Title:
Characterization of Lie higher Derivations on $C^*$-algebras

Author(s):
A. R. Janfada, H. Saidi and M. Mirzavaziri
CHARACTERIZATION OF LIE HIGHER DERIVATIONS ON
C*-ALGEBRAS

A. R. JANFADA, H. SAIDI AND M. MIRZAVAZIRI

(Communicated by Ali Abkar)

Abstract. Let \( A \) be a \( C^* \)-algebra and \( Z(A) \) the center of \( A \). A sequence \( \{L_n\}_{n=0}^{\infty} \) of linear mappings on \( A \) with \( L_0 = I \), where \( I \) is the identity mapping on \( A \), is called a Lie higher derivation if \( L_n[x,y] = \sum_{i+j=n} [L_i x, L_j y] \) for all \( x, y \in A \) and all \( n \geq 0 \). We show that \( \{L_n\}_{n=0}^{\infty} \) is a Lie higher derivation if and only if there exist a higher derivation \( \{D_n\}_{n=0}^{\infty} : A \to A \) and a sequence of linear mappings \( \{\Delta_n\}_{n=0}^{\infty} : A \to Z(A) \) such that \( \Delta_0 = 0, \Delta_n([x,y]) = 0 \) and \( L_n = D_n + \Delta_n \) for every \( x, y \in A \) and all \( n \geq 0 \).

Keywords: Lie Derivations, Lie Higher derivations.


1. Introduction

Let \( A \) be an algebra and \([x,y] = xy - yx\) the commutator (the Lie product) of the elements \( x, y \in A \). A linear mapping \( d : A \to A \) is called a derivation if \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in A \). A linear mapping \( l : A \to A \) is called a Lie derivation if \( l([x,y]) = [l(x),y] + [x,l(y)] \) for all \( x, y \in A \). Clearly, every derivation is a Lie derivation. Johnson [4] proved that every continuous Lie derivation from a \( C^* \)-algebra \( A \) into a Banach \( A \)-module \( M \) is standard, that is, can be decomposed as the form \( d + \delta \), where \( d : A \to M \) is a derivation and \( \delta \) is a linear map from \( A \) into the center of \( M \) vanishing at each commutator. Mathieu and Villena [7] showed that every Lie derivation (without continuity) on a \( C^* \)-algebra is standard. In [11], Qi and Hou proved that the same is true for Lie derivations of nest algebras on Banach spaces. For other results, see [3].

A sequence \( \{D_n\}_{n=0}^{\infty} \) of linear mappings from \( A \) into \( A \) with \( D_0 = I \) is called a higher derivation if \( D_n(xy) = \sum_{i+j=n} D_i(x)D_j(y) \) for all \( x, y \in A \) and all \( n \geq 0 \). Let \( d \) be a derivation on \( A \) and define the sequence \( \{D_n\}_{n=0}^{\infty} \) of linear mappings...
mappings on $\mathcal{A}$ by $D_0 = I$ and $D_n = \frac{d^n}{n!}$. Then the Leibnitz rule ensures that $\{D_n\}_{n=0}^\infty$ is a higher derivation. Higher derivations were introduced by Hasse and Schmidt [1], and algebraists sometimes call them Hasse-Schmidt derivations. In [8], higher derivations are applied to study generic solving of higher differential equations. For more information about higher derivations and its applications see [2], [5], and [14]. The last author [9] characterized higher derivations in terms of derivations. A sequence $\{L_n\}_{n=0}^\infty$ of linear mappings from $\mathcal{A}$ into $\mathcal{A}$ with $L_0 = I$ is called a Lie higher derivation if

$$L_n[x,y] = \sum_{i+j=n}[L_i x, L_j y]$$

for all $x,y \in \mathcal{A}$ and all $n \geq 0$. Clearly, every higher derivation is a Lie higher derivation but the converse is not true in general. Let $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$ and $\{D_n\}_{n=0}^\infty$ be a higher derivation on $\mathcal{A}$. For any $n \geq 0$, let $L_n = D_n + \Delta_n$, where $\{\Delta_n : \mathcal{A} \to \mathcal{Z}(\mathcal{A})\}_{n=0}^\infty$ is a sequence of linear mappings such that $\Delta_0 = 0$ and $\Delta_n([x,y]) = 0$ for each $x,y \in \mathcal{A}$ and all $n \geq 0$. It is easily checked that $\{L_n\}_{n=0}^\infty$ is a Lie higher derivation and not a higher derivation if $\Delta_n \neq 0$ for some $n$. Lie higher derivations of the above form are called proper.

The natural problem that one considers in this context is whether or not every Lie higher derivation is proper. In [10], the author discussed the properties of Lie higher derivations. In [12], Qi and Hou showed that each Lie higher derivation is proper on nest algebras. In [6], Li and Shen proved that the same is true for triangular algebras. In this paper we are going to show that every Lie higher derivation on $C^*$-algebras is standard. This is an extension of Johnson’s result in [4].

2. Main results

In the following theorem we give a representation of Lie higher derivations in terms of Lie derivations.

**Theorem 2.1.** Let $\{L_n\}_{n=0}^\infty$ be a Lie higher derivation on $\mathcal{A}$. Then there exists a sequence $\{l_n\}_{n=0}^\infty$ of Lie derivations on $\mathcal{A}$ such that for every $n \geq 1$, we have

$$L_n = \sum_{i=1}^{n} \left( \sum_{j=1}^{r_j=n} \left( \prod_{j=1}^{i-r_j} \frac{1}{r_1 + r_2 + \cdots + r_i} l_{r_1} l_{r_2} \cdots l_{r_i} \right) \right),$$

where the inner summation is taken over all positive integers $r_j$ with $\sum_{j=1}^{i} r_j = n$.

**Proof.** Let $\{L_n\}_{n=0}^\infty$ be a Lie higher derivation on $\mathcal{A}$. First we show that there exists a sequence $\{l_n\}_{n=1}^\infty$ of Lie derivations on $\mathcal{A}$ such that $(n+1)L_{n+1} = \sum_{k=0}^{n} L_k L_{n-k}$ for every $n \geq 0$. We use induction on $n$. For $n = 0$, We have

$$L_1([x,y]) = \sum_{i+j=1} [L_i x, L_j y] = [L_1 x, y] + [x, L_1 y].$$
Thus if \( l_1 := L_1 \), then \( l_1 \) is a Lie derivation on \( \mathcal{A} \). Suppose that \( l_k \) is defined and is a Lie derivation for \( k \leq n \) and \( (r + 1)L_{r+1} = \sum_{k=0}^{r} l_{k+1}L_{r-k} \), for \( r \leq n \).

Define \( l_{n+1} := (n + 1)L_{n+1} - \sum_{k=0}^{n-1} l_{k+1}L_{n-k} \). We are going to show that \( l_{n+1} \) is a Lie derivation on \( \mathcal{A} \). For \( x, y \in \mathcal{A} \), we have

\[
l_{n+1}([x, y]) = (n + 1)L_{n+1}([x, y]) - \sum_{k=0}^{n-1} l_{k+1}L_{n-k}([x, y])
\]

\[
= \sum_{i+j=n+1} (n + 1)[L_i x, L_j y] - \sum_{k=0}^{n-1} l_{k+1} \left( \sum_{r+s=n-k} [L_r x, L_s y] \right)
\]

\[
= \sum_{i+j=n+1} (i + j)[L_i x, L_j y] - \sum_{k=0}^{n-1} \sum_{r+s=n-k} [l_{k+1}L_r x, L_s y] + [L_r x, l_{k+1}L_s y]
\]

\[
= \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{k=0}^{n-1} \sum_{r+s=n-k} [l_{k+1}L_r x, L_s y] + [L_r x, l_{k+1}L_s y]
\]

\[
= K_1 + K_2.
\]

Note that for \( 0 \leq k \leq n - 1 \) and \( r + s = n - k \) if we put \( u := r + k \), then \( 0 \leq u \leq n, u + s = n, 0 \leq k \leq u \) and \( k \neq n \). Thus by \( L_0 = I \), we have

\[
K_1 = \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{k=0}^{n-1} \sum_{r+s=n-k} [l_{k+1}L_r x, L_s y]
\]

\[
= \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{u=0}^{n} \sum_{k=0, k+u+s=n, k \neq n} [l_{k+1}L_u x, L_s y]
\]

\[
= \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{u=0}^{n-1} \sum_{k=0, k+u+s=n, k \neq n} [l_{k+1}L_u x, L_s y]
\]

\[- \sum_{k=0}^{n-1} [l_{k+1}L_{n-k} x, L_0 y].
\]
Thus

\[ K_1 + \sum_{k=0}^{n-1} [l_{k+1}L_{n-k}x, y] \]

\[ = \sum_{i+j=n+1} i[L_ix, L_jy] - \sum_{u=0}^{n-1} \sum_{k=0, u+s=n, k \neq n}^u [l_{k+1}L_{u-k}x, L_sy] \]

\[ = \sum_{u=0, u+s=n}^n (u+1)[L_{u+1}x, L_sy] - \sum_{u=0}^{n-1} \sum_{k=0, u+s=n, k \neq n}^u [l_{k+1}L_{u-k}x, L_sy] \]

\[ = [(n+1)L_{n+1}x, y] + \sum_{u=0, u+s=n}^{n-1} [((u+1)L_{u+1}x - \sum_{k=0}^u l_{k+1}L_{u-k}x), L_sy]. \]

The second equality above is obtained by replacing \( i, j \) by \( u+1 \) and \( r \), respectively, in the first summation. By our assumption we have \((u+1)L_{u+1}x - \sum_{k=0}^u l_{k+1}L_{u-k}x = 0\) for \( 0 \leq u \leq n-1 \) and \( x \in M \). Therefore,

\[ K_1 = [(n+1)L_{n+1}x, y] - \sum_{k=0}^{n-1} [l_{k+1}L_{n-k}x, y] \]

\[ = [((n+1)L_{n+1}x - \sum_{k=0}^u l_{k+1}L_{n-k}x), y] \]

\[ = [l_{n+1}x, y]. \]

By a similar argument we have

\[ K_2 = \sum_{i+j=n+1} j[L_ix, L_jy] - \sum_{k=0}^{n-1} \sum_{r+s=n-k}^u [L_rx, l_{k+1}L_sy] \]

\[ = [x, l_{n+1}y]. \]

Thus

\[ l_{n+1}(x, y) = K_1 + K_2 = [l_{n+1}x, y] + [x, l_{n+1}y]. \]

Whence \( l_{n+1} \) is a derivation and clearly, \((n+1)L_{n+1} = \sum_{k=0}^n l_{k+1}L_{n-k}\). Now, Theorem 2.3 of [9] implies that for \( n \geq 1 \) we have

\[ L_n = \sum_{i=1}^n \left( \sum_{j=1}^{r_j=n} \left( \Pi_{j=1}^i \frac{1}{r_1 + r_2 + \ldots + r_i} \right) l_{r_1}l_{r_2}\ldots l_{r_i} \right), \]

where the inner summation is taken over all positive integers \( r_j \) with \( \sum_{j=1}^i r_j = n \).

The following lemma is our key to prove our main result.
Lemma 2.2. (\cite{13}) Every derivation on a C*-algebra annihilates its center.

Theorem 2.3. Let \( \mathcal{A} \) be a C*-algebra and \( \{L_n\}_{n=0}^{\infty} \) a sequence of linear mappings from \( \mathcal{A} \) into \( \mathcal{A} \) with \( L_0 = I \). Then \( \{L_n\}_{n=0}^{\infty} \) is a Lie higher derivation if and only if there exist a higher derivation \( \{D_n : \mathcal{A} \to \mathcal{A}\}_{n=0}^{\infty} \) and a sequence of linear mappings \( \{\Delta_n : \mathcal{A} \to Z(\mathcal{A})\}_{n=0}^{\infty} \) such that \( \Delta_0 = 0 \), \( \Delta_n([x,y]) = 0 \) and \( L_n = D_n + \Delta_n \) for every \( x, y \in \mathcal{A} \) and all \( n \geq 0 \).

Proof. Let \( \{L_n\}_{n=0}^{\infty} \) be a Lie higher derivation on \( \mathcal{A} \). Define \( \Delta_0, d_0 : \mathcal{A} \to \mathcal{A} \) by \( \Delta_0 = 0 \) and \( d_0 = I \). Lie derivations \( l_i \) satisfying (2.1) can be decomposed as \( d_{r_i} + \delta_{r_i} \), where \( d_{r_i} : \mathcal{A} \to \mathcal{A} \) is a derivation and \( \delta_{r_i} : \mathcal{A} \to Z(\mathcal{A}) \) vanishes at each commutator, see \cite{7}. Therefore, for \( n \geq 1 \) we have

\[
L_n = \sum_{i=1}^{n} \left( \sum_{\sum_j r_j = n} \left( \prod_{j=1}^{i} \frac{1}{r_1 + r_2 + \cdots + r_i} \right) \left( d_{r_1} + \delta_{r_1} \right) \left( d_{r_2} + \delta_{r_2} \right) \cdots \left( d_{r_i} + \delta_{r_i} \right) \right)
\]

If we define

\[
D_n = \sum_{i=1}^{n} \left( \sum_{\sum_j r_j = n} \left( \prod_{j=1}^{i} \frac{1}{r_1 + r_2 + \cdots + r_i} \right) d_{r_1} d_{r_2} \cdots d_{r_i} \right),
\]

then Theorem 2.5 of \cite{9} implies that \( \{D_n\}_{n=0}^{\infty} \) is a higher derivation. Clearly, \( \Delta_1 = \delta_1 \) and by Lemma 2.2 for \( n \geq 2 \) we have

\[
\Delta_n = \sum_{i=1}^{n} \left( \sum_{\sum_j r_j = n} \left( \prod_{j=1}^{i} \frac{1}{r_1 + r_2 + \cdots + r_i} \right) \delta_{r_1} \delta_{r_2} \cdots \delta_{r_{i-1}} \delta_{r_i} \right) + \sum_{i=2}^{n} \left( \sum_{\sum_j r_j = n} \left( \prod_{j=1}^{i} \frac{1}{r_1 + r_2 + \cdots + r_i} \right) \delta_{r_1} \delta_{r_2} \cdots \delta_{r_{i-1}} d_{r_i} \right).
\]

Therefore, \( \Delta_n : \mathcal{A} \to Z(\mathcal{A}) \) is defined for every \( n \geq 0 \), \( \Delta_n([x,y]) = 0 \) and \( L_n = D_n + \Delta_n \) for every \( x, y \in \mathcal{A} \) and all \( n \geq 0 \). The converse is easy to verify. \(\square\)

References


Characterization Lie higher Derivations on $C^*$-algebras


(Ali Reza Janfada) DEPARTMENT OF SCIENCE, UNIVERSITY OF BIRJAND, P.O. BOX 414, BIRJAND 9717851367, BIRJAND, IRAN
E-mail address: ajanfada@birjand.ac.ir

(Hossein Saidi) DEPARTMENT OF SCIENCE, UNIVERSITY OF BIRJAND, P.O. BOX 414, BIRJAND 9717851367, BIRJAND, IRAN
E-mail address: saidi_math8287@yahoo.com

(Madjid Mirzavaziri) DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P.O. BOX 1159, MASHHAD 91775, MASHHAD, IRAN
E-mail address: mirzavaziri@gmail.com