Caputo-type fuzzy fractional derivatives and applied the modified fractional Euler method for solving fuzzy fractional initial value problems. Mazandarani and Najariyan in [10,11] introduced respectively Type 2 fuzzy derivative and Type 2 fuzzy fractional derivative.

Fuzzy control systems have been implemented in many industrial applications, since uncertainly is inherited most dynamic systems. Nevertheless, there are few systematic procedures available for analysis and design of such dynamical systems.

Diamond and Kloeden [6] showed the existence of fuzzy optimal control for the system $\dot{\tilde{x}}(t)=a(t)\odot\tilde{x}(t)\oplus\tilde{u}(t)$, $\tilde{x}(0)=\tilde{x}_0$, where admissible pair $\tilde{p}=[\tilde{x}(.),\tilde{u}(.)]$ is nonempty compact interval-valued function on $\mathcal{F}(\mathbb{R})$. Diamond and Kloeden interpreted a fuzzy differential equation as a family of differential inclusions $\dot{x}^\alpha(t)\in a(t)x^\alpha(t)+u^\alpha(t)$, $x^\alpha(0)\in x_0^\alpha$, $0\le\alpha\le 1$.

In [14,15] authors presented a new technique for solving linear time invariant control systems with fuzzy parameters and fuzzy boundary conditions respectively.

In this paper, fuzzy linear time varying control systems and fuzzy optimal control problems with fuzzy boundary conditions governed by fuzzy differential equations are studied. By using Heaviside functions, generalized Hukuhara derivative and Pontryagin Maximum Principle (PMP) (see Chapter 6 of [18]), a new approach for solving these kind of control systems is presented.

2. Preliminaries

Let $\mathcal{F}(\mathbb{R})$ be the set of fuzzy numbers, i.e., normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets defined over the real line. To describe our method used in this paper, we need the following definitions, lemma and theorems. For $0\le\alpha\le1$, denote

$$\tilde{M}^\alpha=\begin{cases}\{t\in\mathbb{R}|\tilde{M}(t)\ge\alpha\}, & 0<\alpha\le1,\\ \mathrm{cl}\{t\in\mathbb{R}|\tilde{M}(t)>0\}, & \alpha=0,\end{cases}$$

where $\tilde{M}^\alpha$ is α-level set.

If $\tilde{M}\in\mathcal{F}(\mathbb{R})$, then $\tilde{M}$ is fuzzy convex, so $\tilde{M}^\alpha$ is closed and bounded function in $\mathbb{R}$, i.e. $\tilde{M}^\alpha=[\underline{M}^\alpha,\overline{M}^\alpha]$, where $\underline{M}^\alpha=\inf\{t\in\mathbb{R}:\tilde{M}(t)\ge\alpha\}>-\infty$ and $\overline{M}^\alpha=\sup\{t\in\mathbb{R}:\tilde{M}(t)\ge\alpha\}<\infty$.

Definition 1 (*See Chalco-Cano and Roman-Flores [5]*). Zadeh's extension principle. Let Z be a Cartesian product of universes, that is $Z=Z_1\times Z_2\times\cdots\times Z_r$, and $\tilde{\mu}_1,\tilde{\mu}_2,\ldots,\tilde{\mu}_r$ be r fuzzy sets in $Z_1, Z_2, \ldots, Z_r$, respectively and Y is a given space. Each function $f: Z \to Y$ induces a corresponding function $\tilde{f}: \mathcal{F}(Z_1) \times \mathcal{F}(Z_2) \times \cdots \times \mathcal{F}(Z_r) \to \mathcal{F}(Y)$ (i.e., $\tilde{f}$ is a function mapping fuzzy sets in Z to fuzzy sets in Y) defined for each fuzzy set $\tilde{\mu}$ in Z by

$$\tilde{f}(\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_r)(y) = \begin{cases} \sup\limits_{(z_1, z_2, \ldots, z_r) = f^{-1}(y)} \min\{\tilde{\mu}_1(z_1), \tilde{\mu}_2(z_2), \ldots, \tilde{\mu}_r(z_r)\}, & f^{-1}(y) \neq \varnothing, \\ 0, & f^{-1}(y) = \varnothing, \end{cases}$$

where f^{-1} is the inverse of f.

The function $\tilde{f}$ is said to be obtained from f by the extension principle.

For $\tilde{M}, \tilde{V} \in \mathcal{F}(\mathbb{R})$ and $\gamma \in \mathbb{R}$ addition $\tilde{M} \oplus \tilde{V}$, multiplication $\tilde{M} \otimes \tilde{V}$ and scalar multiplication $\gamma \odot \tilde{M}$ are defined respectively by $[\tilde{M} \oplus \tilde{V}]^\alpha = [\tilde{M}]^\alpha + [\tilde{V}]^\alpha = [\underline{M}^\alpha + \underline{V}^\alpha, \overline{M}^\alpha + \overline{V}^\alpha]$, $[\tilde{M} \otimes \tilde{V}]^\alpha = [min\{\underline{M}^\alpha \underline{V}^\alpha, \underline{M}^\alpha \overline{V}^\alpha, \overline{M}^\alpha \underline{V}^\alpha, \overline{M}^\alpha \overline{V}^\alpha\}, max\{\underline{M}^\alpha \underline{V}^\alpha, \underline{M}^\alpha \overline{V}^\alpha, \overline{M}^\alpha \underline{V}^\alpha, \overline{M}^\alpha \overline{V}^\alpha\}]$ and $[\gamma \odot \tilde{M}]^\alpha = \gamma \cdot [\tilde{M}]^\alpha$ for all $\alpha \in [0,1]$.

The metric space is given by the Hausdorff distance as follows:

$$D: \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \longrightarrow \mathbb{R}^+ \cup \{0\},$$
$$D(\tilde{M}, \tilde{V}) = \sup_{\alpha \in [0,1]} \max\{|\underline{M}^\alpha - \underline{V}^\alpha|, |\overline{M}^\alpha - \overline{V}^\alpha|\}.$$

Definition 2 (*Bede and Stefanini [1]*). Let $\tilde{M}, \tilde{V} \in \mathcal{F}(\mathbb{R})$, the generalized Hukuhara difference (gH-difference) of two fuzzy numbers $\tilde{M}$ and $\tilde{V}$ is the fuzzy number $\tilde{Z} \in \mathcal{F}(\mathbb{R})$, if it exists, such that

$$\tilde{M} \ominus_{gH} \tilde{V} = \tilde{Z} \Longleftrightarrow \begin{cases} \text{(i)} & \tilde{M} = \tilde{V} \oplus \tilde{Z}, \\ \text{or (ii)} & \tilde{V} = \tilde{M} \ominus \tilde{Z}, \end{cases} \tag{1}$$

where $\tilde{M} \ominus \tilde{Z} = \tilde{M} \oplus ((-1) \odot \tilde{Z})$.

It is easy to show that (i) and (ii) are both valid if and only if $\tilde{Z}$ is a crisp number.

In terms of α-cuts we have $[\tilde{M} \ominus_{gH} \tilde{V}]^\alpha = [\min\{\underline{M}^\alpha - \underline{V}^\alpha, \overline{M}^\alpha - \overline{V}^\alpha\}, \max\{\underline{M}^\alpha - \underline{V}^\alpha, \overline{M}^\alpha - \overline{V}^\alpha\}]$. Each of following conditions guarantees the existence of $\tilde{Z} = \tilde{M} \ominus_{gH} \tilde{V} \in \mathcal{F}(\mathbb{R})$ (see [1] for more details)

(a) $\underline{Z}^\alpha = \underline{M}^\alpha - \underline{V}^\alpha$ and $\overline{Z}^\alpha = \overline{M}^\alpha - \overline{V}^\alpha$ with $\underline{Z}^\alpha$ increasing, $\overline{Z}^\alpha$ decreasing, $\underline{Z}^\alpha \leq \overline{Z}^\alpha$, for all $\alpha \in [0,1]$,
(b) $\overline{Z}^\alpha = \underline{M}^\alpha - \underline{V}^\alpha$ and $\underline{Z}^\alpha = \overline{M}^\alpha - \overline{V}^\alpha$ with $\underline{Z}^\alpha$ increasing, $\overline{Z}^\alpha$ decreasing, $\underline{Z}^\alpha \leq \overline{Z}^\alpha$, for all $\alpha \in [0,1]$.

Definition 3 (*Bede and Stefanini [1]*). Let $\tilde{f}: T \subset \mathbb{R} \to \mathcal{F}(\mathbb{R})$ and $t_0 \in T$ be a fixed number, then $\tilde{f}$ is called gH-differentiable at $t_0 \in T$ if:

$$\lim_{h \to 0} \frac{\tilde{f}(t_0 + h) \ominus_{gH} \tilde{f}(t_0)}{h} = \tilde{f}'_{gH}(t_0) \in \mathcal{F}(\mathbb{R}).$$

Theorem 1. *Let* $\tilde{f}: T \to \mathcal{F}(\mathbb{R})$ *be a fuzzy function, where* $[\tilde{f}(t)]^\alpha = [\underline{f}^\alpha(t), \overline{f}^\alpha(t)]$. *Suppose that the functions* $\underline{f}^\alpha(t)$ *and* $\overline{f}^\alpha(t)$ *are real-valued functions, differentiable w.r.t.* t, *uniformly w.r.t.* $\alpha \in [0,1]$. *Then the function* $\tilde{f}(t)$ *is gH-differentiable at a fixed* $t \in T$ *if and only if one of the following two cases hold*:

(a) $\underline{f}'^\alpha(t)$ *is increasing*, $\overline{f}'^\alpha(t)$ *is decreasing as functions of* α, *and* $\underline{f}'^1(t) \le \overline{f}'^1(t)$, *or*
(b) $\underline{f}'^\alpha(t)$ *is decreasing*, $\overline{f}'^\alpha(t)$ *is increasing as functions of* α, *and* $\overline{f}'^1(t) \le \underline{f}'^1(t)$.

Also, $\forall \alpha \in [0,1]$ *we have*

$$[f'_{gH}(t)]^\alpha = [\min\{\underline{f}'^\alpha(t), \overline{f}'^\alpha(t)\}, \max\{\underline{f}'^\alpha(t), \overline{f}'^\alpha(t)\}].$$

Proof. See [1]. □

Based on Theorem 1, and [1], if $\underline{f}^\alpha(t)$, and $\overline{f}^\alpha(t)$ are both differentiable, then there are two cases, corresponding to (i) and (ii) of Eq. (1) as follows:

Definition 4 (*Bede and Stefanini [1]*). Let $\tilde{f}: T \to \mathcal{F}(\mathbb{R})$ and $t_0 \in T$. If $\underline{f}^\alpha(t)$, $\overline{f}^\alpha(t)$ are both differentiable at t_0, then

- $\tilde{f}$ is called (i)-gH-differentiable at t_0 if $[\tilde{f}'_{gH}(t_0)]^\alpha = [\underline{f}'^\alpha(t_0), \overline{f}'^\alpha(t_0)]$, $\forall \alpha \in [0,1]$,
- $\tilde{f}$ is called (ii)-gH-differentiable at t_0 if $[\tilde{f}'_{gH}(t_0)]^\alpha = [\overline{f}'^\alpha(t_0), \underline{f}'^\alpha(t_0)]$, $\forall \alpha \in [0,1]$.

Now consider the following fuzzy linear dynamical system:

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{f}(t, \tilde{x}(t)), \\ \tilde{x}(t_0) = \tilde{x}_0, \end{cases} \tag{2}$$

where $\tilde{x}: [t_0, t_f] \to \mathcal{F}(\mathbb{R})$, $\tilde{f}: [t_0, t_f] \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ are fuzzy functions and $\tilde{x}_0$ is a fuzzy initial condition.

Theorem 2. *Suppose that* $\tilde{f}: [t_0, t_f] \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ *is obtained by Zadeh*'s *extension principle from a continuous function* $f: [t_0, t_f] \times \mathbb{R} \to \mathbb{R}$ *such that* $(t, x(t)) \mapsto f(t, x(t))$. *Then*:

1. *If f is nondecreasing with respect to the second argument, then the fuzzy solution of Eq.* (2) *is obtained using* (i)-*gH-differentiability*.
2. *If f is nonincreasing with respect to the second argument, then the fuzzy solution of Eq.* (2) *is obtained using* (ii)-*gH-differentiability*.

Proof. Proof of this theorem is similar to Theorem 4 from [5]. □

In the next section, by referring to Theorem 2, attempt to solve a time varying fuzzy controlled system with fuzzy boundary conditions.

3. Fuzzy time varying controlled system with fuzzy conditions

Consider the following fuzzy time varying controlled system with fuzzy boundary conditions:

$$\begin{cases} \dot{\tilde{x}}(t) = a(t) \odot \tilde{x}(t) + b(t) \odot \tilde{u}(t), \\ \tilde{x}(t_0) = \tilde{x}_0, \\ \tilde{x}(t_f) = \tilde{x}_f, \end{cases} \tag{3}$$

The task is to carry the controlled system from the initial point $\tilde{x}(t_0) = \tilde{x}_0$ to final target $\tilde{x}(t_f) = \tilde{x}_f$, using suitable fuzzy control function $\tilde{u}(t)$, where $a(t)$ and $b(t)$ are given time varying functions. Suppose $S(t)$ is the Heaviside function, as

$$S(t) = \begin{cases} 1, & t > 0, \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0. \end{cases}$$

The α-cuts of fuzzy controlled system (3) is as follows:

$$\begin{cases} \dot{X}^{\alpha}(t) = A(t)X^{\alpha}(t) + B(t)U^{\alpha}(t), \\ X^{\alpha}(t_0) = X_0^{\alpha}, \\ X^{\alpha}(t_f) = X_f^{\alpha}, \end{cases} \tag{4}$$

where $X^{\alpha}(t) = [\underline{x}^{\alpha}(t)\ \overline{x}^{\alpha}(t)]^T$, $\dot{X}^{\alpha}(t) = [\dot{\underline{x}}^{\alpha}(t)\ \dot{\overline{x}}^{\alpha}(t)]^T$, $U^{\alpha}(t) = [\underline{u}^{\alpha}(t)\ \overline{u}^{\alpha}(t)]^T$, $X^{\alpha}(t_j) = [\underline{x}^{\alpha}(t_j)\ \overline{x}^{\alpha}(t_j)]^T$ for $j = 0$ or f. The entries of the matrices $A(t)$ and $B(t)$ in Eq. (4) using Theorem 2 are determined respectively from $a(t)$ and $b(t)$ as follows:

$$A(t) = \begin{bmatrix} a(t) & 0 \\ 0 & a(t) \end{bmatrix}, \tag{5}$$

and

$$B(t) = \begin{bmatrix} S(a(t)b(t))b(t) & S(-a(t)b(t))b(t) \\ S(-a(t)b(t))b(t) & S(a(t)b(t))b(t) \end{bmatrix}. \tag{6}$$

The actual solution of Eq. (4) in $[t_0, t_f]$ can be found from the following theorem.

Theorem 3. *The controlled system* (4), *in* $[t_0, t_f]$, *has the following solution*:

$$X^{\alpha}(t) = e^{\int A(t)\,dt}\left(wX_0^{\alpha} + \int_{t_0}^{t} e^{(-\int A(\xi)\,d\xi)} B(\xi)U^{\alpha}(\xi)\,d\xi \right), \tag{7}$$

where $w = (e^{-\int A(t)\,dt})_{t = t_0}$.

Proof. Since

$$\dot{X}^{\alpha}(t) = A(t)X^{\alpha}(t) + B(t)U^{\alpha}(t),$$

then

$$\dot{X}^{\alpha}(t) - A(t)X^{\alpha}(t) = B(t)U^{\alpha}(t). \tag{8}$$

Multiply Eq. (8) by $\exp(-\int A(t)\,dt)$ gives

$$e^{-\int A(t)\,dt}\dot{X}^{\alpha}(t)-e^{-\int A(t)\,dt}A(t)X^{\alpha}(t)=e^{-\int A(t)\,dt}B(t)U^{\alpha}(t), \tag{9}$$

but,

$$\frac{d}{dt}\left(e^{-\int A(t)\,dt}X^{\alpha}(t)\right)=e^{-\int A(t)\,dt}\dot{X}^{\alpha}(t)-e^{-\int A(t)\,dt}A(t)X^{\alpha}(t). \tag{10}$$

To show Eq. (10), we know that

$$\begin{aligned}\frac{d}{dt}\left(e^{(-\int A(t)\,dt)}X^{\alpha}(t)\right)&=\lim_{h\to 0}\frac{e^{\left(-\int A(t+h)\,d(t+h)\right)}X^{\alpha}(t+h)-e^{(-\int A(t)\,dt)}X^{\alpha}(t)}{h}\\&=\lim_{h\to 0}\frac{e^{\left(-\int A(t)+h\frac{dA(t)}{dt}\right)\,dt}\left(X^{\alpha}(t)+h\dot{X}^{\alpha}(t)\right)-e^{(-\int A(t)\,dt)}X^{\alpha}(t)}{h}\\&=e^{-\int A(t)\,dt}\lim_{h\to 0}\frac{e^{-hA(t)}-I}{h}X^{\alpha}(t)+\lim_{h\to 0}\frac{e^{-\int(A(t)+hdA(t)/dt)\,dt}h\dot{X}^{\alpha}(t)}{h}\\&=e^{-\int A(t)\,dt}\lim_{h\to 0}\lim_{k\to+\infty}\left(-A(t)+\frac{A(t)^2h}{2!}+\cdots+\frac{(-1)^kA(t)^kh^{k-1}}{k!}\right)X^{\alpha}(t)\\&\quad+\lim_{h\to 0}e^{-\int A(t)\,dt}\dot{X}^{\alpha}(t)=e^{\left(-\int A(t)\,dt\right)}\dot{X}^{\alpha}(t)-e^{\left(-\int A(t)\,dt\right)}A(t)X^{\alpha}(t).\end{aligned}$$

Similarly, one can show that

$$\frac{d}{dt}\left(\int_{t_0}^{t}\left(e^{-\int A(\xi)\,d\xi}B(\xi)U^{\alpha}(\xi)\,d\xi+c\right)=e^{-\int A(t)\,dt}B(t)U^{\alpha}(t). \tag{11}$$

By considering Eqs. (9)–(11) we have

$$\frac{d}{dt}\left(e^{\left(-\int A(t)\,dt\right)}X^{\alpha}(t)\right)=\frac{d}{dt}\int_{t_0}^{t}\left(e^{-\int A(\xi)\,d\xi}B(\xi)U^{\alpha}(\xi)\,d\xi+c\right),$$

i.e.

$$e^{\left(-\int A(t)\,dt\right)}X^{\alpha}(t)=\int_{t_0}^{t}(e^{-\int A(\xi)\,d\xi}B(\xi)U^{\alpha}(\xi)\,d\xi+c).$$

Substituting $t=t_0$, gives $wX^{\alpha}(t_0)=c$. Then

$$e^{(-\int A(t)\,dt)}X^{\alpha}(t)=\int_{t_0}^{t}e^{-\int A(\xi)\,d\xi}B(\xi)U^{\alpha}(\xi)\,d\xi+wX^{\alpha}(t_0),$$

or

$$X^{\alpha}(t)=e^{\int A(t)\,dt}wX_0^{\alpha}+\int_{t_0}^{t}e^{\left(\int A(t)\,dt-\int A(\xi)\,d\xi\right)}B(\xi)U^{\alpha}(\xi)\,d\xi,$$

that shows Eq. (7).

4. Fuzzy optimal control time varying system with fuzzy conditions

Consider the following fuzzy optimal control system with time varying coefficients and fuzzy boundary conditions:

$$\begin{aligned}
&\text{Min} \quad \int_{t_0}^{t_f} \tilde{u}(t) \otimes \tilde{u}(t)\, dt,\\
&\text{s.t.}\\
&\quad \dot{\tilde{x}}(t) = a(t) \odot \tilde{x}(t) \oplus b(t) \odot \tilde{u}(t),\\
&\quad \tilde{x}(t_0) = \tilde{x}_0,\\
&\quad \tilde{x}(t_f) = \tilde{x}_f.
\end{aligned} \tag{12}$$

In Eq. (12) the task is to carry the controlled system from the initial point $\tilde{x}(t_0) = \tilde{x}_0$ to final target $\tilde{x}(t_f) = \tilde{x}_f$, using suitable fuzzy control function $\tilde{u}(t)$ such that the quadratic cost functional be minimized, where $a(t)$ and $b(t)$ are suitable given time varying functions.

In applying the obtained results of previous section to solve the optimal fuzzy controlled system (12), this system is equivalent to the following non-fuzzy optimal control system:

$$\begin{aligned}
&\text{Min} \quad \int_{t_0}^{t_f} U^\alpha(t)(U^\alpha(t))^T\, dt,\\
&\quad \text{s.t.}\\
&\quad \dot{X}^\alpha(t) = A(t)X^\alpha(t) + B(t)U^\alpha(t),\\
&\quad X^\alpha(t_0) = X_0^\alpha,\\
&\quad X^\alpha(t_f) = X_f^\alpha,
\end{aligned} \tag{13}$$

where $A(t)$ and $B(t)$ are described by Eqs. (5) and (6) respectively. Now we can prove the following theorem:

Theorem 4. *Define the Hamiltonian function*

$$H(X^\alpha, U^\alpha, \psi, t) = -U^\alpha(t)(U^\alpha(t))^T + \psi^T(A(t)X^\alpha(t) + B(t)U^\alpha(t)).$$

The necessary conditions for the pair $(X^{\alpha*}, U^{\alpha*})$ *be an optimal solution for Eq.* (13) *are that there exists vector function* ψ *such that satisfies the following co-state differential equation*:

$$\dot{\psi} = -\frac{\partial H}{\partial X^{\alpha*}},$$

and the Hamiltonian function is maximized in $U^{\alpha*}$, *i.e.*

$$\frac{\partial H}{\partial U^{\alpha*}} = 0.$$

Proof. Suppose $\underline{x}^\alpha(t) = x_1^\alpha(t)$, $\overline{x}^\alpha(t) = x_2^\alpha(t)$, $\underline{u}^\alpha(t) = u_1^\alpha(t)$, $\overline{u}^\alpha(t) = u_2^\alpha(t)$ and $f_0(u_1^\alpha, u_2^\alpha) = u_1^{\alpha 2}(t) + u_2^{\alpha 2}(t)$, and $f_k(x_k^\alpha, u_1^\alpha, u_2^\alpha)$ be the k th row of $A(t)X^\alpha(t) + B(t)U^\alpha(t)$ for $k=1,2$, and $\dot{\mathcal{J}} = \int_{t_0}^{t_f} f_0(u_1^\alpha, u_2^\alpha)\, dt = \int_{t_0}^{t_f} U^\alpha(t)(U^\alpha(t))^T\, dt$. Let $u_1^{\alpha*}(t)$ and $u_2^{\alpha*}(t)$ be the optimal controls and $x_1^{\alpha*}(t)$, $x_2^{\alpha*}(t)$ the corresponding optimal states. Consider a small variation of $u_1^{\alpha*}(t)$, as $u_1^\alpha(t) = u_1^{\alpha*}(t) + \delta u_1^\alpha(t)$ and $u_2^{\alpha*}(t)$, as $u_2^\alpha(t) = u_2^{\alpha*}(t) + \delta u_2^\alpha(t)$ with corresponding path $(x_1^{\alpha*}(t) + \delta x_1^\alpha(t), x_2^{\alpha*}(t) + \delta x_2^\alpha(t))$. This will not arrive at x_{1f}^α, and x_{2f}^α at t_f but at a slightly different time $t_f + \delta t$. The end conditions give

$$x_k^\alpha(t_f + \delta t) = x_{kf}^\alpha, \quad k = 1, 2.$$

As usual in variational arguments we are in the first instance interested only in first-order effects, and from the end conditions we deduce that

$$\delta x_k^\alpha(t_f) + \dot{x}_k^\alpha(t_f)\delta t = 0, \quad k = 1, 2.$$

Now if we use the sate equations we obtain

$$\delta x_k^\alpha(t_f) = -f_k(t_f)\delta t,$$

where $f_k(t_f) = f_k(x_k^\alpha(t_f), u_1^\alpha(t_f), u_2^\alpha(t_f))$. The consequent change $\Delta\mathcal{J}$ in $\mathcal{J}$ is

$$\begin{aligned}\Delta\mathcal{J} &= \int_{t_0}^{t_f+\delta t} f_0\left(u_1^{\alpha*} + \delta u_1^\alpha, u_2^{\alpha*} + \delta u_2^\alpha\right) dt - \int_{t_0}^{t_f} f_0\left(u_1^{\alpha*}, u_2^{\alpha*}\right) dt \\ &= \int_{t_0}^{t_f} \left\{\frac{\partial f_0}{\partial u_1^\alpha}\delta u_1^\alpha + \frac{\partial f_0}{\partial u_2^\alpha}\delta u_2^\alpha\right\} dt + f_0\left(t_f\right)\delta t + O\left((\delta u_1^\alpha)^2\right) + O\left((\delta u_2^\alpha)^2\right),\end{aligned}$$

where $f_0(t_f)$ is the value of f_0 at $t = t_f$ and the derivatives in the integrand are evaluated on the optimal trajectory. Let $\delta\mathcal{J}$ denote the first variation. If $u_1^{\alpha*}$ and $u_2^{\alpha*}$ are optimal, it is necessary that the first variation $\delta\mathcal{J}$ is zero, so

$$\int_{t_0}^{t_f} \left\{\frac{\partial f_0}{\partial u_1^\alpha}\delta u_1^\alpha + \frac{\partial f_0}{\partial u_2^\alpha}\delta u_2^\alpha\right\} dt + f_0\left(t_f\right)\delta t = 0,$$

on optimal state for all variations.

We simply need to introduce two Lagrange multipliers $\psi_1(t)$ and $\psi_2(t)$. Now consider the pair of integrals

$$\varphi_k = \int_{t_0}^{t_f} \psi_k(t)(\dot{x}_k^\alpha - f_k(x_k^\alpha, u_1^\alpha, u_2^\alpha))\, dt, \quad k = 1, 2.$$

With respect to Theorem 4.1 of [18] one can see that $\delta\varphi_k = 0$ since $\varphi_k = 0$ for all u_1^α and u_2^α. The calculation is straightforward giving

$$\delta\varphi_k = \int_{t_0}^{t_f} \psi_k(t)\left\{-\frac{\partial f_k}{\partial x_k^\alpha}\delta x_k^\alpha - \frac{\partial f_k}{\partial u_1^\alpha}\delta u_1^\alpha - \frac{\partial f_k}{\partial u_2^\alpha}\delta u_2^\alpha + \frac{d}{dt}\left(\delta x_k^\alpha\right)\right\} dt.$$

Now

$$\int_{t_0}^{t_f} \psi_k(t)\frac{d}{dt}\left(\delta x_k^\alpha\right) dt = -f_k\left(t_f\right)\psi_k\left(t_f\right)\delta t - \int_{t_0}^{t_f} \dot{\psi}_k\left(\delta x_k^\alpha\right) dt.$$

The condition that $\delta\mathcal{J} = 0$ can now be replaced by the condition that $\delta\mathcal{J} + \delta\varphi_1 + \delta\varphi_2 = 0$. And if we introduce the Hamiltonian function:

$$H = -f_0(u_1^\alpha, u_2^\alpha) + \psi_1 f_1(x_1^\alpha, u_1^\alpha, u_2^\alpha) + \psi_2 f_2(x_2^\alpha, u_1^\alpha, u_2^\alpha),$$

then one can obtain the contention of the theorem easily. □

5. Numerical results

This section presents a comparative example between the approach proposed in this paper and that introduced in [1]. Moreover, two examples are presented to show how the proposed approach is able to solve fuzzy time varying control systems.

Example 1. Consider the following fuzzy control system:

$$\begin{cases} \dot{\tilde{x}}(t) = (\sin(2\pi t) \odot \tilde{x}(t)) \oplus \tilde{u}(t), \\ \tilde{x}(0) = (1,2,3). \end{cases} \tag{14}$$

For comparing the proposed approach with the approach based on (i) and (ii)-gH-differentiability [1], the control function is supposed $\tilde{u}(t) = (1, 1.2, 1.5)$. Obtaining the solution of Eq. (14) results in solving following differential equations systems:

$$\begin{cases} \dot{\underline{x}}^\alpha(t) = (\sin(2\pi t) \odot \underline{x}^\alpha(t)) \oplus (1.2\alpha + (1-\alpha)), & t \in \left[\frac{k}{2}, \frac{2k+1}{2}\right], \\ \dot{\overline{x}}^\alpha(t) = (\sin(2\pi t) \odot \overline{x}^\alpha(t)) \oplus (1.2\alpha + 1.5(1-\alpha)), & t \in \left[\frac{k}{2}, \frac{2k+1}{2}\right], \\ \dot{\underline{x}}^\alpha(t) = (\sin(2\pi t) \odot \overline{x}^\alpha(t)) \oplus (1.2\alpha + (1-\alpha)), & t \in \left[\frac{2k+1}{2}, k+1\right], \\ \dot{\overline{x}}^\alpha(t) = (\sin(2\pi t) \odot \underline{x}^\alpha(t)) \oplus (1.2\alpha + 1.5(1-\alpha)), & t \in \left[\frac{2k+1}{2}, k+1\right], \\ \underline{x}^\alpha(0) = 2\alpha + (1-\alpha), \\ \overline{x}^\alpha(0) = 2\alpha + 3(1-\alpha), \\ k = 0, 1, \ldots \end{cases} \tag{15}$$

provided that $\tilde{x}(t)$ is (i)-gH-differentiable,

$$\begin{cases} \dot{\underline{x}}^\alpha(t) = (\sin(2\pi t) \odot \overline{x}^\alpha(t)) \oplus (1.2\alpha + 1.5(1-\alpha)), & t \in \left[\frac{k}{2}, \frac{2k+1}{2}\right], \\ \dot{\overline{x}}^\alpha(t) = (\sin(2\pi t) \odot \underline{x}^\alpha(t)) \oplus (1.2\alpha + (1-\alpha)), & t \in \left[\frac{k}{2}, \frac{2k+1}{2}\right], \\ \dot{\underline{x}}^\alpha(t) = (\sin(2\pi t) \odot \underline{x}^\alpha(t)) \oplus (1.2\alpha + 1.5(1-\alpha)), & t \in \left[\frac{2k+1}{2}, k+1\right], \\ \dot{\overline{x}}^\alpha(t) = (\sin(2\pi t) \odot \overline{x}^\alpha(t)) \oplus (1.2\alpha + (1-\alpha)), & t \in \left[\frac{2k+1}{2}, k+1\right], \\ \underline{x}^\alpha(0) = 2\alpha + (1-\alpha), \\ \overline{x}^\alpha(0) = 2\alpha + 3(1-\alpha), \\ k = 0, 1, \ldots \end{cases} \tag{16}$$

provided that $\tilde{x}(t)$ is (ii)-gH-differentiable. Based on proposed approach, fuzzy time varying system (14) is transformed to

$$\left[\dot{\underline{x}}^\alpha(t) \dot{\overline{x}}^\alpha(t)\right] = \begin{bmatrix} \sin(2\pi t) & 0 \\ 0 & \sin(2\pi t) \end{bmatrix} \begin{bmatrix} \underline{x}^\alpha(t) \\ \overline{x}^\alpha(t) \end{bmatrix} + \begin{bmatrix} S(\sin(2\pi t)) & S(-\sin(2\pi t)) \\ S(-\sin(2\pi t)) & S(\sin(2\pi t)) \end{bmatrix} \begin{bmatrix} 1.2\alpha + (1-\alpha) \\ 1.2\alpha + 1.5(1-\alpha) \end{bmatrix}. \tag{17}$$

Fig. 1 shows the solutions of differential equations systems (15)–(17). In the figure the blue, green and red curves are the solutions of Eqs. (15), (16) and (17) for $\alpha = 0$, respectively. The crisp solution has been shown by the black curve.

What Fig. 1 shows is that $\text{len}(\tilde{x})(t) = \overline{x}^0(t) - \underline{x}^0(t)$ is a monotonic function based on (i)-gH-differentiability and (ii)-gH-differentiability. In simple terms, uncertainty is increasing, i.e. $\lim_{t\to+\infty}\text{len}(\tilde{x})(t) = +\infty$, or vanishing, i.e. $\lim_{t\to+\infty}\text{len}(\tilde{x})(t) = 0$. However, these are not the cases in this example. The output of Eq. (14) is a periodic function in which the uncertainty neither will be infinite nor vanished necessarily that can be found in the solution corresponding to the proposed approach.

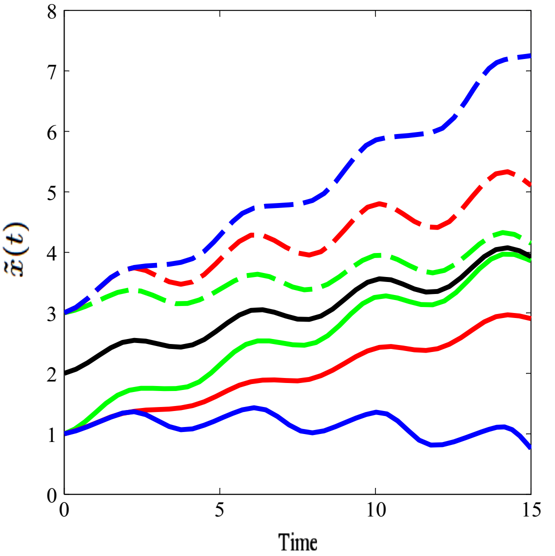

Fig. 1. The solution using (i)-gH- differentiability (blue curves), the solution using (ii)-gH- differentiability (green curves), and the solution based on proposed approach (red curves). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

Example 2. Consider the following fuzzy controlled system:

$$\begin{cases} \dot{\tilde{x}}(t) = ((2t-1) \odot \tilde{x}(t)) \oplus (\sin t \odot \tilde{u}(t)), \\ \tilde{x}(0) = (1,2,3), \\ \tilde{x}(2) = (-1,0,1). \end{cases}$$

Using Eq. (4) we have

$$\left[\dot{\underline{x}}^{\alpha}(t)\dot{\overline{x}}^{\alpha}(t)\right] = \begin{bmatrix} (2t-1) & 0 \\ 0 & (2t-1) \end{bmatrix} \begin{bmatrix} \underline{x}^{\alpha}(t) \\ \overline{x}^{\alpha}(t) \end{bmatrix} + \begin{bmatrix} S((2t-1)\sin t)\sin t & S(-(2t-1)\sin t)\sin t \\ S(-(2t-1)\sin t)\sin t & S((2t-1)\sin t)\sin t \end{bmatrix} \begin{bmatrix} \underline{u}^{\alpha}(t) \\ \overline{u}^{\alpha}(t) \end{bmatrix}.$$

By considering Theorem 3 and Eq. (7) we can deduce the following equation:

$$\begin{bmatrix} \underline{x}^{\alpha} \\ \overline{x}^{\alpha} \end{bmatrix} = \exp\left(\begin{bmatrix} t^2-t & 0 \\ 0 & t^2-t \end{bmatrix}\right)\left(\begin{bmatrix} 2\alpha+(1-\alpha) \\ 2\alpha+3(1-\alpha) \end{bmatrix} + \int_0^t \left\{\exp\left(\begin{bmatrix} \xi-\xi^2 & 0 \\ 0 & \xi-\xi^2 \end{bmatrix}\right) \begin{bmatrix} S((2\xi-1)\sin(\xi))\sin\xi & S(-(2\xi-1)\sin(\xi))\sin\xi \\ S(-(2\xi-1)\sin(\xi))\sin\xi & S((2\xi-1)\sin(\xi))\sin\xi \end{bmatrix} \begin{bmatrix} \underline{u}^{\alpha}(\xi) \\ \overline{u}^{\alpha}(\xi) \end{bmatrix}\right\} d\xi\right).$$

We need to mention that the control function $\underline{u}^{\alpha}(t)$ and $\overline{u}^{\alpha}(t)$ are assumed to be respectively $a(1-2t)$ and $b(1-2t)$, where we find a and b such that to control the system from initial point $\tilde{x}(0) = (1,2,3)$ to final point $\tilde{x}(2) = (-1,0,1)$.

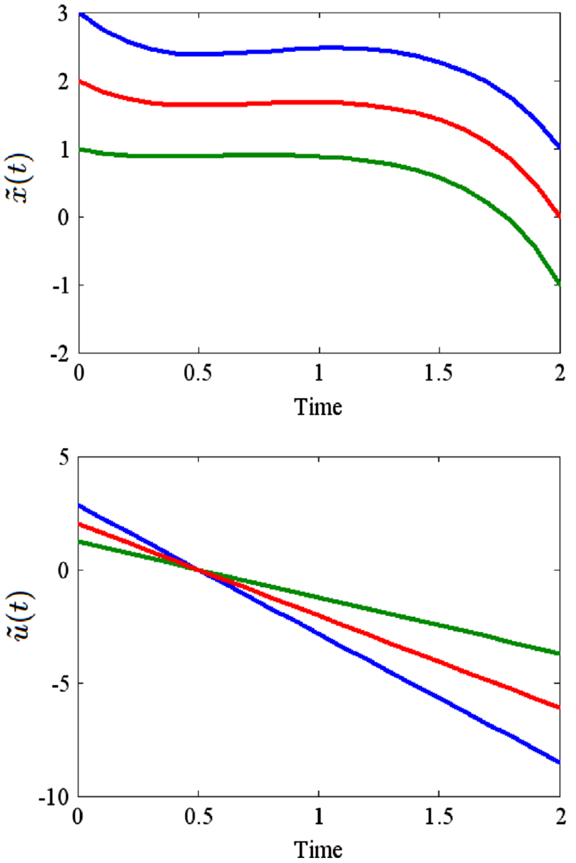

Fig. 2. The state and control functions for Example 2. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

The fuzzy state and control functions are given in Fig. 2, where the red curve is the center of fuzzy functions for $\alpha = 1$, the blue curves are the upper bound, and the green curves are the lower bound for fuzzy control and state functions for $\alpha = 0$.

Example 3. Consider the following optimal fuzzy controlled system:

$$\text{Min} \quad \int_0^2 \tilde{u}(t) \otimes \tilde{u}(t)\, dt,$$

s.t.

$$\dot{\tilde{x}}(t) = ((2t-1) \odot \tilde{x}(t)) \oplus (\sin t \odot \tilde{u}(t)),$$
$$\tilde{x}(0) = (1, 2, 3),$$
$$\tilde{x}(2) = (-1, 0, 1).$$

Using Eq. (13) it is enough to solve the following optimal control system:

$$
\begin{aligned}
&\text{Min} \quad \int_0^2 (\underline{u}^{\alpha 2}(t) + \overline{u}^{\alpha 2}(t))\, dt, \\
&\quad \text{s.t.} \\
&\quad \begin{bmatrix} \dot{\underline{x}}^{\alpha}(t) \\ \dot{\overline{x}}^{\alpha}(t) \end{bmatrix} = \begin{bmatrix} (2t-1) & 0 \\ 0 & (2t-1) \end{bmatrix} \begin{bmatrix} \underline{x}^{\alpha}(t) \\ \overline{x}^{\alpha}(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} S((2t-1)\sin t)\sin t & S(-(2t-1)\sin t)\sin t \\ S(-(2t-1)\sin t)\sin t & S((2t-1)\sin t)\sin t \end{bmatrix} \begin{bmatrix} \underline{u}^{\alpha}(t) \\ \overline{u}^{\alpha}(t) \end{bmatrix}, \\
&\quad \underline{x}^{\alpha}(0) = 2\alpha + (1-\alpha), \\
&\quad \overline{x}^{\alpha}(0) = 2\alpha + 3(1-\alpha), \\
&\quad \underline{x}^{\alpha}(2) = -(1-\alpha), \\
&\quad \overline{x}^{\alpha}(2) = (1-\alpha).
\end{aligned}
$$

By Theorem 4 one can find the Hamiltonian as follows:

$$
\begin{aligned}
H = &-(\underline{u}^{\alpha 2} + \overline{u}^{\alpha 2}) + \psi_1((2t-1)\underline{x}^{\alpha} + S((2t-1)\sin t)\underline{u}^{\alpha} + S(-(2t-1)\sin t)\overline{u}^{\alpha}) \\
&+ \psi_2((2t-1)\overline{x}^{\alpha} + S(-(2t-1)\sin t)\underline{u}^{\alpha} + S((2t-1)\sin t)\overline{u}^{\alpha}).
\end{aligned}
$$

So

$$
\begin{bmatrix} \underline{u}^{*\alpha} \\ \overline{u}^{*\alpha} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} S((2t-1)\sin t)\sin t & S(-(2t-1)\sin t)\sin t \\ S(-(2t-1)\sin t)\sin t & S((2t-1)\sin t)\sin t \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.
$$

where ψ_1 and ψ_2 are derived by solving the following co-state differential equation:

$$
\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = -\begin{bmatrix} (2t-1) & 0 \\ 0 & (2t-1) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.
$$

Thus

$$
\begin{bmatrix} \underline{u}^{*\alpha}(t) \\ \overline{u}^{*\alpha}(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} S((2t-1)\sin t)\sin t & S(-(2t-1)\sin t)\sin t \\ S(-(2t-1)\sin t)\sin t & S((2t-1)\sin t)\sin t \end{bmatrix} \exp\left(\begin{bmatrix} (t-t^2) & 0 \\ 0 & (t-t^2) \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
$$

Now the states of optimal fuzzy control are calculated from the following system:

$$
\begin{aligned}
\begin{bmatrix} \underline{x}^{*\alpha}(t) \\ \overline{x}^{*\alpha}(t) \end{bmatrix} = &\exp\left(\begin{bmatrix} t^2-t & 0 \\ 0 & t^2-t \end{bmatrix} \right) \left(\begin{bmatrix} 2\alpha + (1-\alpha) \\ 2\alpha + 3(1-\alpha) \end{bmatrix} + \int_0^t \left\{ \exp\left(\begin{bmatrix} \xi-\xi^2 & 0 \\ 0 & \xi-\xi^2 \end{bmatrix} \right) \right. \right. \\
&\left. \left. \begin{bmatrix} S((2\xi-1)\sin(\xi))\sin\xi & S(-(2\xi-1)\sin(\xi))\sin\xi \\ S(-(2\xi-1)\sin(\xi))\sin\xi & S((2\xi-1)\sin(\xi))\sin\xi \end{bmatrix} \begin{bmatrix} \underline{u}^{*\alpha}(\xi) \\ \overline{u}^{*\alpha}(\xi) \end{bmatrix} \right\} d\xi \right).
\end{aligned}
$$

The fuzzy state and control functions are given in Fig. 3 where the red curve is the center for $\alpha = 1$, the blue curves are the upper bounds, and the green curves are the lower bounds for fuzzy control and state functions for $\alpha = 0$.

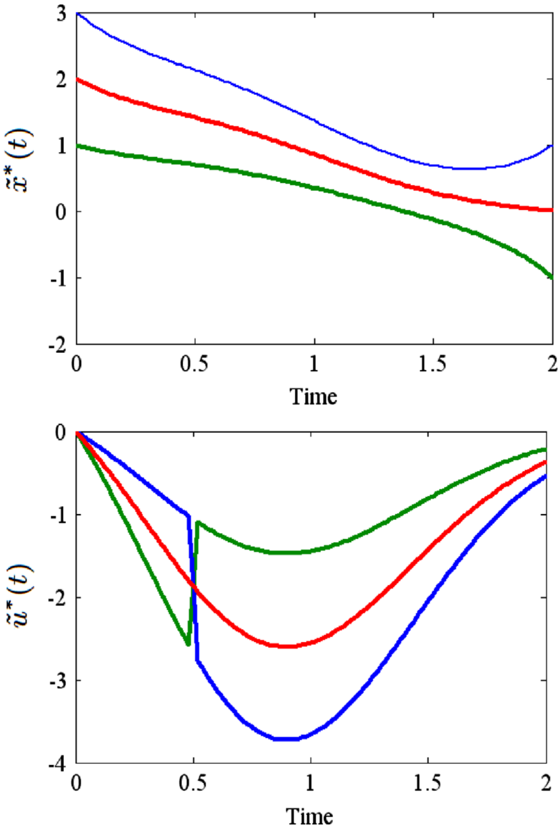

Fig. 3. The state and control functions for Example 3. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

6. Conclusion

We studied fuzzy linear time varying controlled systems and fuzzy optimal control problems, by minimizing the objective value function subject to a fuzzy differential equation with fuzzy boundary conditions. In this sequel, we used α-cut sets of fuzzy system and Heaviside function while applying Pontryagin Maximum Principle and co-state differential equations as well. The gH-derivative is considered because Hukuhara derivative does not always exist. Three numerical examples are given to support the reliability and efficiency of the proposed method.

We need to mention that in this sequel using generalized Hukuhara derivative, a new technique has been presented to solve a class of fuzzy optimal control system, where the coefficients of system are time dependent, and so their signs may change as time goes on. However, in other methods (as the method presented in [2]), the coefficients are considered positive or negative all the time, otherwise, one needs to use alternatively the concepts of (i)-gH-differentiablity and (ii)-gH-differentiablity as the signs of system coefficients change.

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