KINDS OF DERIVATIONS ON HILBERT C\(^*\)-MODULES AND THEIR OPERATOR ALGEBRAS

HOSSEIN SAIDI, ALI REZA JANFADA, AND MADJID MIRZAVAZIRI

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Abstract. Let \( \mathcal{M} \) be a Hilbert \( C^* \)-module. A linear mapping \( d : \mathcal{M} \to \mathcal{M} \) is called a derivation if
\[
d(\langle x, y \rangle_z) = \langle dx, y \rangle_z + \langle x, dy \rangle_z + \langle x, y \rangle dz
\]
for all \( x, y, z \in \mathcal{M} \). We give some results for derivations and automatic continuity of them on \( \mathcal{M} \). Also, we will characterize generalized derivations and strong higher derivations on the algebra of compact operators and adjointable operators of Hilbert \( C^* \)-modules, respectively.

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1. INTRODUCTION AND PRELIMINARIES

Let \( \mathcal{A} \) be a \( C^* \)-algebra. A pre-Hilbert \( \mathcal{A} \)-module \( \mathcal{M} \) is a left \( \mathcal{A} \)-module equipped with a sesquilinear form \( \langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A} \) which satisfies the following axioms for all \( x, y \in \mathcal{M} \) and \( a \in \mathcal{A} \):

1. \( \langle x, x \rangle \geq 0 \);
2. \( \langle x, x \rangle = 0 \iff x = 0 \);
3. \( \langle x, y \rangle^* = \langle y, x \rangle \);
4. \( \langle ax, y \rangle = \langle x, \bar{a}y \rangle \).

For every \( x \in \mathcal{M} \), set \( \| x \| = \|\langle x, x \rangle \|^{1/2} \). A pre-Hilbert \( \mathcal{A} \)-module \( \mathcal{M} \) which is complete with respect to this norm is called a Hilbert \( \mathcal{A} \)-module. For example, a complex Hilbert space \( \mathcal{H} \) is a Hilbert \( C^* \)-module over the \( C^* \)-algebra of complex numbers or a \( C^* \)-algebra \( \mathcal{A} \) is a Hilbert \( C^* \)-module over \( \mathcal{A} \) by \( \langle a, b \rangle = ab^* \), for all \( a, b \in \mathcal{A} \). A linear mapping \( T : \mathcal{M} \to \mathcal{M} \) is called an operator if \( T \) is continuous and \( \mathcal{A} \)-linear (i.e. \( T(ax) = aT(x) \) for all \( a \in \mathcal{A} \) and \( x \in \mathcal{M} \)). By \( \text{End}(\mathcal{M}) \), we denote the set of all operators on \( \mathcal{M} \). A mapping \( T : \mathcal{M} \to \mathcal{M} \) is called adjointable if there exists a mapping \( T^* : \mathcal{M} \to \mathcal{M} \) such that \( \langle Tx, y \rangle = \langle x, T^*y \rangle \) for all \( x, y \in \mathcal{M} \). As a well-known result, every adjointable mapping \( T : \mathcal{M} \to \mathcal{M} \) is an operator. The set of all adjointable mappings on \( \mathcal{M} \) is denoted by \( \text{End}^*(\mathcal{M}) \) which is a \( C^* \)-algebra under the usual operator norm. For \( x, y \in \mathcal{M} \), define \( \theta_{x, y} : \mathcal{M} \to \mathcal{M} \) by \( \theta_{x, y}(z) = \langle z, y \rangle x \), for all \( z \in \mathcal{M} \). Clearly, \( \theta_{x, y} \in \text{End}^*(\mathcal{M}) \) with \( \theta_{x, y}^* = \theta_{y, x} \).

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Note that $\theta_{x,y}$ is quite different from rank one projections in Hilbert spaces. For example we cannot infer $x = 0$ or $y = 0$ from $\theta_{x,y} = 0$. We denote by $\mathcal{K}(\mathcal{M})$ the closed linear span of $\{\theta_{x,y} : x, y \in \mathcal{M}\}$. The elements of $\mathcal{K}(\mathcal{M})$ are called \textit{compact operators}. This concept of compact operators is different from compact operators in the usual sense. However, this concept coincides with the concept of usual compact operators when we choose a Hilbert $C^*$-module. Set $I = \text{span}\{<x, y> : x, y \in \mathcal{M}\}$. It is easy to see that $I$ is a $*$-bi-ideal of $\mathcal{A}$. An important class of Hilbert $C^*$-modules are \textit{full} modules. A Hilbert $C^*$-module $\mathcal{M}$ is called full if $\mathcal{T} = \mathcal{A}$, where $\mathcal{T}$ is the norm closure of $I$ in $\mathcal{A}$. For example, $\mathcal{A}$ is a full $\mathcal{A}$-module. It is well-known that the derivations on Banach algebras are the generators of certain dynamical systems. A linear mapping $\phi : \mathcal{M} \to \mathcal{M}$ is called a \textit{homomorphism} if $\phi(><x, y>z) = <\phi x, \phi y> \phi z$ for all $x, y, z \in \mathcal{M}$. A dynamical system on $\mathcal{M}$ is strongly continuous one-parameter family $(u_t)_{t \in \mathbb{R}}$ of homomorphisms. A linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a \textit{derivation} if $d(><x, y>z) = <dx, y>z + <x, dy>z + <x, y>dz$ for all $x, y, z \in \mathcal{M}$, see [1] and [2]. In [1], Abbaspour and Skeide proved that a $C_0$-group $u = (u_t)_{t \in \mathbb{R}}$ is a dynamical system if and only if its generator is a derivation and every derivation on full Hilbert $C^*$-module $\mathcal{M}$ is a generalized derivation i.e. there exists a derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that $d(ax) = \delta(a)x + ad(x)$ for all $a \in \mathcal{A}$ and $x \in \mathcal{M}$. Also, they proved that every derivation on full Hilbert $C^*$-modules extends as a $*$-derivation to the linking algebra. In this paper, we consider derivations on Hilbert $C^*$-modules and give some results about adjointable derivations and automatic continuity of them.

Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a linear mapping. A $\sigma$-\textit{derivation} is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. If $\sigma = I$, where $I$ is the identity operator on $\mathcal{A}$, then $d$ is a derivation. A generalized derivation on $\mathcal{A}$ is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that there exists a derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. In [7], P. Li, D. Han and W. S. Tang proved that every derivation on $\text{End}^*(\mathcal{M})$ is inner if $\mathcal{A}$ is commutative and unital. In section 3, we will characterize generalized derivations on $\mathcal{K}(\mathcal{M})$ without commutativity condition. Suppose that $\{d_n\}_{n=0}^{\infty}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{A}$. It’s called a \textit{higher derivation} if $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$ for all $a, b \in \mathcal{A}$ and all $n \geq 0$. If $d_0 = I$, $\{d_n\}_{n=0}^{\infty}$ is called a \textit{strong higher derivation}. Let $\delta$ be a derivation on $\mathcal{A}$ and define the sequence $\{d_n\}_{n=0}^{\infty}$ on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{d^n}{n!}$ for every $n \geq 1$. By Leibnitz rule, $\{d_n\}_{n=0}^{\infty}$ is a higher derivation on $\mathcal{A}$. Higher derivations were introduced by Hasse and Schmidt [4] and algebraists sometimes call them Hasse-Schmidt derivations. For a higher derivation obviously, $d_0$ is a homomorphism and $d_1$ is a $d_0$-derivation in the sense of [11]. Therefore, higher derivations are the generalizations of homomorphisms and derivations. In [12], higher derivations
are applied to study generic solving of higher differential equations. For more information about higher derivations and its applications see [5], [6], [9] and [10]. The last author in [10], characterized the strong higher derivations in terms of derivations. In section 4 we give a characterization of higher derivation on \( End^*(\mathcal{M}) \) with use of elements whose product is in \( \mathcal{K}(\mathcal{M}) \).

2. Derivations on Hilbert \( C^* \)-modules

Let \( \mathcal{M} \) be a Hilbert \( C^* \)-module. Recall that a linear mapping \( d : \mathcal{M} \to \mathcal{M} \) is called a derivation if
\[
d(<x,y,z>) = <dx,y>z + <x,dy>z + <x,y>dz
\]
for all \( x,y,z \in \mathcal{M} \). Note that if \( d : \mathcal{M} \to \mathcal{M} \) is an adjointable map with \( d^* = -d \), then \( d \) is a derivation. But the converse is not true. For example suppose that \( H \) is a Hilbert space. Set \( u_0 \in B(H) \) such that \( u^* = -u \) and \( u \) is not in the center of \( B(H) \). Define \( d : B(H) \to B(H) \) by \( d(v) = u_0v - vvu_0 \) for every \( v \in B(H) \). It is easy to see that \( d \) is a derivation on \( B(H) \) as a \( B(H) \)-module but \( d \) is not adjointable. Otherwise, \( d \) is \( \mathcal{A} \)-linear and therefore,
\[
u_0vv - vvu_0 = d vv = d(vv) = vdu_0v - vvu_0
\]
for every \( v \in B(H) \). This implies that \( u_0 \) is in the center of \( B(H) \), which is a contradiction. Let \( M \) be a full Hilbert \( C^* \)-module. Note that if there exists \( a \in \mathcal{A} \) such that \( ax = o \) for every \( x \in \mathcal{M} \), then \( a = o \). Therefore, we have the following theorem:

**Theorem 1.** Let \( \mathcal{M} \) be a full Hilbert \( C^* \)-module. Then \( d \in End^*(\mathcal{M}) \) is a derivation if and only if \( d^* = -d \).

**Proof.** Suppose that \( d \in End^*(\mathcal{M}) \) is a derivation. Then \( (<dx,y> + <x,dy>)z = 0 \) for all \( x,y,z \in \mathcal{M} \). Hence \( d^* = -d \). The converse is trivial. \( \square \)

A set of non-zero elements \( \{x_i\}_{i \in I} \subseteq \mathcal{M} \) is called a standard basis for \( \mathcal{M} \) if the reconstruction formula \( x = \sum_{i \in I} <x,x_i>x_i \) holds for every \( x \in \mathcal{M} \). Let
\[
L_n(\mathcal{A}) = \{(a_1,a_2,\ldots,a_n) : a_i \in \mathcal{A}, 1 \leq i \leq n\}.
\]
Then \( L_n(\mathcal{A}) \) a Hilbert \( C^* \)-module over \( C^* \)-algebra \( \mathcal{A} \) with module product
\[
a(a_1,a_2,\ldots,a_n) = (aa_1,aa_2,\ldots,aa_n)
\]
and inner product
\[
< (a_1,a_2,\ldots,a_n), (b_1,b_2,\ldots,b_n) > = a_1b_1^* + a_2b_2^* + \cdots + a_nb_n^*
\]
for every \( a \in \mathcal{A} \) and \( (a_1,a_2,\ldots,a_n),(b_1,b_2,\ldots,b_n) \in L_n(\mathcal{A}) \), see [8]. If \( \mathcal{A} \) is unital then \( L_n(\mathcal{A}) \) has standard basis \( \{e_i\}_{i=1}^n \) such that \( e_i = (0,0,\ldots,1_{i-th},\ldots,0) \) and \( 1 \leq i \leq n \). In [3], Bakic and proved that every Hilbert \( C^* \)-module over the \( C^* \)-algebra of the compact operators possesses a standard basis.

**Theorem 2.** Let \( \mathcal{M} \) have a standard basis and \( d \in End^*(\mathcal{M}) \). Then \( d \) is a derivation if and only if \( d^* = -d \).
Proof. Let \( \{x_i\}_{i \in I} \) be a standard basis for \( \mathcal{M} \) and \( d \in \text{End}^* (\mathcal{M}) \) be a derivation. Then \( d(x) = \sum_{i \in I} \langle x, x_i \rangle > 0 dx_i \). On the other hand,

\[
\begin{align*}
d x &= \sum_{i \in I} < dx, x_i > x_i + \sum_{i \in I} < x, dx_i > x_i + \sum_{i \in I} < x, x_i > dx_i \\
&= dx + \sum_{i \in I} < d^* x, x_i > x_i + \sum_{i \in I} < x, x_i > dx_i \\
&= dx + d^* x + dx.
\end{align*}
\]

So, \( d^* = -d \).

Lemma 1. Let \( \mathcal{M} \) be a full Hilbert \( C^* \)-module over unital \( C^* \)-algebra \( A \). Then there exist \( x_1, \ldots, x_n \in \mathcal{M} \) such that \( \sum_{i=1}^n < x_i, x_i > = 1 \).

Proof. See [8]. □

A Hilbert \( C^* \)-module \( \mathcal{M} \) over \( C^* \)-algebra \( A \) is called simple if the only closed submodules of \( \mathcal{M} \) over \( A \) are \( \{0\} \) and \( \mathcal{M} \). For example, let \( H \) be a Hilbert space and \( \mathcal{K}(H) \) denotes the the algebra of compact operator on \( H \). Then \( \mathcal{K}(H) \) is a simple Hilbert \( C^* \)-module over itself.

Theorem 3. Let \( \mathcal{M} \) be a full and simple Hilbert \( C^* \)-module over the unital \( C^* \)-algebra \( A \) and \( d \) be a derivation on \( \mathcal{M} \) with closed range. Then \( d \) is continuous or surjective.

Proof. Define the separating space \( S(d) = \{ y \in \mathcal{M} : \exists \{x_n\} \to 0 \text{ in } \mathcal{M} \text{ such that } dx_n \to y \} \). As a well-known result \( S(d) \) is a closed subspace of \( \mathcal{M} \). By lemma 1, there exist \( x_1, \ldots, x_m \) such that \( \sum_{i=1}^m < a x_i, x_i > = 1 \). Therefore, \( a = \sum_{i=1}^m < a x_i, x_i > \) for all \( a \in A \). For \( z \in S(d) \) there exists a sequence \( z_n \to 0 \) such that that \( dz_n \to z \). Hence

\[
d(az_n) = \sum_{i=1}^m < adx_i, x_i > z_n + \sum_{i=1}^m < ax_i, dx_i > z_n + \sum_{i=1}^m a < x_i, x_i > dz_n \to az.
\]

(2.1)

This implies that \( S(d) \) is a submodule of \( \mathcal{M} \). Since \( \mathcal{M} \) is simple, \( S(d) = \{0\} \) or \( S(d) = \mathcal{M} \). If \( S(d) = \{0\} \), by closed graph theorem, \( d \) is continuous. If \( S(d) = \mathcal{M} \) by (2.1), \( A \mathcal{M} \subseteq \overline{Im(d)} \). Since \( A \) is unital \( A \mathcal{M} = \mathcal{M} \). Therefore, \( \overline{Im(d)} = Im(d) = \mathcal{M} \) and \( T \) is surjective. □

Lemma 2. Let \( \mathcal{M} \) be a Hilbert \( C^* \)-module over unital \( C^* \)-algebra \( A \). Then \( I \mathcal{M} = \mathcal{M} \).

Proof. Clearly, \( I \mathcal{M} \subseteq \mathcal{M} \). let \( z \in \mathcal{M} \) and set

\[
x = \lim_{n \to \infty} \frac{1}{n} + < z, z > 1/3 - 1 z.
\]

One can see that \( z = < x, x > x \) and therefore, \( I \mathcal{M} = \mathcal{M} \). For more detail see [8]. □
Theorem 4. Let \( \mathcal{M} \) be a Hilbert \( C^* \)-module over unital \( C^* \)-algebra \( A \). Suppose that \( \mathcal{M} \) is a simple \( I \)-module and \( d \) be a derivation on \( \mathcal{M} \) with closed range. Then \( d \) is continuous or surjective.

Proof. Let \( a \in I \) and \( z \in S(d) \). So there exist a sequence 
\[
z_n \rightarrow 0, \ \ x_1, x_2, \cdots, x_m, \ \ y_1, y_2, \cdots, y_m \in \mathcal{M}
\]
for some \( m \in \mathbb{N} \) such that \( dz_n \rightarrow z \) and \( a = \sum_{i=1}^{m} < x_i, y_i > \). But
\[
d(az_n) = \sum_{i=1}^{m} < dx_i, y_i > z_n + \sum_{i=1}^{m} < x_i, dy_i > z_n + \sum_{i=1}^{m} adz_n \rightarrow az. \quad (2.2)
\]
This implies that \( S(d) \) is a submodule of \( \mathcal{M} \). Therefore, \( S(d) = \{0\} \) or \( S(d) = \mathcal{M} \). If \( S(d) = \{0\} \), by closed graph theorem, \( d \) is continuous. If \( S(d) = \mathcal{M} \), by (2.2), \( I \mathcal{M} \subseteq Tm(d) \). Therefore, by lemma 2, \( Im(d) = \mathcal{M} \) and \( T \) is surjective. \( \square \)

3. Characterization of Generalized Derivations on the Algebra of Compact Operators

Let \( A \) be an algebra. Recall that a derivation on \( A \) is a linear mapping \( \delta : A \rightarrow A \) such that \( \delta(ab) = a\delta(b) + \delta(a)b \) for all \( a, b \in A \). A generalized derivation on \( A \) is a linear mapping \( d : A \rightarrow A \) such that there exists a derivation \( \delta : A \rightarrow A \) such that \( d(ab) = d(a)b + \delta(b) \) for all \( a, b \in A \). Recall that a linear mapping \( \Pi : A \rightarrow A \) is called a left multiplier if \( \Pi(ab) = \Pi(a)b \) for all \( a, b \in A \). For a generalized derivation \( d \), set \( \Pi = d - \delta \). One can easily see that \( \Pi \) is a left multiplier. Let \( d : A \rightarrow A \) be a linear mapping. As a well-known result \( d \) is a generalized derivation if and only if there exist a derivation \( \delta : A \rightarrow A \) and left multiplier \( \Pi : A \rightarrow A \) such that \( d = \delta + \Pi \).

Theorem 5. Let \( \mathcal{M} \) be a full Hilbert \( C^* \)-module over unital \( C^* \)-algebra \( A \). Then linear mapping \( \Pi : \mathcal{K}(\mathcal{M}) \rightarrow \mathcal{K}(\mathcal{M}) \) is a left multiplier if and only if there exists \( T \in \text{End}(\mathcal{M}) \) such that \( \Pi(A) = TA \) for all \( A \in \mathcal{K}(\mathcal{M}) \).

Proof. Let \( T \in \text{End}(\mathcal{M}) \). Define \( \Pi : \mathcal{K}(\mathcal{M}) \rightarrow \mathcal{K}(\mathcal{M}) \) by \( \Pi(A) = TA \), for every \( A \in \mathcal{K}(\mathcal{M}) \). Clearly \( \Pi \) is a left multiplier. Conversely, since \( \mathcal{M} \) is full, by lemma 1, there exist \( x_1, \cdots, x_n \) such that \( \sum_{i=1}^{n} < x_i, x_i > = 1 \). Define \( T : \mathcal{M} \rightarrow \mathcal{M} \) by
\[
T(x) = \sum_{i=1}^{n} \Pi(\theta_{x, x_i}) x_i,
\]
for every \( x \in \mathcal{M} \). For every \( A \in \mathcal{K}(\mathcal{M}) \) we have
\[
TA(x) = \sum_{i=1}^{n} \Pi(\theta_{Ax, x_i}) x_i = \sum_{i=1}^{n} \Pi(A \theta_{x, x_i}) x_i = \sum_{i=1}^{n} \Pi(A)(\theta_{x, x_i}) x_i
\]
\[
\Pi(A)x = \Pi(A)x.
\]
So \(\Pi(A) = TA\). \(T\) is obviously a continuous linear mapping. To show that \(T \in \text{End}(\mathcal{M})\) it’s remain to show that \(T\) is \(\mathcal{A}\)-linear. Now suppose that \(a \in \mathcal{A}, x \in \mathcal{M}\) and \(A \in \mathcal{K}(\mathcal{M})\) We have,
\[
\Pi(A)(ax) = TA(ax) = T(aA(x))
\]
On the other hand
\[
\Pi(A)(ax) = a\Pi(A)(a) = aTA(x)
\]
and so \(T(aA(x)) = aTA(x)\) for every \(a \in \mathcal{A}, x \in \mathcal{M}\) and \(A \in \mathcal{K}(\mathcal{M})\). Now lemma 2 implies that \(T\) is \(\mathcal{A}\)-linear. 

**Definition 1.** By \(L_0(\mathcal{M})\) we denote the set of all linear mapping \(A\) on \(\mathcal{M}\) such that \(AB - CA \in \mathcal{K}(\mathcal{M})\) for all \(B, C \in \mathcal{K}(\mathcal{M})\). Clearly \(\text{End}^*(\mathcal{M}) \subset L_0(\mathcal{M})\).

**Theorem 6.** Let \(\mathcal{M}\) be a full Hilbert \(C^*\)-module over unital \(C^*\)-algebra \(\mathcal{A}\). Then linear mapping \(\delta : \mathcal{K}(\mathcal{M}) \rightarrow \mathcal{K}(\mathcal{M})\) is a derivation if and only if there exists \(T \in L_0(\mathcal{M})\) such that \(\delta(A) = TA - AT\) for all \(A \in \mathcal{K}(\mathcal{M})\).

**Proof.** Let \(T \in L_0(\mathcal{M})\). Define \(\delta : \mathcal{K}(\mathcal{M}) \rightarrow \mathcal{K}(\mathcal{M})\) by \(\delta(A) = TA - AT\), for every \(A \in \mathcal{K}(\mathcal{M})\). Clearly, \(\delta\) is a derivation. Conversely, since \(\mathcal{M}\) is full by lemma 2 there exist \(x_1, \ldots, x_n\) such that \(\sum_{i=1}^{n} <x_i, x_i> = 1\). Define \(T : \mathcal{M} \rightarrow \mathcal{M}\) by
\[
T(x) = \sum_{i=1}^{n} \delta(\theta_{x,x_i})x_i,
\]
for every \(x \in \mathcal{M}\). For every \(A \in \mathcal{K}(\mathcal{M})\) we have
\[
TA(x) = \sum_{i=1}^{n} \delta(A\theta_{x,x_i})x_i
= \sum_{i=1}^{n} \delta(A\theta_{x,x_i})x_i
= \sum_{i=1}^{n} \delta(A)(\theta_{x,x_i})x_i + \sum_{i=1}^{n} A\delta(\theta_{x,x_i})x_i
= \delta(A)x + AT(x).
\]
\[\square\]
Theorem 7. Let \( \mathcal{M} \) be a full Hilbert \( \mathcal{C}^* \)-module over unital \( \mathcal{C}^* \)-algebra \( \mathcal{A} \) and \( d : \mathcal{K}(\mathcal{M}) \rightarrow \mathcal{K}(\mathcal{M}) \) be a linear mapping. Then \( d \) is a generalized derivation if and only if there exist \( T_1 \in \mathcal{L}_0(\mathcal{M}) \) and \( T_2 \in \text{End}(\mathcal{M}) \) such that \( d(A) = T_1 A - AT_1 + T_2 A \) for every \( A \in \mathcal{K}(\mathcal{M}) \).

4. Characterization of higher derivation on the algebra of adjointable operators

Let \( \mathcal{A} \) be an algebra and suppose that \( \{d_n\}_{n=0}^{\infty} \) is a sequence of linear mappings from \( \mathcal{A} \) into \( \mathcal{A} \). It’s called a higher derivation if

\[
d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)
\]

for all \( a, b \in \mathcal{A} \) and all \( n \geq 0 \). If \( d_0 = I \), \( \{d_n\}_{n=0}^{\infty} \) is called a strong higher derivation.

If (4.1) holds for all \( x, y \in \mathcal{A} \) and \( n = 0, 1, 2, \ldots, m \), it is called a higher derivation of rank \( m \). Now we are going to give a characterization of strong higher derivations in terms of operators whose product is compact.

Theorem 8. Let \( \mathcal{M} \) be a full Hilbert \( \mathcal{C}^* \)-module over the unital \( \mathcal{C}^* \)-algebra \( \mathcal{A} \). Let \( \{d_n : \text{End}^*(\mathcal{M}) \rightarrow \text{End}^*(\mathcal{M})\}_{n=0}^{\infty} \) be a sequence of linear mappings such that \( d_0 = I \). Then \( \{d_n\}_{n=0}^{\infty} \) is a strong higher derivation if and only if \( d_n(AB) = \sum_{i=0}^{n} d_i(A)d_{n-i}(B) \) for all \( A, B \in \text{End}^*(\mathcal{M}) \) such that \( AB \in \mathcal{K}(\mathcal{M}) \) and all \( n \geq 1 \).

Proof. By lemma 1, there exist \( x_1, \ldots, x_n \) such that \( \sum_{i=1}^{n} < x_i, x_i > = 1 \). Let \( x_i \) for some \( 1 \leq i \leq n, x \in \mathcal{M}, A, B \in \text{End}^*(\mathcal{M}) \), and \( m \geq 1 \) be arbitrary elements. Since \( \mathcal{K}(\mathcal{M}) \) is a two sided ideal in \( \text{End}^*(\mathcal{M}) \),

\[
d_m(A\theta_{x,x_i}) = \sum_{i=0}^{m} d_i(A)d_{m-i}(\theta_{x,x_i})
\]

and

\[
d_m(AB\theta_{x,x_i}) = d_m(AB)\theta_{x,x_i} + \sum_{i=0}^{m-1} d_i(AB)d_{m-i}(\theta_{x,x_i}).
\]

On the other hand,

\[
d_m(AB\theta_{x,x_i}) = d_m(A)B\theta_{x,x_i} + \sum_{i=0}^{m-1} d_i(A)d_{m-i}(B\theta_{x,x_i}).
\]

Take \( m = 1 \). By comparing these equalities, we obtain

\[
d_1(AB)\theta_{x,x_i} = d_1(A)B\theta_{x,x_i} + Ad_1(B)\theta_{x,x_i}.
\]
So
\[ d_1(AB)x = \sum_{i=1}^{n} d_1(AB) < x_i, x_i > x = \sum_{i=1}^{n} d_1(A)B < x_i, x_i > x + \sum_{i=1}^{n} Ad_1(B) < x_i, x_i > x = d_1(A)Bx + Ad_1(B)x. \]

This implies that \( d_1 \) is a derivation. As an induction suppose that \( \{d_0, d_1, \cdots, d_m\} \) is a higher derivation of rank \( m \). By induction, we get
\[ d_{m+1}(AB\theta_{x,x_i}) = d_{m+1}(AB)\theta_{x,x_i} + \sum_{i=0}^{m} d_i(AB)d_{m+1-i}(\theta_{x,x_i}) = d_{m+1}(AB)\theta_{x,x_i} + \sum_{i=0}^{m} \sum_{j=0}^{m-i} d_j(A)d_{j-1}(B)d_{m+1-i-j}(\theta_{x,x_i}) \]

and by (4.2),
\[ d_{m+1}(AB\theta_{x,x_i}) = d_{m+1}(A)B\theta_{x,x_i} + \sum_{i=0}^{m-i} d_i(A)d_{m+1-i}(B\theta_{x,x_i}) = d_{m}(A)B\theta_{x,x_i} + \sum_{i=0}^{m-i} \sum_{j=0}^{m-i} d_j(A)d_{j-1}(B)d_{m+1-i-j}(\theta_{x,x_i}). \]

One can see that
\[ \sum_{i=0}^{m} \sum_{j=0}^{m-i} d_j(A)d_{j-1}(B)d_{m+1-i-j}(\theta_{x,x_i}) = \sum_{i=0}^{m-i} \sum_{j=0}^{m-i} d_j(A)d_{j-1}(B)d_{m+1-i-j}(\theta_{x,x_i}). \]

Therefore,
\[ d_{m+1}(AB)\theta_{x,x_i} = \sum_{i=0}^{m+1} d_i(A)d_{m+1-i}(B)\theta_{x,x_i}. \]

So
\[ d_{m+1}(AB)x = \sum_{i=0}^{m+1} d_i(A)d_{m+1-i}(B)x. \]

will imply that \( \{d_0, d_1, \cdots, d_{m+1}\} \) is a higher derivation of rank \( m + 1 \). \( \square \)
References


Authors’ addresses

Hossein Saidi
University of Birjand, Department of Mathematics, P. O. Box 97175-615, Birjand, Iran
E-mail address: hosseinsaidi@birjand.ac.ir

Ali Reza Janfada
University of Birjand, Department of Mathematics, P. O. Box 97175-615, Birjand, Iran
E-mail address: ajanjfada@birjand.ac.ir

Madjid Mirzavaziri
Ferdowsi University of Mashhad, Department of Pure Mathematics, P. O. Box 91775-1159, Mashhad, Iran
Current address: Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Mashhad, Iran
E-mail address: mirzavaziri@um.ac.ir