A NEW CONSTRUCTION OF MULTIWAVELETS WITH COMPOSITE DILATIONS

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ABSTRACT

Consider an affine system $\mathcal{A}_{ab}(\Psi)$ with composite dilations $D_a, D_b,$ in which $a \in A, b \in B, A, B \subseteq GL_n(\mathbb{R})$ and $\Psi \in L^2(\mathbb{R}^n)$. It can be made an orthonormal $AB$-multiwavelet $\Psi$ or a parsval frame $AB$-wavelet $\Psi$, by choosing appropriate sets $A$ and $B$. In this paper, we construct an orthonormal $AB$-multiwavelet that arises from $AB$-multiresolution analysis. Our construction is useful since the group $B$ is shear group. More generally, we give a parsval frame $AB$-wavelet.

Indexing terms/Keywords

Wavelet with composite dilation; orthonormal basis; parsval frame; multiwavelet.

SUBJECT CLASSIFICATION

Mathematics Subject Classification: Primary 43A15, Secondary 42C40.
1 Introduction and Preliminaries

A collection of the form

\[ A_{\Lambda}(\Psi) = \{ D_{a}T_{k}\Psi : a \in A, k \in \mathbb{Z}^{n} \} \]

is an Affine system. If \( A_{\Lambda}(\Psi) \) is an orthonormal basis or, more generally a parsval frame for \( L^{2}(\mathbb{R}^{n}) \), then \( \Psi \) is called an \( A \)-wavelet or parsval frame \( A \)-wavelet, respectively. The Affine system \( A_{\Lambda}(\Psi) \) where \( \Psi = \chi_{\Omega} \), for some measurable set \( \Omega \subseteq \mathbb{R}^{n} \), is called minimally supported in frequency (MSF) system. If \( \Psi \) is a parsval frame \( A \)-wavelet for \( L^{2}(S)^{*} \), the corresponding function \( \Psi \) is called an MSF wavelet for \( L^{2}(S)^{*} \), in which \( L^{2}(S)^{*} = \{ f \in L^{2}(\mathbb{R}^{n}) : supp \hat{f} \subseteq S \} \), for some measurable set \( S \subseteq \mathbb{R}^{n} \). Fang and Wang in [6] introduce the MSF wavelets, which are studied also in [12], [13]. In particular, Dai and Larson in [2] consider a special kind of MSF wavelets \( \Psi \), which satisfy \( \hat{\Psi} = \chi_{\Omega} \) for some measurable sets \( \Omega \) in \( \mathbb{R}^{n} \). They prove that such a \( \Psi(x) \) is a wavelet with dilation set \( D = \{ 2^{n} : n \in \mathbb{Z} \} \) and translation set \( L = \mathbb{Z} \) if and only if

1. The sets \( \{ \Omega + \lambda : \lambda \in \mathbb{Z} \} \) is a tiling of \( \mathbb{R}^{n} \).
2. The sets \( \{ 2^{n}\Omega : n \in \mathbb{Z} \} \) is a tiling of \( \mathbb{R}^{n} \).

The result is later extended to higher dimensions in [3] for \( L = \mathbb{Z}^{n} \) and \( D = \{ A^{n} : n \in \mathbb{Z} \} \), where \( A \) is any expanding \( n \times n \) matrix. One can show in [10], [15], that \( \Psi \) is an orthonormal basis \( A \)-wavelet for \( L^{2}(S)^{*} \) if and only if \( \mathbb{R}^{n} = \bigcup_{k \in \mathbb{Z}^{n}}(\Omega + k) \) and \( S = \bigcup_{a \in A}(\Omega a^{-1}) \) where the unions are disjoint up to a set of measure zero. Also this result explain in [16], for \( L^{2}(\mathbb{R}^{n}) \). The construction and the study of orthonormal bases and parsval frames is of major importance in several areas of mathematics and applications, recently. The motivation for this study comes partly from signal processing, where such bases are useful in image compression and feature extraction. ([5], [8]).

To be more precise, we need to fix some notation. Throughout this paper, we shall consider the points \( x \in \mathbb{R}^{n} \) to be column vectors, i.e., \( x = \begin{pmatrix} x_{1} \\ \vdots \\ x_{n} \end{pmatrix} \), and the points \( \xi \in \mathbb{R}^{n} \) of the frequency domain to be row vectors, i.e., \( \xi = (\xi_{1}, \ldots, \xi_{n})^{T} \). A vector \( x \) multiplying \( a \) on the left is a row vector. Thus, \( ax \in \mathbb{R}^{n} \) and \( \xi a \in \mathbb{R}^{n} \).

The Fourier transform of \( f \) is defined as

\[ \hat{f}(\xi) = \int_{\mathbb{R}^{n}} f(x)e^{-2\pi i \xi x} dx, \]

where \( \xi \in \mathbb{R}^{n} \), and the invers Fourier transform is

\[ f(x) = \int_{\mathbb{R}^{n}} \hat{f}(\xi)e^{2\pi i \xi x} d\xi. \]
Let $L^2(\mathbb{R}^n)$ be the space of all square integrable functions on $\mathbb{R}^n$. It is well known that a countably family $\{e_j : j \in J\}$ in $L^2(\mathbb{R}^n)$ is a frame if there exist constants $0 < \alpha \leq \beta < \infty$ satisfying
\[
\alpha \| f \|_2^2 \leq \sum_{j \in J} |\langle f, e_j \rangle|^2 \leq \beta \| f \|_2^2
\]
for all $f \in L^2(\mathbb{R}^n)$. The family $\{e_j \}_{j \in J}$ is called a normalize tight frame or parsval frame if $\alpha = \beta = 1$. Therefore, if $\{e_j \}_{j \in J}$ is a parsval frame in $L^2(\mathbb{R}^n)$, then
\[
\| f \|_2^2 = \sum_{j \in J} |\langle f, e_j \rangle|^2
\]
for each $f \in L^2(\mathbb{R}^n)$. This is equivalent to reproducing formula
\[
v = \sum_{j \in J} \langle f, e_j \rangle e_j
\]  
for all $f \in L^2(\mathbb{R}^n)$, where the series (1) converges in the norm of $L^2(\mathbb{R}^n)$. Equation (1) shows that a parsval frame provides a basis-like representation. In general, a parsval frame need not be a basis. For more details about frames see [4],[14].

For the reader’s convenience we recall some basic concept of tiling set and packing set. The subspace $L$ in $\mathbb{R}^n$ is a lattice if $L = AZ^n$, where $A \in GL_n(\mathbb{R})$. Given a measurable set $\Omega \subseteq \mathbb{R}^n$ and a lattice $L$ in $\mathbb{R}^n$, it is to be said $\Omega$ tiles $\mathbb{R}^n$ by $L$ translation, or $\Omega$ is a fundamental domain of $L$ if the following properties hold:

1. $\bigcup_{l \in L} (\Omega + l) = \mathbb{R}^n$ a.e.,
2. $\mu((\Omega + l) \cap (\Omega + l')) = 0$ for any $l \neq l' \in L$.

It is called $\Omega$ packs $\mathbb{R}^n$ by $L$ translation if only (ii) holds. Equivalently, $\Omega$ tiles $\mathbb{R}^n$ by $L$ if and only if
\[
\sum_{j \in L} \chi_{\Omega}(x-l) = 1 \text{ for a.e. } x \in \mathbb{R}^n,
\]
and $\Omega$ packs $\mathbb{R}^n$ by $L$ if and only if
\[
\sum_{j \in L} \chi_{\Omega}(x-l) \leq 1 \text{ for a.e. } x \in \mathbb{R}^n.
\]
Clearly, $\mu(\Omega) = |\text{det} A|$ if $\Omega$ tiles by $L$, and $\mu(\Omega) \leq |\text{det} A|$ if $\Omega$ packs by $L$. Furthermore, if $\Omega$ packs $\mathbb{R}^n$ by $L$ and $\mu(\Omega) = |\text{det} A|$, then $\Omega$ necessarily tiles $\mathbb{R}^n$ by $L$. We refer the reader to [11] for more details about lattice tiling.

In general, Blanchard in [1] considers the definition of tiling sets, for an arbitrary group $G$. Let $G$ be a group acting from right on a measurable set $S \subseteq \mathbb{R}^n$. Then $\Omega$ is a $G$-tiling set for $S$, if

1. $\bigcup_{g \in G} \Omega g = S$ a.e.,
2. $\mu(\Omega g_1 \cap \Omega g_2) = 0$ for $g_1 \neq g_2 \in G$.

In this note, we construct an admissible wavelet $\Psi$, that it arise from $AB$-multiresolution analysis. Also, we give, more generally, a parsval frame for $L^2(\mathbb{R}^2)$.  

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2 Main Result

In this section our notation will be the same as before. We first recall an \( AB \)-affine system and \( AB \)-MRA. Then we construct some examples of \( AB \)-affine system, which are an orthonomal basis or parsval frame of \( L^2(R^2) \).

Let \( A \) and \( B \) be a countable subset of \( GL_n(R) \). A collection of the form
\[
A_{AB}(\psi) = \{D_aT_k:\psi: k \in \mathbb{Z}^n, a \in A, b \in B \},
\]
is called Affine systems with composite dilation, or \( AB \)-Affine system, where \( \Psi = \{\psi^1, \ldots, \psi^l\} \subset L^2(\mathbb{R}^n) \), and the operators \( T_k \) and \( D \) are called the translations and dilations, respectively, and defined as follows:
\[
T_kf(x) = f(x-k),
\]
and
\[
D_a f(x) = |\text{det}a|^{1/2} f(a^{-1}x).
\]

If \( A_{AB}(\psi) \) is an orthonormal basis (ON) or, more generally, a parsval frame (PF) for \( L^2(R^n) \), then \( \Psi \) is called an ON \( AB \)-multiwavelet or a PF \( AB \)-multiwavelet, respectively. Let \( C \subset GL_n(R) \) be a countable set containing the identity matrix \( I \) and let \( S \subset \mathbb{R}^n \) be a measurable set. The set \( C \) is called \( S \)-admissible if tiling multiwavelets for \( L^2(S)^\vee \) exist. In case \( S = \mathbb{R}^n \), for simply \( C \) is called admissible (rather than \( \mathbb{R}^n \)-admissible).

Associated with the Affine system with composite dilation, is the following generalization of the classical Multiresolution Analysis, that will be useful to construct more examples of \( AB \) multiwavelets, as well as examples with properties that are of great potential in applications.

Let \( B = \{b^j: j \in \mathbb{Z} \} \) be a collection of invertiable \( 2 \times 2 \) matrices with \( |\text{det}b| = 1 \), in which \( b \in GL_n(R) \), and \( A \) be an invertible \( 2 \times 2 \) matrix with integer enteries. A sequence \( \{V_i\}_{i \in \mathbb{Z}} \) of closed subspaces of \( \mathbb{R}^n \) is called an \( AB \)- Multiresolution Analysis (\( AB \)-MRA) if the following holds:

1. \( D_{b^j}T_kV_o = V_0 \), for any \( j \in \mathbb{Z}, k \in \mathbb{Z}^2 \),
2. \( V_i \subset V_{i+1} \), for each \( i \in \mathbb{Z} \), where \( V_i = D_{b^i}V_0 \),
3. \( \bigcap_{i \in \mathbb{Z}} V_i = \{0\} \) and \( \bigcup_{i \in \mathbb{Z}} V_i = L^2(R^2) \),
4. there exists \( \phi \in L^2(R^2) \) such that \( \Phi_B = \{D_{b^j}T_k\phi: j \in \mathbb{Z}, k \in \mathbb{Z}^2 \} \) is a semi-orthogonal Parsval frame for \( V_0 \); that is, \( \Phi_B \) is a parsval frame for \( V_o \) and in addition, \( D_{b^j}T_k\phi \perp D_{b^j}T_{k'}\phi \) for any \( j \neq j', j, j' \in \mathbb{Z}, k \neq k', k, k' \in \mathbb{Z}^2 \).

The space \( V_0 \) is called an \( AB \) scaling space and the function \( \phi \) is an \( AB \) scaling function for \( V_0 \). If in addition, \( \Phi_B \) is an orthonormal basis, then \( \phi \) is said an ON \( AB \) scaling function. (see [7], [8], [9]).

Now we need to explain a result by an elementary Fourier series argument.
Proposition 2.1  Let $I \subseteq \mathbb{R}^n$ be a measurable set, that $|I| < 1$ and $\hat{\Psi} = \chi_I$. Then, the collection 
$\{F_k = M_k \hat{\Psi} : k \in \mathbb{Z}^n\}$, is a parsval frame for $L^2(I)$, in which $M_k \hat{\Psi}(\xi) = e^{2\pi ik \cdot \xi}$.

Proof. First we show that $f = \sum_{k \in \mathbb{Z}} \langle f, F_k \rangle F_k$, for each $f \in L^2(I)$. Indeed,

$$
\| f - \sum_{k = -n}^{n} \langle f, F_k \rangle F_k \|_{L^2(I)}^2 = \int |f(x) - \sum_{k = -n}^{n} \langle f, F_k \rangle e^{2\pi ik \cdot \psi(\xi)}|^2 \, d\xi \leq \int |f(x) - \sum_{k = -n}^{n} \langle f, e_k \rangle e^{2\pi ik \cdot \psi(\xi)}|^2 \, d\xi \to 0
$$
as $n \to \infty$

we consider, $\| F_k \|_{L^2(I)} = A$. So we have:

$$
\| f \|_{L^2(I)}^2 = \langle f, f \rangle_{L^2(I)} = \langle \sum_{k \in \mathbb{Z}} \langle f, F_k \rangle F_k, \sum_{k \in \mathbb{Z}} \langle f, F_k \rangle F_k \rangle_{L^2(I)} = \sum_{k \in \mathbb{Z}} |\langle f, F_k \rangle|^2 \| F_k \|_{L^2(I)}^2 = A \sum_{k \in \mathbb{Z}} |\langle f, F_k \rangle|^2
$$

After a normalization conclude that, the restriction of the set $\{e^{2\pi ik \cdot \psi} : k \in \mathbb{Z}^n\}$ to $I$, is a parsval frame for $L^2(I)$.

We show that, there exists a relationship between an orthonormal basis and a fundamental domain. Also, there exists a relationship between a parsval frame and packing set. Therefore, we have the following:

Proposition 2.2  Let $\Omega \subseteq \mathbb{R}^n$, be a measurable set and $\hat{\Psi} = \chi_\Omega$, in $L^2(\Omega)$. Then, the collection 
$\{(T_k \psi)^\alpha = e^{2\pi ik \cdot \chi_\Omega} : k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\Omega)$ if and only if $\Omega$ is a fundamental domain.

Proof. Suppose that, the collection $\{(T_k \psi)^\alpha = e^{2\pi ik \cdot \chi_\Omega} : k \in \mathbb{Z}^n\}$, is an orthonormal basis for $L^2(\Omega)$. Then, $\| e^{2\pi ik \cdot \chi_\Omega} \|_{L^2(\Omega)}^2 = 1$. On one hand,

$$
\| e^{2\pi ik \cdot \chi_\Omega} \|_{L^2(\Omega)}^2 = \int_{\Omega} |e^{2\pi ik \cdot \xi}|^2 \chi_\Omega(\xi) \, d\xi = \int_{\Omega} \chi_\Omega(\xi) \, d\xi = \mu(\Omega).
$$

Thus, $\mu(\Omega) = 1$. Therefore, $\Omega$ is a fundamental domain.

Conversely, assume that $\Omega$ is a fundamental domain. As, $\mu((\Omega + k) \cap (\Omega + k')) = 0$, conclude the measure of $\Omega$ cannot be larger than one. Thus, by proposition 2.1, the collection $\{e^{2\pi ik \cdot \chi_\Omega} : k \in \mathbb{Z}^n\}$, is a parsval frame for $L^2(\Omega)$. On one hand, $\Omega$ is a fundamental domain. So, the measure of $\Omega$ is exactly one. Then, $\| e^{2\pi ik \cdot \chi_\Omega} \| = 1$. Therefore, the collection $\{(T_k \psi)^\alpha = e^{2\pi ik \cdot \chi_\Omega} : k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\Omega)$.

Proposition 2.3  Let $\Omega \subseteq \mathbb{R}^n$, be a measurable set and $\hat{\Psi} = \chi_\Omega$, in $L^2(\Omega)$. Then, the collection 
$\{(T_k \psi)^\alpha = e^{2\pi ik \cdot \chi_\Omega} : k \in \mathbb{Z}^n\}$ is a parsval frame for $L^2(\Omega)$ if and only if $\Omega$ is a packing set by translation of $\mathbb{Z}^n$, for $\mathbb{R}^n$. i.e. $\mu((\Omega + k) \cap (\Omega + k')) = 0$ for $k \neq k'$ in $\mathbb{Z}^n$. 
Proof. First let us suppose \( \Omega \) is a packing set by \( Z^n \) translation, for \( \mathbb{R}^n \). So, the measure of the set \( \Omega \) cannot be larger than one. Then, by the proposition 2.1, the collection \( \{ (T_k y)r \} = e^{\frac{2\pi ik}{2} \cdot x} : k \in \mathbb{Z}^n \} \) is a parsval frame for \( L^2(\Omega) \).

Conversely, suppose that \( \{ (T_k y)r \} = e^{\frac{2\pi ik}{2} \cdot x} : k \in \mathbb{Z}^n \} \) is a parsval frame for \( L^2(\Omega) \). Then, the measure of \( \Omega \), cannot be larger than one. Since, by contradiction, if \( | \Omega | > 1 \), then the collection \( \{ e^{2\pi ik} x_k(\xi) : k \in \mathbb{Z}^n \} \) cannot be a parsval frame. Thus, \( \Omega \) is a packing set by translation of \( Z^n \), for \( \mathbb{R}^n \).

We need to stating some basic properties of the translation and dilation operators, that will be used throughout this paper.

**Proposition 2.4** Let

\[ G = \{ U = D_a T_k : (a, k) \in GL_n(\mathbb{R}) \times \mathbb{R}^n \}. \]

\( G \) is a subgroup of the group of unitary operators on \( L^2(\mathbb{R}^n) \). We consider \( \hat{U} \hat{f} = (Uf) \). Then we have:

1. \( D_a T_k = T_k D_a \),
2. \( D_{a_1} D_{a_2} = D_{a_1 a_2} \), for each \( a_1, a_2 \in GL_n(\mathbb{R}) \),
3. for \( U = D_a T_k \), then \( \hat{U} = D_{a^{-1}} M_{-k} \), where \( D_{a^{-1}} \hat{f}(\xi) = | \det a |^{1/2} \hat{f}(\xi a) \),
4. \( \hat{D_a L^2(S)} = L^2(S a^{-1}) \), for measurable set \( S \subset \mathbb{R}^n \), and \( L^2(S) = \{ \hat{f} \in L^2(\mathbb{R}^n) : \text{supp} \hat{f} \subseteq S \}. \)

In the sequel we construct an orthonormal \( AB \)-multiwavelet that arises from \( AB \)-multiresolution analysis. Also, we give a parsval frame \( AB \)-wavelet.

**Example 2.5** Let \( a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \), and \( b = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \). Let \( G = \{ (b^j, k) : j \in \mathbb{Z}, k \in \mathbb{Z}^2 \} \), in which \( b^j = \begin{pmatrix} j+1 & j \\ -j & j+1 \end{pmatrix} \). Then \( G \) is a group with group multiplication:

\[ (b^j, m)(b^i, k) = (b^{i+j}, k + b^j m). \] (2)

The identity element of this group is \((1, 0)\), so we have \((b^j, k)^{-1} = (b^{-j}, -b^j k)\). The multiplication (2) is consistent with the operation that maps \( x \in \mathbb{R}^2 \) into \( b^j (x + k) \in \mathbb{R}^2 \). This is clarified by introducing the unitary representation \( \pi \) of \( G \), acting on \( L^2(\mathbb{R}^2) \), defined by

\[ (\pi(b^j, k)f)(x) = f((b^j, k)^{-1}x) = f(b^{-j}x - k) = (D_b^j T_k f)(x), \] (3)

for \( f \in L^2(\mathbb{R}^2) \). The observation that

\[ (D_b^j T_m)(D_b^i T_k) = (D_b^{i+j} T_{k+b^j m}), \]
where \( l, j \in \mathbb{Z}, k, m \in \mathbb{Z}^2 \), shows how the group operation (2) is associated with the unitary representation (3).

Let \( S_0 = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2 - \xi_1| \leq 1 \} \) and define

\[
V_0 = L^2(S_0)^\vee = \{ f \in L^2(\mathbb{R}^2) : \text{supp} \hat{f} \subset S_0 \}.
\]

For all \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}^2 \), we have

\[
(\pi(b^j, k)f)(\xi) = (D^j_k f)(\xi) = e^{-2\pi i \xi \cdot k} \hat{f}(\xi b^j),
\]

and, \( \xi b^j = (\xi_1, \xi_2)b^j = (j \xi_1 + \xi_2, j \xi_1 - \xi_2) \). Then the action of \( b^j \) maps the bias strip domain \( S_0 \) into itself. So the condition \( (i) \) of \( AB \)-MRA has been proved. Thus the space \( V_0 \) is invariant under the action of the operators \( \pi(b^j, k) \).

Let \( S_i = S_0 a^i = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2 - \xi_1| \leq 2^i \} \),

and

\[
V_i = \{ f \in L^2(\mathbb{R}^2) : \text{supp} \hat{f} \subset S_i \}.
\]

We can see that the space \( \{ V_i \}_{i \in \mathbb{Z}} \) satisfy the following properties :

\begin{enumerate}
\item \( V_i \subset V_{i+1}, i \in \mathbb{Z} \);
\item \( D^i_k : V_i = V_{i+1} \);
\item \( \cap_{i \in \mathbb{Z}} V_i = \{ 0 \} \);
\item \( \cup_{i \in \mathbb{Z}} V_i = L^2(\mathbb{R}^2) \).
\end{enumerate}

Consider \( A = \{ a^i : i \in \mathbb{Z} \} \), \( B = \{ b^j : j \in \mathbb{Z} \} \), and \( U = U_1 \cup U_2 \), where \( U_1 \) is a triangle with vertices at \((0,0),(-1,0),(0,1)\), and \( U_2 = \{ \xi \in \mathbb{R}^2 : -\xi \in U_1 \} \). Define \( \varphi \) by \( \hat{\varphi}(\xi) = \chi_U(\xi) \). A simple computation shows that \( U \) is a fundamental domain of \( \mathbb{Z}^2 \) and a \( B \)-tiling region for \( S_0 \), too. That is, \( \mathbb{R}^2 = \bigcup_{k \in \mathbb{Z}^2} (U + k) \) and \( S_0 = \bigcup_{j \in \mathbb{Z}} (U b^j) \), where the unions are disjoint up to a set of measure zero.

Therefore, \( \Phi_B = \{ D_j^k \varphi : b \in B, k \in \mathbb{Z}^2 \} \) is an orthonormal basis of \( V_0 \) and \( \varphi \) is a scaling function of \( V_0 \). Since the dilation operator \( D^i_0 \) is a unitary, thus the collection \( \{ D^i_{-k} D^i_0 T_k \varphi : j \in \mathbb{Z}, k \in \mathbb{Z}^2 \} \), \( i \in \mathbb{Z} \) is an orthonormal basis of \( V_i \). Thus \( \{ V_i \}_{i \in \mathbb{Z}} \) is an \( AB \)-MRA with scaling function \( \varphi \).

In order to have an orthonormal wavelet system, we must be obtained an orthogonal complement of \( V_0 \) in \( V_1 \). Let \( W_0 \) be an orthogonal complement \( V_0 \) in \( V_1 \), that is, \( V_1 = V_0 \oplus W_0 \). By the standard MRA wavelet construction, if we find an orthogonal basis for \( W_0 \), then we have a wavelet system. Since \( V_0 = L^2(S_0)^\vee \) and \( V_1 = L^2(S_1)^\vee \) so we have

\[
L^2(S_i)^\vee = L^2(S_0 a)^\vee = L^2((S_0 a \setminus S_0) \cup S_0)^\vee = L^2(S_0 a \setminus S_0)^\vee \oplus L^2(S_0)^\vee.
\]

Then we define \( W_0 = L^2(S_0 a \setminus S_0)^\vee = L^2(S_1 \setminus S_0)^\vee \). We set :

\[
R_0 := S_1 \setminus S_0 = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : 1 \leq |\xi_2 - \xi_1| \leq 2 \},
\]

then
We shall now explain how to construct an $AB$-multiwavelet generated by three mutually orthogonal functions $\psi^1, \psi^2, \psi^3$ of norm 1. To do this, define the following subsets of $R_0 = S_1 \setminus S_0$:

$$E_i = E_i^+ \cup E_i^-, E_2 = E_2^+ \cup E_2^-, E_3 = E_3^+ \cup E_3^-,$$

where

$$E_1^+ = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : -2 \leq \xi_1 \leq -1, 0 \leq \xi_2 \leq \xi_1 + 2 \},$$

$$E_2^+ = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : -1 \leq \xi_1 \leq 0, \xi_1 + 1 \leq \xi_2 \leq 1 \},$$

$$E_3^+ = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : -1 \leq \xi_1 \leq 0, 1 \leq \xi_2 \leq \xi_1 + 2 \},$$

and

$$E_i^- = \{ \xi \in \mathbb{R}^2 : -\xi \in E_i^+ \}, i = 1, 2, 3.$$

We then define $\psi^l, l = 1, 2, 3$, by setting $\hat{\psi}^l = \chi_{E_l}, l = 1, 2, 3$. Notice that each set $E_l$ is a fundamental domain of $\mathbb{Z}^2$, that is, the function $\{ e^{-2\pi i k \cdot \xi} : k \in \mathbb{Z}^2 \}$, restricted to $E_l$ form an orthonormal basis of $L^2(E_l)$. It follows that the collection $\{ e^{-2\pi i k \cdot \xi} : k \in \mathbb{Z}^2 \}$ is an orthonormal basis of $L^2(E_l), l = 1, 2, 3$. A simple direct calculation shows that the sets $\{ E_j b^{-j} : j \in \mathbb{Z}, l = 1, 2, 3 \}$ are a partition of $R_0$, that is,

$$\bigcup_{l=1}^3 \bigcup_{j=\mathbb{Z}} E_l b^{-j} = R_0, \quad (5)$$

where the union is disjoint.
But the dilations $D^j_b$ are unitary operators. Hence they maps an orthonormal basis into an orthonormal basis. Thus for each $j \in \mathbb{Z}$, the set $\{e^{-2\pi i \mathbf{b}^j l} (\mathcal{H}^j) : k \in \mathbb{Z}^2 \}$ is an orthonormal basis for $L^2(E,b^{-i})$. It follows from (5), that

$$L^2(R_0) = \bigoplus_{i=1}^3 \bigoplus_{j \in \mathbb{Z}} L^2(E,b^{-i}).$$

(6)

Since, for each fixed $j \in \mathbb{Z}$, $b^j$ maps $\mathbb{Z}^2$ into itself, the collection $\{e^{-2\pi i \mathbf{b}^j l} (\mathcal{H}^j) : k \in \mathbb{Z}^2 \}$ is equal to the collection $\{e^{-2\pi i \mathbf{b}^j l} (\mathcal{H}^j) : k \in \mathbb{Z}^2 \}$. It follows from (6), that the collection

$$\{e^{-2\pi i \mathbf{b}^j l} (\mathcal{H}^j) : k \in \mathbb{Z}^2, j \in \mathbb{Z}, l = 1,2,3 \}$$

is an orthonormal basis of $L^2(R_0)$. Thus, by taking the inverse Fourier transform, we have that $\{D^j_b T_k \mathcal{H}^j : j \in \mathbb{Z}, k \in \mathbb{Z}^2, l = 1,2,3 \}$ is an orthonormal basis of $W_0 = L^2(R_0)^\vee$. In order to obtain the desired ON $AB$-affine system for $L^2(R_1)$, we apply the dilations $D^j_a$, $i \in \mathbb{Z}$, to the orthonormal basis. The dilations operators $D^j_a$, for each $i \in \mathbb{Z}$, maps $R_0$ into $R_i$, in which

$$R_i = R_0 a^i = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : 2^i \leq |\xi_2 - \xi_1| \leq 2^{i+1} \},$$

and we have $\bigcup_{i \in \mathbb{Z}} R_i = \mathbb{R}^2$, where the unions are disjoint. Using the unitary operators $D^j_a$, for each $i \in \mathbb{Z}$, thus the set $\{D^j_a T_k \mathcal{H}^j : k \in \mathbb{Z}^2, j \in \mathbb{Z}, l = 1,2,3 \}$ is an orthonormal basis of $L^2(R_i)^\vee = W_i$. Since the spaces $L^2(R_i)$ (and thus the spaces $W_i$) are mutually orthogonal, it follows that the system

$$\{D^j_a \mathcal{H}^j : k \in \mathbb{Z}^2, i, j \in \mathbb{Z}, l = 1,2,3 \}$$

is an orthonormal basis of $L^2(R_1) = \bigoplus_{i \in \mathbb{Z}} W_i$, that is, $\Psi = \{\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3\}$ is an ON $AB$-multiwavelet. The number of generators of this $AB$-multiwavelet is fixed. Infact, by the next proposition, if we could replace $\Psi$ by $\Phi = \{\phi^1, \ldots, \phi^N\}$, then $L = 3$.

**Proposition 2.6** ([8], [9]): Let $G$ be a countable set and, for each $u \in G$, let $T^u$ be a unitary operator acting on a Hilbert space $H$. Assume that, for each $T^u$, there is a unique $u^* \in G$ such that $T^u = T^{u^*}$. Suppose $\Phi = \{\phi^1, \ldots, \phi^N\}$, $\Psi = \{\psi^1, \ldots, \psi^M\} \subset H$, where $N, M \in \mathbb{N} \cup \{\infty\}$. If $\{T^u \psi^k : u \in G, 1 \leq k \leq N\}$ and $\{T^u \psi^i : u \in G, 1 \leq i \leq M\}$ are each orthonormal basis for $H$, then $N = M$.

The following result establishes the number of generators needed to obtain an orthonormal MRA $AB$-wavelet.

**Theorem 2.7** ([8], [9]): Let $\Psi = \{\psi^1, \ldots, \psi^M\}$ be an orthonormal MRA $AB$-multiwavelet for $L^2(R^n)$, and let $N = |B/\alpha B a^{-1}|$ ( = the order of quotient group $B/\alpha B a^{-1}$). Assume that $|\text{deta}| \in \mathbb{N}$. Then $L = N/|\text{deta}| - 1$.

By using this theorem, we can calculate the number of $AB$-multiwavelet.
Remark 2.8 In example (2.5), the set \( B \) is considered as \( B = \{ b^j : j \in \mathbb{Z} \} \) in which, 
\[
 b^j = \begin{pmatrix}
  j + 1 & j \\
  -j - j + 1 & \end{pmatrix}.
\]
By a simple calculation, we get \( ab^j a^{-1} = b^j \), thus, \( aBa^{-1} = \langle b \rangle \) and it is clearly \( B = \langle b \rangle \). Then \( B/aBa^{-1}; I_{>2} \), thus \( N = \| B/aBa^{-1} \| = 1 \). Therefore, 
\[
 L = N \| \text{det} \| -1 = 1.4 - 1 = 3.
\]
Now we give a parsval frame wavelet with composite dilation from \( AB \)-MRA with a single generator.

Example 2.9 Let \( F = F_1 \cup F_2 \), where \( F_1 \) is a trapezoid with vertices \((-1,0), (-\frac{1}{2},0), (0,\frac{1}{2}), (0,1)\), and \( F_2 = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : -\xi \in F_1 \} \). Suppose that \( S_i, A \) and \( B \) are defined in Example (2.5), and let \( H := S_0 \setminus S_{-1} = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \frac{1}{2} \leq |\xi_2 - \xi_1| \leq 1 \} \). A simple direct computation shows that \( H = \bigcup_{i \in \mathbb{Z}} Fb^j \), where the union is disjoint. It follows from the Plancherel theorem (using the fact that \( F \) is contained inside a fundamental domain) that the function \( \chi_F(\xi) \) satisfies 
\[
 \sum_{k \in \mathbb{Z}} |\langle \hat{f}, e^{2\pi i k \xi} \rangle|^2 = \| \hat{f} \|^2, \text{ for all } \hat{f} \in L^2(F), \text{ and the collection}
\]
\[
\{ D_a^i D_b^j \chi_F(\xi) : k \in \mathbb{Z}, j \in \mathbb{Z} \}
\]
is a parsval frame of \( L^2(H) \). Similarly to the construction above, we have \( \mathbb{R}^2 = \bigcup_{i \in \mathbb{Z}} Ha^i \), where the union is disjoint. Define \( \psi \) by setting \( \psi = \chi_F \). It follows that the system 
\[
\{ D_a^i D_b^j \psi : i, j \in \mathbb{Z}, k \in \mathbb{Z}^2 \},
\]
is a parsval frame of \( L^2(\mathbb{R}^2) \). That is to say the function \( \psi \), is a parsval frame wavelet with composite dilations.

References


