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RELATIVE N-TH NON-COMMUTING GRAPHS OF FINITE GROUPS

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ABSTRACT. Suppose n is a fixed positive integer. We introduce the relative n-th non-commuting graph $\Gamma^n_{H,G}$, associated to the non-abelian subgroup H of group G. The vertex set is $G \setminus C^n_{H,G}$ in which $C^n_{H,G} = \{x \in G : [x, y^n] = 1 \text{ and } [x^n, y] = 1 \text{ for all } y \in H\}$. Moreover, $\{x, y\}$ is an edge if x or y belong to H and $xy^n \neq y^n x$ or $x^n y \neq yx^n$. In fact, the relative n-th commutativity degree, $P_n(H, G)$ the probability that n-th power of an element of the subgroup H commutes with another random element of the group G and the non-commuting graph are the keys to construct such a graph. It is proved that two isoclinic non-abelian groups have isomorphic graphs under special conditions.

1. **Introduction**

Erdös associated a graph Γ to the group G, whose vertex set is G (as a set)and two vertices join whenever they do not commute. He asked whether there is a finite bound for the cardinalities of cliques in Γ, if Γ has no infinite clique. This problem was posed by Neumann in [13] and gave a positive answer to Erdös 's question. Of course, there are some other ways to make a graph associated to a group or semigroup. One may refer to the works of Bertram et al. [4], Grunewald et al. [7],

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Moghadamfar et al. [11] and Williams [14] or recent papers on non-commuting graph, Engel graph and non-cyclic graph in [2], [1] and [3], respectively.

Let H be a subgroup of a finite group G and n a positive integer. Mohd Ali et al. in [12] introduced n-th commutativity degree which is the probability that the n-th power of a random element of G commute with another element. If $n = 1$ then this probability coincide to the commutativity degree which was investigated by Erdös and Turan in [5]. Later Erfanian et al. improved their results to the relative case. They defined the relative n-th commutativity degree of a subgroup H of G as is the probability of commuting the n-th power of an element of the subgroup H with an element of group G. They generalized several facts which is valid for commutativity degree in [6].

By these facts about the graphs and the probability, we led to introduce the relative n-th non-commuting graph. We define the relative n-th non-commuting graph $\Gamma^n_{H,G}$, with vertex set $G \setminus C^n_{H,G}$, where $C^n_{H,G} = \{x \in G : [x, y^n] = 1 \text{ and } [x^n, y] = 1 \ \forall\ y \in H\}$. Moreover, $\{x, y\}$ is an edge if x or y belong to H and $xy^n \neq y^n x$ or $x^n y \neq yx^n$. We discuss about diameter, dominating set, domination number, clique number and planarity of the graph. Third section is organized to state connections between the n-th non-commuting graph and the n-th commutativity degree. We find a lower bound for the number of edges of the n-th non-commuting graph and also we show that this lower bound is sharp for nilpotent groups of class 2. Moreover, we prove that there is no relative n-th non-commuting star graph, n-th non-commuting complete graph and relative n-th noncommuting complete bipartite graph associated to an AC-group. We discuss about non-regularity of the relative n-th non-commuting graph in some special cases as well. Furthermore, it is proved that two isoclinic non-abelian groups are associated to the isomorphic graphs in special cases.

2. **The n-th Non-commuting Graph**

Let us start with the definition of the relative n-th non-commuting graph.

Definition 2.1. *Let n be a fixed positive integer. We assign a graph to a non-abelian subgroup H of group G. The vertex set is $G \setminus C^n_{H,G}$ where $C^n_{H,G} = \{x \in G : [x, y^n] = 1$ and $[x^n, y] = 1$ for all $y \in H\}$ and* *two distinct vertices x and y join together whenever x or y belong to H and* $[x, y^n] \neq 1$ *or* $[x^n, y] \neq 1$ *. We call such a graph relative n-th non-commuting graph and is denoted by* $\Gamma^n_{H,G}$*. If* $H = G$ *then the graph* Γ^n_G *is n-th non-commuting graph.*

It is clear that for $n = 1$ the graph Γ^1_G corresponds to the non-commuting graph denoted by Γ_G in [2]. Note that if H is an abelian subgroup of G or has exponent that divides n then there is no vertex in H but some vertices in $G \setminus H$ may exist and hence $\Gamma^n_{H,G}$ is an empty graph. Similarly, if the exponent of the group G divides n or G is an abelian then Γ^n_G is a null graph. Furthermore, if G is a group which its central factor is elementary abelian p-group of rank s. If $m \equiv n(mod\ p)$ then $\Gamma^m_G = \Gamma^n_G$.

Example 2.2.

(i) *If n is an even or odd number and* $G = D_8$ *then* Γ^n_G *has no vertices or* $\Gamma^n_G = \Gamma_G$*, respectively.*

(ii) *For even or odd number n,* $V(\Gamma^n_{S_3}) = V(\Gamma_{S_3})$*.*

(iii) *Let G be a group of exponent m such that* $n \equiv r\ (mod\ m)$*, then* $\Gamma^n_G = \Gamma^r_G$*.*

Theorem 2.3. *For any non-abelian group G and* $n \in \mathbb{N}$*,* $diam(\Gamma^n_G) \leq 4$*. Moreover, the girth of* Γ^n_G *is less than or equal to 4.*

Proof. Let x and y be two distinct vertices of the graph such that they are not adjacent. We may assume $xy^n = y^nx$ *and* $x^ny = yx^n$. Since x and y are not in C^n_G, there exist $x_1, y_1 \in V(\Gamma^n_G)$ where $\{x, x_1\}$ and $\{y, y_1\}$ are edges. If x and y_1 or y and x_1 are adjacent then $d(x, y) = 2$. Otherwise, ($[x, y_1^n] = 1$ and $[x^n, y_1] = 1$) and ($[x_1, y^n] = 1$ and $[x_1^n, y] = 1$). Moreover, (i) $[x_1, x^n] \neq 1$ or (ii) $[x_1^n, x] \neq 1$ and (iii) $[y_1, y^n] \neq 1$ or (iv) $[y_1^n, y] \neq 1$. Thus, we have four cases. If (i) and (iii) then x_1y_1 is a vertex which is join x and y, hence $d(x, y) = 2$. Suppose (ii) and (iv). If x_1 joins y_1 then $d(x, y) \leq 3$. Assume x_1 and y_1 are not adjacent, then xy is a vertex which joins x_1 and y_1 so $d(x, y) \leq 4$. Now, suppose (i) and (iv). If x_1 and y_1 are adjacent then $d(x, y) \leq 3$. Otherwise we have $x_1y_1^n = y_1^nx_1$ *and* $x_1^ny_1 = y_1x_1^n$ then x_1y is a vertex which is adjacent to y_1 and x because $[x_1y, y_1^n] \neq 1$ and $[x_1y, x^n] \neq 1$ and this means $d(x, y) \leq 3$. Similarly, for the case (ii) and (iii) if x_1 and y_1 are adjacent then $d(x, y) \leq 3$. Otherwise, xy_1 is a vertex adjacent to x_1 and y so $d(x, y) \leq 3$.

In order to compute the girth, let $\{x, y\}$ be an edge, if $x^ny \neq yx^n$ then xy

is a vertex which joins x. Consider two cases, if $xy^n \neq y^n x$ then clearly xy and y are adjacent so we have cycle with length 3. If $xy^n = y^n x$ then for vertices x and y there exist x_1 and y_1 respectively. Now, without loss of generality suppose $y = x_1$ and $[x^n, y] \neq 1$. If x joins y_1 then we have a triangle, so suppose not. Consider $[y^n, y_1] \neq 1$ consequently yy_1 is a vertex that joins y and x, a triangle obtained. Moreover, if $[y, y_1^n] \neq 1$ then xy is a vertex which joins x and y_1 so we have a square. Suppose $y_1 \neq x$ and $x_1 \neq y$ with $xy^n = y^n x$. If y joins x_1 or x and y_1 are adjacent then we get a triangle so suppose not. We have (i) $[x_1, x^n] \neq 1$ or (ii) $[x_1^n, x] \neq 1$ and (iii) $[y_1, y^n] \neq 1$ or (iv) $[y_1^n, y] \neq 1$. Now, consider four cases similar as the argument for the diameter. Note that, if (ii) and (iii) then xy_1 is a vertex joins y. Suppose x_1 and y_1 are not adjacent, so xy_1 join x_1 and we make a cycle of length 4. Furthermore, if (ii), (iv) and x_1 and y_1 are not adjacent, then $\{x, y, y_1, xy\}$ is a cycle of length 4. For the case, (i) and (iv) in the worst conditions i.e. $\{x_1, y_1\}$, $\{x, y_1\}$ and $\{x_1, y\}$ are not edges then xy is a vertex which joins y_1 and x so we have a square. Case (i) and (iii), follows by similar method that $x_1 y_1$ is a vertex which joins x and y. □

In particular, $\mathrm{diam}(\Gamma_G) = 2$ and the girth of Γ_G is 3 (see [2]). We define $C_G^n(S) = \{y \in G : [s, y^n] = 1 \text{ and } [s^n, y] = 1 \text{ for all } s \in S\}$, where S is the subset of G.

Now, let us deal with the dominating set of the graph. The following results are generalizations of Remark 2.5, Proposition 2.12 part (1), Remark 2.13 and Proposition 2.14 in [2] so we prefer to omit some of the proofs.

Proposition 2.4. *If $\{x\}$ is a dominating set for $\Gamma_{H,G}^n$ where $x \in H$ and n is an odd number, then $x^2 = 1$, $C_{H,G}^n = 1$ and $C_G^n(x) = \langle x \rangle$.*

Proof. If $x^2 \neq 1$, then x^{-1} is not adjacent to x. If $C_{H,G}^n$ contains a non-trivial element g then $[g, x^n] = [g, x] = 1$ and $[g^n, x] = 1$. Therefore $t = gx$ is a vertex which does not join x. Because for vertex x, there exists $y \in H$ such that (i) $xy^n \neq y^n x$ or (ii) $x^n y \neq y x^n$. If (i) or (ii), then $t \in V(\Gamma_{H,G}^n)$ because $[t, y^n] \neq 1$ or $[t^n, y] \neq 1$, respectively. Furthermore, t is not adjacent to x, since clearly $[g^n x^n, x] = 1$ and $[t, x^n] = [gx, x^n] = 1$. For the last part, if $g \in C_G^n(x)$ is distinct from x and 1, then as it is not in $C_{H,G}^n$, it is a vertex not adjacent to x and we get a contradiction. □

In the above proposition if $H = G$ we can deduce similar result for Γ^n_G when n is odd. Furthermore, if $\{x\}$ is a dominating set for Γ^n_G when n is odd, then $Z(G) = 1$. For a subgroup H of G, $S \subseteq H \cap V(\Gamma^n_{H,G})$ is a dominating set of $\Gamma^n_{H,G}$ if and only if $C^n_G(S) \subseteq C^n_{H,G} \cup S$. Again, by considering $H = G$ similar assertion follows for Γ^n_G. It is easy to verify that $V(\Gamma^n_{H,G})$ is a dominating set for Γ^n_G whenever $C^n_{H,G} = 1$. If S is an dominating set for Γ^n_H then it is a dominating set for $\Gamma^n_{H,G}$ whenever $C^n_{H,G} = C^n_G(S)$.

Proposition 2.5. *Let G be a finite non-abelian group and $\omega(\Gamma)$ denotes the clique number of the graph Γ.*

(i) *For any non-abelian subgroup H of G, $\omega(\Gamma^n_H) \leq \omega(\Gamma^n_G)$.*
(ii) *For any non-abelian factor group G/N of G, $\omega(\Gamma^n_{G/N}) \leq \omega(\Gamma^n_G)$.*

Proof. The first part follows easily. For (ii) assume that $\{g_1N, \cdots, g_sN\}$ are the vertices for the maximal complete subgraph of $\Gamma^n_{G/N}$. As $[g_i, g_j^n] \notin N$ or $[g_i^n, g_j] \notin N$ we conclude that $g_i^n g_j \neq g_j g_i^n$ or $g_j^n g_i \neq g_i g_j^n$. Thus, $\{g_1, \cdots, g_s\}$ is a complete subgraph of Γ^n_G. □

By the above proposition it follows that

$$\omega(\Gamma^n_{PSL(m,R)}) \leq \omega(\Gamma^n_{PGL(m,R)}) \leq \omega(\Gamma^n_{GL(m,R)}).$$

Obviously, $C^n_G \subseteq C^n_{H,G}$ and $\Gamma^n_{H,G}$ is an induced subgraph of Γ^n_G. Moreover, Γ^n_H is an induced subgraph of $\Gamma^n_{H,G}$ and $\Gamma^n_{H,G}$ itself is an induced subgraph of $\Gamma^1_{H,G}$. It is clear that $Z(G) \subseteq C^n_G$ and if x joins y in Γ^n_G they are adjacent in Γ_G, consequently Γ^n_G is an induced subgraph of Γ_G. Recall from [2] that Γ_G is planar if and only if $G \cong S_3$ or D_8 or Q_8. Hence, if $G \cong S_3$ or D_8 or Q_8 then the n-th non-commutating graph is planar.

A group is called an AC-group if the centralizer of the set of non-central elements is abelian. Let G be an AC-group. As $C^n_G \subseteq C^n_G(x) \subseteq C_G(x^n)$ for a vertex x, elements of C^n_G commute with each other, we deduce $C^n_G, C^n_G(x)$ are subgroups of G such that $C^n_G(x) = C_G(x^n)$ and $C^n_G = Z(G)$. Hence, $\Gamma^n_G = \Gamma_G$ in this case. Thus Γ^n_G which is associated to the AC-group G has the exact properties of the non-commuting graph Γ_G (see [2]).

Theorem 2.6. *Let G be an AC-group. Non-null graph Γ^n_G is planar if and only if $G \cong S_3$ or D_8 or Q_8.*

Example 2.7. *The n-th non-commutating graph* $\Gamma^n_{S_3,S_4}$ *is not planar. Because* K_5 *with vertices* $\{(1\ 2),(1\ 2\ 3),(2\ 3\ 4),(1\ 2\ 4),(1\ 3\ 4)\}$ *and* $\{(1\ 2),(1\ 3),(2\ 3),(1\ 2\ 3),(1\ 2\ 3\ 4)\}$ *are its subgraph whenever* n *is an even or odd number respectively. Thus,* $\Gamma^n_{S_3,S_m}$ *is not planar, for* $m \geq 4$.

3. n-th Non-commuting Graphs and $P_n(G)$

In this section, by the use of probability theory and graph theory obtain some results for some graphs. Therefore, we should recall some necessary definitions. For a non-abelian group G the relative n-th commutativity degree is the probability of commuting the n-th power of an element of a subgroup H with an element of the group G. It is defined by the following ratio

$$P_n(H,G) = \frac{|\{(x,y) \in H \times G : [x^n, y] = 1\}|}{|H||G|},$$

(see [6] for more details). If $H = G$ then the above probability is called n-th commutativity degree and denoted by $P_n(G)$. Furthermore, $P_1(G) = d(G)$ which is called the commutativity degree. Assume

$$A = \{(x,y) \in H \times G : [x^n, y] = 1\}$$

and $B = \{(x,y) \in H \times G : [x^n, y] = 1 \textit{ and } [y^n, x] = 1\}$. In general $B \subseteq A$ so we can deduce

$$|E(\Gamma^n_{H,G})| \geq |H||G|(1 - P_n(H,G)) - \frac{|H|^2}{2}(1 - P_n(H)).$$

It is clear that if $H = G$ then $|E(\Gamma^n_G)| \geq (|G|^2 - |G|^2 P_n(G))/2$.

An element x of a group G is called n-Bell, if $[x^n, y] = [x, y^n]$ for all $y \in G$. G is called n-Bell if all its elements are n-Bell. In this case $|A| = |B|$ so the bound for the number of edges is sharp. Furthermore, for all nilpotent groups of class 2 we have $|A| = |B|$.

Every graph has at most $m(m-1)/2$ edges, where m is the number of vertices. Thus, for a nilpotent group of class 2 like G follows

$$P_n(G) \geq \frac{2|C^n_G|}{|G|} + \frac{1}{|G|} - \frac{|C^n_G|^2}{|G|^2} - \frac{|C^n_G|}{|G|^2}.$$

If G_1 and G_2 are two nilpotent groups of class 2, $|C^n_{G_1}| = |C^n_{G_2}|$ and $\Gamma^n_{G_1} \simeq \Gamma^n_{G_2}$ then clearly $P_n(G_1) = P_n(G_2)$.

Proposition 3.1. *Let G be a non-abelian group whose central factor is an elementary abelian p-group of rank s where p is prime. If p does not divide n then*

$$|E(\Gamma_G^n)| \geq \frac{(|G| - |C_G^n|)}{2}\frac{(p-1)|G|}{p}$$

Proof. Degree-edge formula $2|E(\Gamma_G^n)| = \sum_{x \in V(\Gamma_G^n)} deg(x)$, implies

$$2|E(\Gamma_G^n)| = (|G| - |C_G^n|)|G| - \sum_{x \in V(\Gamma_G^n)} |C_G^n(x)|.$$

Moreover for every non-central element $t \in G$ we have

$$p^s = [G : Z(G)] = [G : C_G(t)][C_G(t) : Z(G)] \leq [G : C_G(t)]p^{s-1}$$

Therefore, $|C_G(t)| \leq |G|/p$. Clearly, $C_G^n(x) \subseteq C_G(x^n)$ and the assertion follows. □

In the above theorem if p divides n there is no vertex for this graph. Easily one can define a one to one correspondence map between $C_G^n(x)$ and $C_G^n(x^g)$ for $g \in G$. Thus, the degree of vertices of Γ_G^n in the same conjugacy class are equal. By the fact that $|C_G^n(x)| \leq |C_G(x^n)| \leq |G|/2$ for an arbitrary vertex x, it follows that Γ_G^n is Hamiltonian graph.

Theorem 3.2. *There is no relative n-th non-commuting complete graph associated to an AC-group G.*

Proof. Suppose $\Gamma_{H,G}^n$ is a relative n-th non-commuting complete graph. Therefore $|G| - |C_G^n(h)| = |G| - |C_{H,G}^n| - 1$ for a vertex $h \in H$. Consequently $|C_G^n(h)| = 2$ and $|h| = 2$. Thus H is abelian so there is no vertices in H which is a contradiction. □

By the same argument it follows that there is no n-th non-commuting complete graph associated to an AC-group G.

Theorem 3.3. *There is no n-th non-commuting complete graph Γ_G^n associated to a non-abelian nilpotent group G of class 2 with $C_G^n = 1$.*

Proof. Assume that we have such a graph. Thus $P_n(G) = 3/|G| - 2/|G|^2$. On the other hand $k(G)/|G| = d(G) \leq P_n(G)$ where $k(G)$ is number of conjugacy classes (see [8]). Hence, $k(G) \leq 2$ that implies G is an abelian group, which is a contradiction. □

Theorem 3.4. *There is no relative n-th non-commuting star graph associated to an AC-group G and its subgroup H.*

Proof. on the contrary suppose that $\Gamma^n_{H,G}$ is the relative n-th non-commuting star graph. Assume $h \in H$ is the unique vertex of degree $(|V(\Gamma^n_{H,G})|-1)$. Then we conclude that $C^n_{H,G} = 1$. On the other hand, for a vertex $g \in G \setminus H$ we have $|C^n_H(g)| = |H| - 1$ so $[H : C^n_H(g)] = |H|/(|H| - 1)$ which is impossible. Secondly, assume $g \in G \setminus H$ is the unique vertex of degree $(|V(\Gamma^n_{H,G})| - 1)$ and all the other vertices, for instance $h \in H$, have degree 1. Then we have $|G| = |C^n_G(h)| + 1$. Therefore, $|C^n_{H,G}| = 1$ and similarly, $[G : C^n_G(h)] = |G|/(|G| - 1)$ which is impossible. Hence such a graph does not exist. □

Furthermore, in Theorem 3.4 if G is an AC-group which is nilpotent of class 2, then not only the associated n-th non-commuting graph is not star but also there is no group with $P_n(G) = 1 - 2/|G| + 4/|G|^2$ whenever $|C^n_G| = 1$.

Theorem 3.5. *There is no n-th non-commuting complete bipartite graph associated to an AC-group G.*

Proof. Suppose Γ^n_G is an n-th non-commuting complete bipartite graph, where vertices are partitioned in to two disjoint sets V_1 and V_2 such that $|V_1| + |V_2| = |G| - |C^n_G|$. We should have $deg(x) = |G| - |C^n_G(x)| \leq (|G| - |C^n_G|)/2$, $|C^n_G|q = |C^n_G(x)|$ for some $q \in \mathbb{Z}$ and $x \in V(\Gamma^n_G)$. Hence $|G| \leq |C^n_G|(2q - 1)$ and so $[G : C^n_G(x)] \leq (2 - (1/q)) < 2$ which is a contradiction. □

We claim that there is no relative n-th non-commuting complete bipartite graph. Otherwise, the only possibility is to have two disjoint sets V_1 and V_2 such that any of the sets V_1, V_2 contains vertices of H. Since all vertices of H are not adjacent so if $h \in V_1$ then $[h^n, x] = 1$ and $[x^n, h] = 1$ for all $x \in H \setminus C^n_{H,G}$ which implies that $h \in C^n_{H,G}$ and it a contradiction.

We finish this section by some interesting results about the non-regularity of the n-th non-commuting graph. Note that H is a non-trivial proper subgroup of G.

Theorem 3.6. *There is no relative n-th non-commuting m-regular graph associated to an AC-group G, where m is an square free positive odd integer.*

Proof. Suppose $\Gamma^n_{H,G}$ is a relative n-th non-commuting graph which is m-regular and $P = \{p_1, p_2, \cdots, p_k\}$ is the set of distinct odd primes which factorize m. If $h \in H$ is a vertex of the relative n-th non-commuting $\Gamma^n_{H,G}$ then $m = deg(h) = |C^n_G(h)|([G : C^n_G(h)] - 1)$, $|C^n_G(h)| = \prod_{p_i \in S} p_i$ and

$([G : C^n_G(h)] - 1) = \prod_{p_j \in S^c} p_j$, where S and S^c are complement subsets of $P^* = P \cup \{1\}$ such that $|C^n_G(h)| \neq 1$. Thus $|G| = \prod_{p_i \in S} p_i(\prod_{p_j \in S^c} p_j + 1)$. By similar argument about the degree of a vertex belongs to $G \setminus H$ it follows that $|H| = \prod_{p_i \in T} p_i(\prod_{p_j \in T^c} p_j + 1)$, where T and T^c are complement subsets of P^*. Since H is a subgroup of G we get that $\prod_{p_i \in T \setminus T \cap S} p_i(\prod_{p_j \in T^c} p_j + 1)$ divides $(\prod_{p_j \in S^c} p_j + 1)$ which is impossible. □

Theorem 3.7. *The graph $\Gamma^n_{H,G}$ associated to an AC-group G is not $2k$-regular graph, where k is an square free positive odd integer.*

Proof. By similar method of previous theorem, we obtain several cases for the orders of H and G which none of them is valid. □

If $\Gamma^n_{H,G}$ is a graph associated to an AC-group G of odd order then the degrees of its vertices are even numbers. By Theorem 3.7 it is not $2k$-regular graph, for an square free positive odd integer k. It is not 2^r-regular graph because otherwise we have $2^r = |C^n_G(h)|([G : C^n_G(h)] - 1)$ for $h \in H \cap V(\Gamma^n_{H,G})$. Since $|C^n_G(h)| \neq 1$ it follows that $|C^n_G(h)| = 2^\alpha$, $1 \leq \alpha \leq r$ and it is a contradiction. We guess $\Gamma^n_{H,G}$ is not regular at all when the order of G is odd.

4. n-th Non-commuting Graph and Isoclinism

We are going to attempt on a known conjecture which states that if two isomorphic graphs are associated to the groups G and H, then the groups G and H are isomorphic as well or at least their orders are equal. We show that two non-abelian isoclinic groups under an extra condition, are associated to some isomorphic graphs.

Proposition 4.1. *Let G be a non-abelian AC-group and $\Gamma^n_G \cong \Gamma^n_{S_3}$ where $n \not\equiv 0$ (mod 6). Then $G \cong S_3$.*

Proof. By graph isomorphism $|G| - |C^n_G| = |S_3| - |C^n_{S_3}|$. Obviously, if $n \equiv i$ (mod 6) for all $1 \leq i \leq 5$ then $C^n_{S_3} = 1$ and $|G| - |C^n_G| = 5$. Since G is an AC-group $|C^n_G| \mid 5$ so $|C^n_G|$ is 1 or 5. If $|C^n_G| = 5$ then $|G| = 10$ and this implies that $G = D_{10}$. As $\Gamma^n_{S_3}$ is not null it follows that $n \not\equiv 0$ (mod 10), because otherwise it is against the graph isomorphism. If $n \equiv j$ (mod 10), for all $1 \leq j \leq 9$ then $C^n_{D_{10}} = 1$ and this is the contradiction. Hence, $|C^n_G| = 1$ and result follows. □

Proposition 4.2. *Let G be a non-abelian AC-group and $\Gamma_G^n \cong \Gamma_{D_{2m}}^n$ where n is an odd number and $m \geq 4$. Then $|G| = |D_{2m}|$.*

Proof. Suppose $D_{2m} = \langle a, b : a^m = b^2 = 1, a^b = a^{-1}\rangle$. First, suppose that m is an odd number. By a simple computation on elements we can find $C_{D_{2m}}^n = 1$ and so $|G| - |C_G^n| = 2m - 1$. Moreover, $C_{D_{2m}}^n(b) = \{e, b\}$ implies $|G| - |C_G^n(b')| = 2m - 2$ for some element $b' \in G$. Consequently, $|C_G^n(b')| - |C_G^n| = 1$. Hence, $|C_G^n| = 1$ and this completes the proof in this case. Now, let m be an even number. By a similar method as in the previous case, $C_{D_{2m}}^n = \{1, a^{m/2}\}$ and so $2m - 2 = |G| - |C_G^n|$. Furthermore, by the fact that $C_{D_{2m}}^n(b) = \{e, b, a^{m/2}, a^{m/2}b\}$ follows $|G| - |C_G^n(b')| = 2m - 4$, for some element $b' \in G$. Consequently, $|C_G^n(b')| - |C_G^n| = 2$. Thus $|C_G^n|$ is 1 or 2. It is easy to see that $C_{D_{2m}}^n(a) = \{1, a, a^2, \cdots, a^{m-1}\}$ and similarly we have $m - 2 = |C_G^n(a')| - |C_G^n|$ for an element $a' \in G$. Now, if $|C_G^n| = 1$ then $|C_G^n(a')| = m - 1$. Since G is an AC-group we should have $m - 1 \mid |G| = 2m - 1$ which is a contradiction. Hence $|G| = 2m$ as required. □

Hall [9] introduces the concept of isoclinism which is an equivalence relation on the class of all groups.

Definition 4.3. *Let G_1 and G_2 be two groups. Then the pair (α, β) is called a n-isoclinism from G_1 to G_2 whenever*

(i) *$\alpha : G_1/Z_n(G_1) \to G_1/Z_n(G_1)$ is an isomorphism, where $Z_n(G_1)$ and $Z_n(G_2)$ are the n-th term of the upper central series of G_1 and G_2, respectively.*

(ii) *β is an isomorphism from $\gamma_{n+1}(G_1)$ to $\gamma_{n+1}(G_2)$ with the law $[g_{11}, \cdots, g_{1\ n+1}] \mapsto [g_{21}, \cdots, g_{2\ n+1}]$, where $g_{2j} \in \alpha(g_{1j}Z_n(G_1))$. If there is such a pair (α, β) with the above properties then we say that G_1 and G_2 are n-isoclinic and is denoted by $G_1 \underset{n}{\sim} G_2$.*

If G_1 and G_2 are 1-isoclinic, then denote it by the abbreviate form $G_1 \sim G_2$ and is it called isoclinism (see [10] for more details).

Theorem 4.4. *If G and H are two non-abelian groups which are isoclinic and $|Z(H)| = |C_H^n| = |Z(G)| = |C_G^n|$. Then $\Gamma_G^n \simeq \Gamma_H^n$.*

Proof. Since $Z(H) \subseteq C_H^n$ and $Z(G) \subseteq C_G^n$, we conclude that $Z(H) = C_H^n, Z(G) = C_G^n$. There is an isomorphism $\phi : G/Z(G) \to H/Z(H)$ such that $g_iZ(G) \mapsto h_iZ(H)$ where $\{g_1, \cdots, g_t\}, \{h_1, \cdots, h_t\}$ are transversal sets for $Z(G)$ and $Z(H)$ in G and H, respectively. Furthermore, one can define the bijection $f : Z(H) = C_H^n \to Z(G) = C_G^n$ and by using isomorphism ϕ we deduce $|V(\Gamma^n_G)| = |V(\Gamma^n_H)|$. Hence the map $\psi : G \setminus C^n_G \to H \setminus C^n_H$ can be constructed such that $x = g_i z \mapsto h_i f(z)$ which is well-defined bijection and preserves edges. □

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