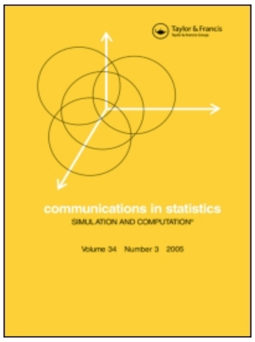

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Two-sample prediction for progressively Type-II censored Weibull lifetimes

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ABSTRACT

Prediction on the basis of censored data has an important role in many fields. This article develops a non-Bayesian two-sample prediction based on a progressive Type-II right censoring scheme. We obtain the maximum likelihood (ML) prediction in a general form for lifetime models including the Weibull distribution. The Weibull distribution is considered to obtain the ML predictor (MLP), the ML prediction estimate (MLPE), the asymptotic ML prediction interval (AMLPI), and the asymptotic predictive ML intervals of the sth-order statistic in a future random sample (Y_s) drawn independently from the parent population, for an arbitrary progressive censoring scheme. To reach this aim, we present three ML prediction methods namely the numerical solution, the EM algorithm, and the approximate ML prediction. We compare the performances of the different methods of ML prediction under asymptotic normality and bootstrap methods by Monte Carlo simulation with respect to biases and mean square prediction errors (MSPEs) of the MLPs of Y_s as well as coverage probabilities (CP) and average lengths (AL) of the AMLPIs. Finally, we give a numerical example and a real data sample to assess the computational comparison of these methods of the ML prediction.

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1. Introduction

Prediction of a future random variable is valuable in the analysis of lifetime data. Based on a complete random sample, prediction problems have been discussed in the literature; see, for example, Aitchison and Dunsmore (1975), Geisser (1993), and references therein. Fernandez (2000) considered the Bayesian prediction problem for an independent future sample from the Rayleigh distribution based on Type-II double censoring. Ali Mousa and Jaheen (2002) considered the two-parameter Burr Type-XII model for obtaining Bayesian prediction in a two-sample problem on the basis of progressively censored data. Kundu and Howlader (2010) presented the Bayesian prediction for the inverse Weibull distribution under Type-II censoring schemes.

According to Aitchison and Dunsmore (1975), "an essential feature of statistical prediction analysis is that it involves two experiments e and f (two-sample prediction). From the information which we gain from a performance of e, the informative experiment, we wish to make some reasoned statement concerning the performance of f, the future experiment.

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In order that e should provide information on f there must be some link between these two experiments."

Due to an applicable example from the book of Aitchison and Dunsmore (1975), "let we have the survival times (weeks) of 20 patients with a certain type of carcinoma and receiving treatment of preoperative radiotherapy followed by radical surgery. On the basis of this information what can appropriately be said about the future of a new patient with this type of carcinoma and assigned to this form of treatment?"

In this example, "the informative experiment e consists of recording the survival times of the 20 patients already treated. The future experiment f consists of treating the new patient similarly and recording his survival time. If no change in the treatment has been made since the conducting of e, then e and f consist respectively of 20 replicates and a single replicate of the same basic trial (record the survival time of a treated patient) and are independent."

Censoring is usual in lifetime data due to time and cost restrictions. In statistics, engineering and medical research, censoring arises when exact lifetimes are only partially known. There are various types of censoring such as Type-II censoring, doubly Type-II censoring, random censoring, and progressive censoring. In this article, we consider a progressive Type-II right censoring scheme. Based on the progressively Type-II censored data, many authors developed statistical inference and prediction for future observations (failure times). For example, Cohen (1963) and Cohen and Norgaard (1977) studied statistical inference for several failure time distributions based on progressively Type-II censored data. Also, see Thomas and Wilson (1972), Cacciari and Motanari (1987), and Viveros and Balakrishnan (1994).

A comprehensive review of theory, methods, and applications of the progressive censoring can be seen in Balakrishnan and Aggarwala (2000). Also, recently a book from Balakrishnan and Cramer (2014) offers a thorough and updated guide to the theory and methods of progressive censoring along with its practical applications to reliability and survival analysis.

Bayesian prediction and inference for the Pareto distribution based on progressive censoring is discussed by Ali Mousa (2001). Balakrishnan et al. (2001) computed bounds for means and variances of progressively Type-II censored order statistics. In addition, Ali Mousa and Al-Sagheer (2005) obtained the Bayesian two-sample prediction bounds with progressive Type-II censoring for the Rayleigh model.

Also, Soliman et al. (2011) considered point and interval Bayesian predictions based on general progressively Type-II censored data from Weibull model under symmetric and asymmetric loss functions. In addition, they obtained prediction bounds for the future observations (two-sample prediction) based on this type of censored samples.

Huang and Wu (2012) derived ML estimators and the Bayes estimators for the parameters of Weibull distribution under squared error loss as well as the Bayes prediction intervals for future observations in the one- and two-sample cases with the data that are progressively type II censored.

Based on progressively Type II censored sampling, Jung and Chung (2013) suggested a very general form of Bayesian prediction bounds from two parameters exponentiated Weibull distribution using the proper general prior density.

Recently, Ghafoori et al. (2011) obtained the Bayesian two-sample prediction bounds as well as the Bayes predictive estimations for a future progressively Type-II censored sample in a general form of lifetime models under the SEL function.

Some prediction studies have also been considered based on the ML approach. For instance, based on a general multiple Type-II censored sample from a shifted exponential distribution, Raqab (2004) introduced a simple approximation to one of prediction likelihood equations, derived approximate predictors of missing failure times. Basak et al. (2006) considered the problem of predicting time to failure of units censored in a progressively censored sample from an absolutely continuous population. The best linear unbiased predictors (BLUPs), the ML predictors (MLPs), and the approximate MLPs of units censored in a progressively censored sample for the Pareto distribution were presented by Raqab et al. (2010).

In this article, we focus on obtaining the MLP and the AMLPI of the sth order statistic in a future random sample drawn from the parent population independently. Therefore, we use the numerical ML solution, the EM algorithm, and the approximate ML for prediction of the future progressively Type II censored order statistics in the Weibull model as a special case of the general class introduced by AL-Hussaini (1999). For the Weibull distribution discussed here, both parameters are assumed to be unknown. The rest of the article is organized as follows.

The predictive likelihood equations in the general class are presented in Section 2. We derive the MLP of y_s and the MLPEs of unknown parameters in the Weibull distribution using the numerical ML prediction, the EM algorithm, and the approximate ML prediction in Section 3. Section 4 then considers how to obtain the AMLPI of y_s and the asymptotic predictive ML intervals of unknown parameters in more details, while in Section 5 a simulation study as well as two illustrative examples are given to assess the proposed performance of the procedures. Finally, Section 6 is devoted to our conclusion and some expressions.

2. Predictive likelihood equations in a general lifetime model

Let $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ be the progressively Type-II censored ordered statistics from a sample of size n with progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ from a continuous distribution. Also, suppose that $\underline{x} = (x_1, x_2, \ldots, x_m)$ be an observed progressively Type-II censored sample with scheme $(R_1, R_2, \ldots, R_m)$. The joint pdf of the sample is (Balakrishnan and Aggarwala, 2000, p. 8)

$$f(\underline{x}; \underline{\theta}) = A \prod_{i=1}^{m} f(x_i)\Big(1 - F(x_i)\Big)^{R_i}, \tag{1}$$

where $A = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\ldots(n - R_1 - R_2 - \ldots - R_{m-1} - m + 1)$, is a normalizing constant, $f(x_i)$ and $F(x_i)$ are the pdf and the cdf of the parent population, respectively, and $\underline{\theta} \in \Theta$ is a parameter vector and Θ is the parameter space. Suppose that $K_{\underline{\theta}}(x)$ is the cumulative hazard rate of the cdf $F_{\underline{\theta}}(.)$ which is increasing in x and non-negative. Then

$$F_{\underline{\theta}}(x) = 1 - e^{-K_{\underline{\theta}}(x)}, \quad x > 0. \tag{2}$$

For more details, see, AL-Hussaini (1999). Substituting (2) into (1), the likelihood function reduces to

$$L(\underline{\theta}; \underline{x}) = A \exp\left\{ \sum_{j=1}^{m} \Big(\ln(K'_{\underline{\theta}}(x_j)) - (R_j + 1)K_{\underline{\theta}}(x_j) \Big) \right\}, \tag{3}$$

where A is given by (1). According to Ali Mousa and AL-Sagheer (2005), assume that $Y_{1:M:N}^{(S_1,S_2,\ldots,S_M)}, Y_{2:M:N}^{(S_1,S_2,\ldots,S_M)}, \ldots, Y_{M:M:N}^{(S_1,S_2,\ldots,S_M)}$ is another (unobserved) independent set of progressively Type-II right censored ordered statistics of size M from a sample of size N with progressive censoring scheme $(S_1, S_2, \ldots, S_M)$. The first sample is considered as an "informative" (past) sample, whereas the second sample is considered as the "future" sample. Now, assume that Y_s represents the sth order statistic in the future sample of size M, $1 \leq s \leq M$. The problem of prediction is very important in practice such as for determining optimal experiments. For more details, see Aitchison and Dunsmore (1975). The rest of this article is devoted to non-Bayesian prediction of Y_s for a future sample.

For the general lifetime model (2) with a vector of parameters $\underline{\theta}$, the pdf of Y_s, $1 \leq s \leq M$ is obtained as (see Balakrishnan and Aggarwala, 2000, p. 26)

$$h(y_s; \underline{\theta}) = C_{s-1} \sum_{i=1}^{s} a_i \exp\left\{ \ln(K'_{\underline{\theta}}(y_s)) - \gamma_i K_{\underline{\theta}}(y_s) \right\}, \tag{4}$$

where

$$\gamma_i = \sum_{j=i}^{M} (S_j + 1) = N - \sum_{j=1}^{i-1} (S_j + 1), \;\; C_{s-1} = \prod_{i=1}^{s} \gamma_i, \;\; a_i = \prod_{j=1}^{s} \frac{1}{\gamma_j - \gamma_i}, \forall i \neq j, s > 1, \tag{5}$$

and $a_1 = 1$ for $s = 1$. Therefore, from (3) and (4), the predictive likelihood function (PLF) is of the form

$$L(\underline{\theta}, y_s; \underline{x}) = h(y_s; \underline{\theta}) L(\underline{\theta}; \underline{x}) = AC_{s-1} \sum_{i=1}^{s} a_i$$
$$\times \exp\left\{ \sum_{j=1}^{m} \Big(\ln K'_{\underline{\theta}}(x_j) - (R_j + 1) K_{\underline{\theta}}(x_j) \Big) + \ln K'_{\underline{\theta}}(y_s) - \gamma_i K_{\underline{\theta}}(y_s) \right\}. \tag{6}$$

Apart from a constant term, by differentiating with respect to y_s and $\underline{\theta}$ from the predictive log-likelihood function and setting to zero, we can obtain the MLP and the AMLPI of y_s of a future random sample. Also, the MLPEs and asymptotic predictive ML intervals of unknown parameters $\underline{\theta}$ can be found. Unfortunately, these equations do not admit explicit forms. Therefore, we use the numerical ML solution, the EM algorithm prediction, and the approximate ML prediction to estimate them. These methods are considered in more details for the Weibull model in Sections 3 and 4.

3. MLP of Y_s

The Weibull distribution is one of the most popular distributions in reliability and survival analysis. This distribution has been widely used for analyzing lifetime data. Here $\underline{\theta} = (\alpha, \beta)$ and $K_{\underline{\theta}}(x) = \alpha x^{\beta}$, $\alpha, \beta > 0$. The corresponding pdf is

$$f(x; \alpha, \beta) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, \quad x > 0, \; \alpha, \beta > 0. \tag{7}$$

It is worth noting that instead of working with the Weibull model for X, because of complexity, it is often more convenient to work with the equivalent model for the log-lifetime $T = \ln X$ which is an extreme value variable with pdf

$$f(t; \mu, \sigma) = \frac{1}{\sigma} e^{(\frac{t-\mu}{\sigma}) - e^{(\frac{t-\mu}{\sigma})}}, \quad -\infty < t < \infty, \; -\infty < \mu < \infty, \sigma > 0. \tag{8}$$

Namely, T is the extreme value variable with parameters $\sigma = \beta^{-1}$ and $\mu = \beta^{-1} \ln(\alpha^{-1})$. Suppose $T_{1:m:n}^{(R_1,R_2,\ldots,R_m)}$, $T_{2:m:n}^{(R_1,R_2,\ldots,R_m)}$, $\ldots$, $T_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ are progressively Type-II censored ordered statistics from a sample of size n with progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ from the extreme value distribution. Also, let $U_{1:M:N}^{(S_1,S_2,\ldots,S_M)}, U_{2:M:N}^{(S_1,S_2,\ldots,S_M)}, \ldots, U_{M:M:N}^{(S_1,S_2,\ldots,S_M)}$ be another (unobserved) independent progressively Type-II right censored series of ordered statistics of size M from the extreme value sample of size N with progressive censoring scheme $(S_1, S_2, \ldots, S_M)$. Now, U_s represents the sth order statistic in the future sample of size M; $1 \leq s \leq M$.

Assume we observed a progressively Type-II censored extreme value sample with scheme $(R_1, R_2, \ldots, R_m)$, denoted by $\underline{t} = (t_1, t_2, \ldots, t_m)$. The log PLF is obtained as

$$\ln L(\mu, \sigma, u_s; \underline{t}) \propto -(m+1)\ln\sigma + \sum_{j=1}^{m}\left(\frac{t_j-\mu}{\sigma}\right) - \sum_{j=1}^{m}(R_j+1)\exp\left(\frac{t_j-\mu}{\sigma}\right)$$
$$+\left(\frac{u_s-\mu}{\sigma}\right) + \ln\left(\sum_{i=1}^{s} a_i \exp\left\{-\gamma_i \exp\left(\frac{u_s-\mu}{\sigma}\right)\right\}\right). \tag{9}$$

We differentiate with respect to μ, σ and u_s and set to zero, then we have

$$\begin{cases} 1: \dfrac{\partial \ln L(\mu,\sigma,u_s;\underline{t})}{\partial\mu} = -\dfrac{m}{\sigma} + \displaystyle\sum_{j=1}^{m}\frac{(R_j+1)}{\sigma}\exp\left(\frac{t_j-\mu}{\sigma}\right) - \frac{1}{\sigma} \\ \qquad + \dfrac{\sum_{i=1}^{s} a_i\gamma_i \exp\left\{\left(\frac{u_s-\mu}{\sigma}\right) - \gamma_i\exp\left(\frac{u_s-\mu}{\sigma}\right)\right\}}{\sigma\sum_{i=1}^{s} a_i\exp\left\{-\gamma_i\exp\left(\frac{u_s-\mu}{\sigma}\right)\right\}} = 0, \\ 2: \dfrac{\partial \ln L(\mu,\sigma,u_s;\underline{t})}{\partial\sigma} = -\dfrac{(m+1)}{\sigma} - \displaystyle\sum_{j=1}^{m}\left(\frac{t_j-\mu}{\sigma^2}\right) + \sum_{j=1}^{m}(R_j+1)\left(\frac{t_j-\mu}{\sigma^2}\right) \\ \qquad \times exp\left(\frac{t_j-\mu}{\sigma}\right) - \left(\dfrac{u_s-\mu}{\sigma^2}\right) \\ \qquad + \dfrac{\sum_{i=1}^{s} a_i\gamma_i\left(\frac{u_s-\mu}{\sigma^2}\right)\exp\{\left(\frac{u_s-\mu}{\sigma}\right) - \gamma_i\exp\left(\frac{u_s-\mu}{\sigma}\right)\}}{\sum_{i=1}^{s} a_i\exp\{-\gamma_i\exp\left(\frac{u_s-\mu}{\sigma}\right)\}} = 0, \\ 3: \dfrac{\partial \ln L(\mu,\sigma,u_s;\underline{t})}{\partial u_s} = \dfrac{1}{\sigma} - \dfrac{\sum_{i=1}^{s} a_i\gamma_i\exp\{\left(\frac{u_s-\mu}{\sigma}\right) - \gamma_i\exp\left(\frac{u_s-\mu}{\sigma}\right)\}}{\sigma\sum_{i=1}^{s} a_i\exp\{-\gamma_i\exp\left(\frac{u_s-\mu}{\sigma}\right)\}} = 0. \end{cases} \tag{10}$$

By substituting 3: into 1: the MLPE of μ can be simplified

$$\hat{\mu} = \hat{\sigma}\ln\left(\frac{1}{m}\sum_{j=1}^{m}(R_j+1)\exp\{\frac{t_j}{\hat{\sigma}}\}\right), \tag{11}$$

where $\hat{\sigma}$ is the MLPE of σ which is obtained by replacing 3: in 2:

$$\hat{\sigma} = \frac{1}{m+1}\left(\frac{\sum_{j=1}^{m}(R_j+1)t_j\exp(\frac{t_j}{\hat{\sigma}})}{\frac{1}{m}\sum_{j=1}^{m}(R_j+1)\exp(\frac{t_j}{\hat{\sigma}})} - \sum_{j=1}^{m}t_j\right). \tag{12}$$

Also, substituting $\hat{\mu}$ and $\hat{\sigma}$ into 3: we find that the MLP of U_s as

$$\hat{U}_s = \hat{\sigma}\ln\left(\frac{\sum_{i=1}^{s} a_i\exp\left\{-\gamma_i\exp(\frac{\hat{U}_s-\hat{\mu}}{\hat{\sigma}})\right\}}{\sum_{i=1}^{s} a_i\gamma_i\exp\left\{-\gamma_i\exp(\frac{\hat{U}_s-\hat{\mu}}{\hat{\sigma}})\right\}}\right) + \hat{\mu}. \tag{13}$$

It is worth mentioning that it is easy to show that the likelihood equations have a unique solution (see Example 6.1, Lehmann and Casella, 1998). To show this, suppose

$$h(\sigma) = \frac{1}{m+1}\left(\frac{\sum_{j=1}^{m}(R_j+1)t_j \exp(\frac{t_j}{\sigma})}{\frac{1}{m}\sum_{j=1}^{m}(R_j+1)\exp(\frac{t_j}{\sigma})}\right) - \sigma = \frac{1}{m+1}\sum_{j=1}^{m} t_j.$$

And let $p_j = \frac{(R_j+1)\exp(\frac{t_j}{\sigma})}{\sum_{j=1}^{m}(R_j+1)\exp(\frac{t_j}{\sigma})}$, so we have, $h(\sigma) = \frac{m}{m+1}\sum_{j=1}^{m} t_j p_j - \sigma = \frac{1}{m+1}\sum_{j=1}^{m} t_j$. Now, $h'(\sigma) = -\frac{m}{(m+1)\sigma^2}(\sum_{j=1}^{m} t_j^2 p_j - (\sum_{j=1}^{m} t_j p_j)^2) - 1 < 0$ by the Cauchy–Schwarz inequality. Thus, $h(\sigma)$ is decreasing. Since h is continuous and

$$-\infty = \lim_{\sigma\to\infty} h(\sigma) \le \frac{1}{m+1}\sum_{j=1}^{m} t_j \le \lim_{\sigma\to 0} h(\sigma) = \frac{m}{m+1}t_m,$$

we conclude that the likelihood equations have a unique solution. Therefore, we could obtain the MLP of U_s and the predictive ML estimators of μ, σ by usual numerical solution.

Also, by transformations $\hat{Y}_s = \exp(\hat{U}_s)$, $\hat{\beta} = 1/\hat{\sigma}$ and $\hat{\alpha} = \exp(-\hat{\mu}/\hat{\sigma})$, the corresponding MLP of Y_s and the MLPEs of α and β can be found.

3.1. Point prediction based on the EM algorithm

The EM algorithm, originally suggested by Dempster et al. (1977), is a very powerful iterative technique for handling any incomplete or missing data and thus can be used for progressively censored samples. Readers are referred to the book by McLachlan and Krishnan (1997) which provides detailed discussion on the EM literature and its applications. The EM algorithm is an iterative method which is applicable to obtain the ML estimators of parameters.

Here, we will apply the EM algorithm in the progressively censored extreme value sample to find the MLP of Y_s and the MLPEs of α and β in a future random sample.

The EM algorithm has two steps which are known as E-step and M-step. In the E-step, the algorithm replaces any missing data by its expected value, while in the M-step, the log-likelihood function is maximized with the observed data and expected value of the incomplete data (censored data) updating the values of the estimates. By repeating the E- and M-steps, the ML estimators are derived when convergence occurs.

First, let us denote the observed and censored vector as $\underline{T} = (T_{1:m:n}, T_{2:m:n}, \ldots, T_{m:m:n})$, $\underline{Z} = (\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_m)$, respectively, where $\underline{Z}_j = (Z_{j1}, Z_{j2}, \ldots, Z_{jR_j})$, for $j = 1, 2, \ldots, m$. By combining $\underline{T}$ and $\underline{Z}$, we have $\underline{W} = (W_1, W_2, \ldots, W_n)$ which is the complete dataset.

On the basis of $\underline{W}$ and apart from a constant term, the log PLF of the extreme value distribution may be taken in the form of

$$\ln L(\mu, \sigma, u_s; \underline{W}) \propto -(n+1)\ln\sigma + \sum_{i=1}^{n}\left(\frac{w_i - \mu}{\sigma}\right) - \sum_{i=1}^{n}\exp\left(\frac{w_i-\mu}{\sigma}\right) + \left(\frac{u_s-\mu}{\sigma}\right)$$
$$+ \ln\left(\sum_{i=1}^{s} a_i \exp\left\{-\gamma_i \exp\left(\frac{u_s-\mu}{\sigma}\right)\right\}\right). \tag{14}$$

In the E-step, one needs to compute the conditional expectation $E(\ln L(\mu,\sigma,u_s;(\underline{T},\underline{Z})|\underline{T}=\underline{t}))$ which can be given by

$$E\Big(\ln L(\mu,\sigma,u_s;(\underline{T},\underline{Z})|\underline{T}=\underline{t})\Big)\propto-(n+1)\ln\sigma+\sum_{j=1}^{m}\Big(\frac{t_j-\mu}{\sigma}\Big)$$

$$+\sum_{j=1}^{m}R_jE\Big(\frac{Z_{jk}-\mu}{\sigma}|Z_{jk}>t_j\Big)-\sum_{j=1}^{m}\exp\{\frac{t_j-\mu}{\sigma}\}-\sum_{j=1}^{m}R_jE\Big(\exp\{\frac{Z_{jk}-\mu}{\sigma}\}|Z_{jk}>t_j\Big)$$

$$+\frac{u_s-\mu}{\sigma}+\ln\left(\sum_{i=1}^{s}a_i\exp\Big\{-\gamma_i\exp(\frac{u_s-\mu}{\sigma})\Big\}\right),\qquad k=1,2,\ldots,R_j. \tag{15}$$

Remark 3.1. It is obvious that the function $E\big(\ln L(\mu,\sigma,u_s;(\underline{T},\underline{Z})|\underline{T}=\underline{t})\big)$ with respect to μ,σ and u_s is continuous.

In order to obtain the above conditional expectations, we use the theorem 3.2 in Ng et al. (2002). Now, suppose $\zeta=\frac{t_j-\mu}{\sigma}$; $j=1,2,\ldots,m$. Therefore, under progressive censoring in the extreme value distribution, the conditional distribution of $\underline{Z}$ given $\underline{T}$ takes the form

$$\frac{f_W(z_{jk})}{1-F_W(t_{j:m:n})}=\frac{\exp\{\exp(\zeta)\}}{\sigma}\exp\left\{\Big(\frac{z_{jk}-\mu}{\sigma}\Big)-\exp\Big(\frac{z_{jk}-\mu}{\sigma}\Big)\right\},\quad z_{jk}>t_{j:m:n}. \tag{16}$$

Let $\mu(h)$ and $\sigma(h)$ be the MLPEs of parameters μ and σ at the hth stage, then by making the transformation $U=\exp\frac{Z_{jk}-\mu(h)}{\sigma(h)}$; $j=1,2,\ldots,m$, conditional expectations can be seen as (Lehmann and Casella, 1998, pp. 457–461)

$$E_1=E\Big(Z_{jk}|Z_{jk}>t_j;\mu(h),\sigma(h)\Big)=\sigma(h)\int_{\exp(\zeta)}^{\infty}\exp(-u)\ln u\,du+\mu(h), \tag{17}$$

$$E_2=E\left(\exp\left\{\frac{Z_{jk}}{\sigma(h+1)}\right\}|Z_{jk}>t_j;\mu(h),\sigma(h)\right)=\exp\left\{\frac{\mu(h)}{\sigma(h+1)}\right\}\exp\{\exp(\zeta)\}$$

$$\times\Gamma\left(\frac{\sigma(h)}{\sigma(h+1)}+1\right)\left[1-\int_0^{\exp(\zeta)}\frac{u^{\frac{\sigma(h)}{\sigma(h+1)}}}{\Gamma\big(\frac{\sigma(h)}{\sigma(h+1)}+1\big)}\exp(-u)\,du\right]. \tag{18}$$

And finally,

$$E_3=E\left(Z_{jk}\exp\left\{\frac{Z_{jk}}{\sigma(h+1)}\right\}|Z_{jk}>t_j;\mu(h),\sigma(h)\right)=\mu(h)E_2+\sigma(h)\exp\{\exp(\zeta)\}$$

$$\times\exp\left\{\frac{\mu(h)}{\sigma(h+1)}\right\}\left[\int_{\exp(\zeta)}^{\infty}(\ln u)u^{\frac{\sigma(h)}{\sigma(h+1)}}\exp(-u)\,du\right]. \tag{19}$$

In the M-step we maximize Eq. (15) with respect to μ,σ, and u_s. Based on complete data, the MLPEs of parameters μ and σ and the MLP of U_s can be easily obtained. By differentiating with respect to μ,σ, and u_s from (14) and equating to zero, the likelihood equations can be rewritten as

$$\begin{cases} a: \dfrac{\partial \ln L\left(\mu,\sigma,u_s;\underline{W}\right)}{\partial\mu} = -\dfrac{(n+1)}{\sigma} + \dfrac{1}{\sigma}\exp\left\{\left(\dfrac{w_i-\mu}{\sigma}\right)\right\} \\ \qquad + \dfrac{\sum_{i=1}^{s} a_i\gamma_i \exp\{(\frac{u_s-\mu}{\sigma}) - \gamma_i\exp(\frac{u_s-\mu}{\sigma})\}}{\sigma\sum_{i=1}^{s} a_i\exp\{-\gamma_i\exp(\frac{u_s-\mu}{\sigma})\}} = 0, \\ b: \dfrac{\partial \ln L(\mu,\sigma,u_s;\underline{W})}{\partial\sigma} = -\dfrac{(n+1)}{\sigma} - \sum_{i=1}^{n}\left(\dfrac{w_i-\mu}{\sigma^2}\right) + \sum_{i=1}^{n}\left(\dfrac{w_i-\mu}{\sigma^2}\right)\exp\left\{\left(\dfrac{w_i-\mu}{\sigma}\right)\right\} \\ \qquad + \left(\dfrac{u_s-\mu}{\sigma^2}\right)\dfrac{\sum_{i=1}^{s} a_i\gamma_i\exp\{(\frac{u_s-\mu}{\sigma}) - \gamma_i\exp(\frac{u_s-\mu}{\sigma})\}}{\sum_{i=1}^{s} a_i\exp\{-\gamma_i\exp(\frac{u_s-\mu}{\sigma})\}} - \left(\dfrac{u_s-\mu}{\sigma^2}\right) = 0, \\ c: \dfrac{\partial \ln L(\mu,\sigma,u_s;\underline{W})}{\partial u_s} = \dfrac{1}{\sigma} - \dfrac{\sum_{i=1}^{s} a_i\gamma_i\exp\{(\frac{u_s-\mu}{\sigma}) - \gamma_i\exp(\frac{u_s-\mu}{\sigma})\}}{\sigma\sum_{i=1}^{s} a_i\exp\{-\gamma_i\exp(\frac{u_s-\mu}{\sigma})\}} = 0. \end{cases} \tag{20}$$

We substitute c: into a: and the MLPE of μ under a complete sample can be expressed as

$$\tilde{\mu} = \tilde{\sigma}\ln\left(\frac{1}{n}\sum_{i=1}^{n}\exp\left\{\frac{w_i}{\tilde{\sigma}}\right\}\right), \tag{21}$$

where $\tilde{\sigma}$ is the MLPE of σ (under a complete sample) which is given by replacing a:, c: and $\tilde{\mu}$ in b:

$$\tilde{\sigma} = \frac{1}{(n+1)}\left(\frac{\sum_{i=1}^{n} w_i\exp(\frac{w_i}{\tilde{\sigma}})}{\sum_{i=1}^{n}\frac{1}{n}\exp(\frac{w_i}{\tilde{\sigma}})} - \sum_{i=1}^{n} w_i\right). \tag{22}$$

Finally, after substituting (21) and (22) into c: the MLP of U_s under a complete sample can be simplified as

$$\tilde{U}_s = \tilde{\sigma}\ln\left(\frac{\sum_{i=1}^{s} a_i\exp\left\{-\gamma_i\exp(\frac{\tilde{U}_s-\tilde{\mu}}{\tilde{\sigma}})\right\}}{\sum_{i=1}^{s} a_i\gamma_i\exp\left\{-\gamma_i\exp\left(\frac{\tilde{U}_s-\tilde{\mu}}{\tilde{\sigma}}\right)\right\}}\right) + \tilde{\mu}. \tag{23}$$

Now, if $\sigma(h)$ and $\mu(h)$ are the MLPEs of parameters σ and μ at the hth stage, respectively, then from (21)–(23), the MLPEs of σ and μ at the $(h+1)$th iteration are

$$\sigma(h+1) = \frac{1}{(n+1)}\left(\frac{\sum_{j=1}^{m} t_j\exp\{\frac{t_j}{\sigma(h+1)}\} + \sum_{j=1}^{m} R_j E\big(Z_{jk}\exp\{\frac{Z_{jk}}{\sigma(h+1)}\}|Z_{jk} > t_j;\sigma(h),\mu(h)\big)}{\frac{1}{n}\sum_{j=1}^{m}\exp\{\frac{t_j}{\sigma(h+1)}\} + \frac{1}{n}\sum_{j=1}^{m} R_j E\big(\exp\{\frac{Z_{jk}}{\sigma(h+1)}\}|Z_{jk} > t_j;\sigma(h),\mu(h)\big)}\right.$$
$$\left. -\sum_{j=1}^{m} t_j - \sum_{j=1}^{m} R_j E\big(Z_{jk}|Z_{jk} > t_j;\sigma(h),\mu(h)\big)\right), \tag{24}$$

$$\mu(h+1) = \sigma(h+1)\ln\left(\frac{1}{n}\sum_{j=1}^{m}\exp\left\{\frac{t_j}{\sigma(h+1)}\right\} + \frac{1}{n}\sum_{j=1}^{m} R_j\right.$$
$$\left.\times E\left(\exp\left\{\frac{z_{jk}}{\sigma(h+1)}\right\}|z_{jk} > t_j;\sigma(h),\mu(h)\right)\right). \tag{25}$$

In the iterative EM procedure, we will start with $\sigma(0)$ and $\mu(0)$ as the initial values. In the $(h+1)$th iteration of the EM algorithm, the MLPEs of $\sigma(h+1)$ and $\mu(h+1)$ can be computed by substitution of expected values (17), (18), and (19) into (24) and (25), respectively. After convergence and obtaining final MLPEs of μ and σ in the EM algorithm ($\tilde{\mu}$ and $\tilde{\sigma}$), the MLP of U_s is as follows:

$$\tilde{U}_s = \tilde{\sigma} \ln \left(\frac{\sum_{i=1}^{s} a_i \exp\left\{ -\gamma_i \exp(\frac{\tilde{U}_s - \tilde{\mu}}{\tilde{\sigma}}) \right\}}{\sum_{i=1}^{s} a_i \gamma_i \exp\left\{ -\gamma_i \exp(\frac{\tilde{U}_s - \tilde{\mu}}{\tilde{\sigma}}) \right\}} \right) + \tilde{\mu}. \tag{26}$$

Similar to the proof for uniqueness of the likelihood equations (10), we can show that the likelihood equations (20), have a unique solution. Therefore, $\tilde{\mu}$, $\tilde{\sigma}$, and $\tilde{U}_s$ are unique. Then, by the uniqueness of $\tilde{\mu}$, $\tilde{\sigma}$, and $\tilde{U}_s$ and with respect to the continuity of $E\big(\ln L(\mu, \sigma, u_s; (\underline{T}, \underline{Z}) | \underline{T} = \underline{t})\big)$, the convergence of the EM algorithm (see Lehmann and Casella, 1998, Theorem 4.12, p. 460) has been guaranteed.

As mentioned earlier, by substitutions $\tilde{Y}_s = \exp(\tilde{U}_s)$, $\tilde{\beta} = 1/\tilde{\sigma}$ and $\tilde{\alpha} = \exp(-\tilde{\mu}/\tilde{\sigma})$, the MLP of Y_s and the MLPEs of α and β for the Weibull model can be obtained.

3.2. *Point prediction on the basis of the approximate ML*

In this section, we derive the MLPEs of unknown parameters α and β and the MLP of Y_s by approximating the likelihood equations (10), where this approximate ML prediction method does not require any starting values.

In addition, let $t_{i:m:n} = \ln x_{i:m:n}$, $\ddot{z}_j = \ddot{z}_{j:m:n} = \frac{t_{j:m:n} - \mu}{\sigma}$, $g(\ddot{z}_j) = \exp\{\ddot{z}_j - \exp(\ddot{z}_j)\}$, $G(\ddot{z}_j) = 1 - \exp\{-\exp(\ddot{z}_j)\}$, $u_s = u_{s:M:N} = \ln y_{s:M:N}$ $(1 \le s \le M)$, $v = \frac{u_s - \mu}{\sigma}$, and $h(v) = C_{s-1} \sum_{i=1}^{s} a_i \exp\{v - \gamma_i \exp(v)\}$. Suppose we denoted a progressively Type-II censored extreme value sample with scheme $(R_1, R_2, \ldots, R_m)$ by $\underline{\ddot{z}} = (\ddot{z}_1, \ddot{z}_2, \ldots, \ddot{z}_m)$. Under these assumptions, the log PLF (14) is reduced to

$$\ln L(\mu, \sigma, u_s; \underline{\ddot{z}}) \propto -m \ln \sigma + \sum_{j=1}^{m} \ln g(\ddot{z}_j) + \sum_{j=1}^{m} R_j \ln\big[1 - G(\ddot{z}_j)\big] - \ln \sigma + \ln h(v), \tag{27}$$

and obtain

$$\begin{cases} E1: \dfrac{\partial \ln L(\mu, \sigma, u_s; \underline{\ddot{z}})}{\partial \mu} = -\displaystyle\sum_{j=1}^{m} \frac{g'(\ddot{z}_j)}{\sigma g(\ddot{z}_j)} + \sum_{j=1}^{m} R_j \frac{g(\ddot{z}_j)}{\sigma\big[1 - G(\ddot{z}_j)\big]} - \frac{h'(v)}{\sigma h(v)} = 0, \\ E2: \dfrac{\partial \ln L(\mu, \sigma, u_s; \underline{\ddot{z}})}{\partial \sigma} = -\dfrac{(m+1)}{\sigma} - \displaystyle\sum_{j=1}^{m} \frac{g'(\ddot{z}_j)\ddot{z}_j}{\sigma g(\ddot{z}_j)} + \sum_{j=1}^{m} R_j \frac{g(\ddot{z}_j)\ddot{z}_j}{\sigma\big[1 - G(\ddot{z}_j)\big]} - \frac{h'(v)v}{\sigma h(v)} = 0, \\ E3: \dfrac{\partial \ln L(\mu, \sigma, u_s; \underline{\ddot{z}})}{\partial u_s} = \dfrac{h'(v)}{\sigma h(v)} = 0. \end{cases} \tag{28}$$

These likelihood equations in (28) do not have explicit solutions. But we can approximate the functions $\frac{g'(\ddot{z}_j)}{g(\ddot{z}_j)}$ and $\frac{g(\ddot{z}_j)}{[1 - G(\ddot{z}_j)]}$ by expanding them in a Taylor series around the points $\mu_j = G^{-1}(p_j) = \ln(-\ln(1 - p_j))$, where $p_j = 1 - q_j = 1 - \prod_{k=m-j+1}^{m} \alpha_k$; $\alpha_k = \frac{k + \sum_{i=m-k+1}^{m} R_i}{1 + k + \sum_{i=m-k+1}^{m} R_i}$ for $j = 1, 2, \ldots m$, and the function $h(v)$ around $\mu_s = G^{-1}(p_s) = \ln(-\ln(1 - p_s))$, where $p_s = 1 - q_s = 1 - \prod_{k=M-s+1}^{M} \alpha'_k$; $\alpha'_k = \frac{k + \sum_{i=M-k+1}^{M} S_i}{1 + k + \sum_{i=M-k+1}^{M} S_i}$ for $1 \le s \le M$. From Balakrishnan and Aggarwala (2000), we have

$$p_j = 1 - \prod_{k=m-j+1}^{m} \frac{k + \sum_{i=m-k+1}^{m} R_i}{1 + k + \sum_{i=m-k+1}^{m} R_i}; \; j = 1, 2, \ldots, s, \ldots m,$$

$$p_s = 1 - \prod_{k=M-j+1}^{M} \frac{k + \sum_{i=M-k+1}^{M} S_i}{1 + k + \sum_{i=M-k+1}^{M} S_i}; \ 1 \le s \le M.$$

Also, in the extreme value distribution, $G^{-1}(u) = \ln(-\ln(1-u)),\ 0 < u < 1$.

Now, by expanding the functions $\frac{g'(\ddot{z}_j)}{g(\ddot{z}_j)}$ and $\frac{g(\ddot{z}_j)}{[1-G(\ddot{z}_j)]}$ around μ_j and the function $h(v)$ around μ_s and keeping only the first two terms, we may approximate these functions by

$$\begin{cases} Eq1: \dfrac{g'(\ddot{z}_j)}{g(\ddot{z}_j)} \approx \dfrac{g'(\mu_j)}{g(\mu_j)} + (\ddot{z}_j - \mu_j)\Big(\dfrac{g'(\mu_j)}{g(\mu_j)}\Big)' = \alpha_j - \beta_j \ddot{z}_j, \\ Eq2: \dfrac{g(\ddot{z}_j)}{(1-G(\ddot{z}_j))} \approx \dfrac{g(\mu_j)}{(1-G(\mu_j))} + (\ddot{z}_j - \mu_j)\Big(\dfrac{g(\mu_j)}{(1-G(\mu_j))}\Big)' \\ \qquad = \alpha_j^* - \beta_j^* \ddot{z}_j = 1 - \alpha_j + \beta_j \ddot{z}_j, \\ Eq3: \dfrac{h'(v)}{h(v)} \approx \dfrac{h'(\mu_s)}{h(\mu_s)} + (v - \mu_s)\left(\dfrac{h'(\mu_s)}{h(\mu_s)}\right)' = \alpha_s - \beta_s v, \end{cases} \tag{29}$$

where

$$\alpha_j = \frac{g'(\mu_j)}{g(\mu_j)} - \mu_j \left(\frac{g''(\mu_j)}{g(\mu_j)} - \left(\frac{g'(\mu_j)}{g(\mu_j)}\right)^2 \right) = 1 + \ln q_j [1 - \ln(-\ln q_j)], \tag{30}$$

$$\beta_j = \left(\frac{g'(\mu_j)}{g(\mu_j)}\right)^2 - \frac{g''(\mu_j)}{g(\mu_j)} = -\ln q_j, \tag{31}$$

$$1 - \alpha_j = \alpha_j^*, \quad -\beta_j = \beta_j^* \tag{32}$$

$$\begin{aligned} \alpha_s &= \frac{h'(\mu_s)}{h(\mu_s)} - \mu_s \left(\frac{h''(\mu_s)}{h(\mu_s)} - \Big(\frac{h'(\mu_s)}{h(\mu_s)}\Big)^2 \right) \\ &= 1 - \frac{\sum_{i=1}^{s} a_i \gamma_i \exp\{\mu_s - \gamma_i \exp(\mu_s)\}}{\sum_{i=1}^{s} a_i \exp\{-\gamma_i \exp(\mu_s)\}} - \mu_s \Bigg[\frac{\sum_{i=1}^{s} a_i \gamma_i^2 \exp\{2\mu_s - \gamma_i \exp(\mu_s)\}}{\sum_{i=1}^{s} a_i \exp\{-\gamma_i \exp(\mu_s)\}} \\ &\quad - \frac{\sum_{i=1}^{s} a_i \gamma_i \exp\{\mu_s - \gamma_i \exp(\mu_s)\}}{\sum_{i=1}^{s} a_i \exp\{-\gamma_i \exp(\mu_s)\}} - \left(\frac{\sum_{i=1}^{s} a_i \gamma_i \exp\{\mu_s - \gamma_i \exp(\mu_s)\}}{\sum_{i=1}^{s} a_i \exp\{-\gamma_i \exp(\mu_s)\}} \right)^2 \Bigg], \end{aligned} \tag{33}$$

$$\begin{aligned} \beta_s &= \left(\frac{h'(\mu_s)}{h(\mu_s)}\right)^2 - \frac{h''(\mu_s)}{h(\mu_s)} = \left(\frac{\sum_{i=1}^{s} a_i \gamma_i \exp\{\mu_s - \gamma_i \exp(\mu_s)\}}{\sum_{i=1}^{s} a_i \exp\{-\gamma_i \exp(\mu_s)\}} \right)^2 \\ &\quad + \frac{\sum_{i=1}^{s} a_i \gamma_i \exp\{\mu_s - \gamma_i \exp(\mu_s)\}}{\sum_{i=1}^{s} a_i \exp\{-\gamma_i \exp(\mu_s)\}} - \frac{\sum_{i=1}^{s} a_i \gamma_i^2 \exp\{2\mu_s - \gamma_i \exp(\mu_s)\}}{\sum_{i=1}^{s} a_i \exp\{-\gamma_i \exp(\mu_s)\}}. \end{aligned} \tag{34}$$

Using these linear approximations, the approximate predictive log-likelihood equations $E1-E3$ in (28) can be written, respectively

$$-\sum_{j=1}^{m} (\alpha_j - \beta_j \ddot{z}_j) + \sum_{j=1}^{m} R_j (\alpha_j^* - \beta_j^* \ddot{z}_j) - (\alpha_s - \beta_s v) = 0, \tag{35}$$

$$-(m+1) - \sum_{j=1}^{m} (\alpha_j - \beta_j \ddot{z}_j)\ddot{z}_j + \sum_{j=1}^{m} R_j (\alpha_j^* - \beta_j^* \ddot{z}_j)\ddot{z}_j - (\alpha_s - \beta_s v)v = 0, \tag{36}$$

$$\alpha_s - \beta_s v = 0. \tag{37}$$

From (37) and $v = \frac{u_s - \mu}{\sigma}$, the approximate MLP of U_s can be obtained as

$$\check{U}_s = \check{\sigma}\frac{\alpha_s}{\beta_s} + \check{\mu}, \tag{38}$$

where $\check{\sigma}$ and $\check{\mu}$ are the approximate MLPEs of σ and μ, respectively and they can be derived as follows.

Substituting $\ddot{z}_j = \ddot{z}_{j:m:n} = \frac{t_{j:m:n} - \mu}{\sigma}$ into (35), using (37) and after algebraic simplification, we get the approximate MLPE of μ as $\check{\mu} = A_L - \check{\sigma} B_L$, where

$$A_L = \frac{\sum_{j=1}^m t_j \beta j(1 + R_j)}{\sum_{j=1}^m \beta_j(1 + R_j)}, \quad B_L = \frac{\sum_{j=1}^m \alpha_j - \sum_{j=1}^m R_j(1 - \alpha_j)}{\sum_{j=1}^m \beta_j(1 + R_j)}. \tag{39}$$

Similarly, simplifying (36) can be transformed to the approximate predictive likelihood equation form of σ as $(m + 1)\sigma^2 + D_L - F_L = 0$, where

$$D_L = \sum_{j=1}^m \alpha_j(t_j - A_L) - \sum_{j=1}^m R_j(t_j - A_L) - \sum_{j=1}^m R_j\alpha_j(A_L - t_j),$$
$$F_L = \sum_{j=1}^m \beta_j R_j(t_j - A_L)^2 + \sum_{j=1}^m \beta_j(t_j - A_L)^2 > 0. \tag{40}$$

Therefore, (40) yields the approximate MLPE of σ to be

$$\check{\sigma} = \frac{-D_L + \sqrt{D_L^2 + 4(m + 1)F_L}}{2(m + 1)}, \tag{41}$$

which is the only positive root (see Bayat Mokhtari et al., 2011).

4. Asymptotic ML Prediction Interval of Y_s

In this section, we derive the Fisher information based on the predictive likelihood equations in the numerical ML prediction, prediction on the basis of the EM algorithm and the approximate ML prediction. To construct the AMLPI of Y_s and the asymptotic predictive ML intervals of unknown parameters, we use the asymptotic standard normal distribution for large samples (sufficiently large m and n) in the central limit theorem (see Faulkenberry, 1973).

First of all, we investigate the Fisher information matrix of the numerical ML prediction. We know that the pdf of $t_j, \quad j = 1, 2, \ldots, m$ of the first sample under the extreme value distribution is

$$p(t_j|\underline{\theta}) = \frac{C'_{j-1}}{\sigma}\sum_{k=1}^{j} a'_k \exp\left\{\left(\frac{t_j - \mu}{\sigma}\right) - \gamma'_k \exp\left(\frac{t_j - \mu}{\sigma}\right)\right\}, \tag{42}$$

where $\gamma'_k = \sum_{i=k}^m (R_i + 1) = n - \sum_{i=1}^{k-1}(R_i + 1), \quad C'_{j-1} = \prod_{k=1}^j \gamma'_k, \quad a'_k = \prod_{i=1}^j \frac{1}{\gamma_i - \gamma_k}, \forall i \neq k, \quad j = 2, \ldots, m$, and $a'_1 = 1$ for $j = 1$. Also, we have the following relationship between the digamma and the trigamma functions

$$\psi'(t) + \psi^2(t) = \frac{1}{\Gamma(t)}\int_0^\infty \exp(-y)y^{t-1}\ln^2 y\, dy. \tag{43}$$

By making the transformations $\grave{z}_j = \exp(\frac{t_j-\mu}{\sigma})$; $j = 1, 2, \ldots, m$ and $\gamma'_k \grave{z}_j = \grave{u}_j$, we get

$$E\left(\frac{T_j-\mu}{\sigma}\right) = C'_{j-1}\sum_{k=1}^{j}\frac{a'_k}{\gamma'_k}\Big[\psi(1) - \ln\gamma'_k\Big], \tag{44}$$

$$E\left(\left(\frac{T_j-\mu}{\sigma}\right)\exp\left\{\frac{T_j-\mu}{\sigma}\right\}\right) = C'_{j-1}\sum_{k=1}^{j}\frac{a'_k}{(\gamma'_k)^2}\left[\psi(2) - \psi(1)\ln\gamma'_k\right], \tag{45}$$

$$E\left(\exp\left\{\frac{T_j-\mu}{\sigma}\right\}\right) = C'_{j-1}\sum_{k=1}^{j}\frac{a'_k}{(\gamma'_k)^2}, \tag{46}$$

$$E\left(\left(\frac{T_j-\mu}{\sigma}\right)^2\exp\left\{\frac{T_j-\mu}{\sigma}\right\}\right) = C'_{j-1}\sum_{k=1}^{j}\frac{a'_k}{(\gamma'_k)^2}\Big[\psi^2(2) + \psi'(2) + \ln^2\gamma'_k - 2\psi(2)\ln\gamma'_k\Big]. \tag{47}$$

By taking expectation from the second partial derivatives, the elements of the Fisher information matrix can be derived by replacing (44)–(47) in the expectation as

$$I(\underline{\theta}) = -E\Big[\frac{\partial^2 L(\mu,\sigma,u_s;\underline{t})}{\partial\underline{\theta}^2}\Big],$$

where $\underline{\theta} = (\mu,\sigma,u_s)$. For large samples (sufficiently large m and n such that $m/n \to p$ $0 < p < 1$) by Theorems 3.7 (p. 447) and 5.1 (p. 463) from Lehmann and Casella (1998), the random vector $(\hat{\mu}-\mu, \hat{\sigma}-\sigma, \hat{U}_s - U_s)^T$ tends to the multivariate normal distribution with mean vector $(0,0,0)^T$ and the covariance matrix $\sum = I(\underline{\theta})^{-1}$ where $I(\underline{\theta})$ is the fisher information of the available data. Here, we have $[I(\underline{\theta})]^{-1} = [[V^{ij}]]_{i,j=1,2,3}$ and

$$I(\underline{\theta}) = \begin{bmatrix} -E(V_{11}) & -E(V_{12}) & -E(V_{13}) \\ -E(V_{12}) & -E(V_{22}) & -E(V_{23}) \\ -E(V_{21}) & -E(V_{23}) & -E(V_{33}) \end{bmatrix},$$

where V_{ij}, $i, j = 1, 2, 3$ are the second partial derivatives.

Through Monte Carlo simulations, we simulate the CPs

$$P(-1.64 < B_i < 1.64), \quad i = 1, 2, 3,$$

where $B1 = \frac{\hat{\mu}-\mu}{\sqrt{V^{11}}}$, $B2 = \frac{\hat{\sigma}-\sigma}{\sqrt{V^{22}}}$, $B3 = \frac{\hat{U}_s - U_s}{\sqrt{V^{33}}}$. We will compute the AMLPI of U_s and the asymptotic predictive ML intervals of unknown parameters (μ and σ). Thus, if we denote the AMLPI of Y_s and the asymptotic predictive ML intervals of μ and σ by $(l(\hat{\mu}), u(\hat{\mu}))$, $(l(\hat{\sigma}), u(\hat{\sigma}))$ and $(l(\hat{U}_s), u(\hat{U}_s))$, respectively, then $((u(\hat{\mu}))^{-1}, (l(\hat{\mu}))^{-1})$, and $(\exp\{l(\hat{U}_s)\}, \exp\{u(\hat{U}_s)\})$ will be the corresponding asymptotic predictive ML interval of β and the AMLPI of Y_s based on the Weibull distribution, respectively. Also, if $\frac{l(\hat{\mu})}{u(\hat{\sigma})} < \frac{u(\hat{\mu})}{l(\hat{\sigma})}$, then the corresponding asymptotic predictive ML interval of α will be $(\exp\{-\frac{u(\hat{\mu})}{l(\hat{\sigma})}\}, \exp\{-\frac{l(\hat{\mu})}{u(\hat{\sigma})}\})$, otherwise the asymptotic predictive ML interval of α can be obtained as $(\exp\{-\frac{u(\hat{\mu})}{u(\hat{\sigma})}\}, \exp\{-\frac{l(\hat{\mu})}{l(\hat{\sigma})}\})$.

4.1. Interval prediction based on the EM algorithm

One of the advantages of using the EM algorithm is that it gives a measure of information in the censored (missing) data in a natural way through the missing information principle. Louis (1982) developed a procedure for extracting the observed information matrix when the

EM algorithm is used. It can be expressed as observed information = complete information-missing information (see Ng. et al. 2002).

Here, we use this procedure in order to obtain the Fisher information matrix for the EM prediction (the observed information matrix) under progressive Type-II censoring. The complete information matrix in the extreme value distribution, denoted by $I_{\underline{W}}(\underline{\theta})$, is

$$I_{\underline{W}}(\underline{\theta}) = -E\Big[\frac{\partial^2 L(\mu, \sigma, u_s; \underline{W})}{\partial \underline{\theta}^2}\Big], \quad \text{where} \quad \underline{\theta} = (\mu, \sigma, u_s). \tag{48}$$

On the basis of the conditional distribution in (16), the Fisher information matrix in one observation which is censored at the time of the jth failure is

$$I^{(j)}_{\underline{Z}|\underline{T}}(\underline{\theta}) = E\left(\frac{\partial^2 \ln f_{\underline{Z}|\underline{T}}(z_{jk}|t_{j:m:n}, \underline{\theta})}{\partial \underline{\theta}^2}\right)^2 = -E\Big[\frac{\partial^2 \ln f_{\underline{Z}|\underline{T}}(z_{jk}|t_{j:m:n}, \underline{\theta})}{\partial \underline{\theta}^2}\Big], \qquad k = 1, 2, \ldots, R_j. \tag{49}$$

Then, the missing information (the expected information for the conditional distribution of $\underline{Z}$ given $\underline{T}$) can be found as $I_{\underline{Z}|\underline{T}}(\underline{\theta}) = \sum_{j=1}^{m} R_j I^{(j)}_{\underline{Z}|\underline{T}}(\underline{\theta})$.

Therefore, the observed information is

$$I_{\underline{T}}(\underline{\theta}) = I_{\underline{W}}(\underline{\theta}) - I_{\underline{Z}|\underline{T}}(\underline{\theta}). \tag{50}$$

In order to compute the complete information matrix, we will need the expectations $E(\exp(\frac{W_i-\mu}{\sigma}))$, $E(\frac{W_i-\mu}{\sigma})$, $E((\frac{W_i-\mu}{\sigma})\exp(\frac{W_i-\mu}{\sigma}))$ and $E((\frac{W_i-\mu}{\sigma})^2\exp(\frac{W_i-\mu}{\sigma}))$.

The W_i; $i = 1, 2 \ldots, n$ are iid extreme value random variables with parameters μ and σ, and from (43), we easily get

$$\begin{aligned} &E\left(\exp\left(\frac{W_i-\mu}{\sigma}\right)\right) = 1, \quad E\left(\frac{W_i-\mu}{\sigma}\right) = \psi(1), \\ &E\left(\left(\frac{W_i-\mu}{\sigma}\right)\exp\left(\frac{W_i-\mu}{\sigma}\right)\right) = \psi(2), \end{aligned} \tag{51}$$

$$\text{and} \quad E\left(\left(\frac{W_i-\mu}{\sigma}\right)^2\exp\left(\frac{W_i-\mu}{\sigma}\right)\right) = \psi'(2) + \psi^2(2). \tag{52}$$

If we take expectation from the second partial derivatives and replace (51) and (52) in the taken expectation, we can obtain the elements of the complete information matrix as $I_{\underline{W}}(\underline{\theta}) = -E[\frac{\partial^2 L(\mu,\sigma,u_s;\underline{W})}{\partial \underline{\theta}^2}]$, where $\underline{\theta} = (\mu, \sigma, u_s)$. From (16), we can write

$$\begin{aligned} \ln f_{\underline{Z}|\underline{T}}(z_{jk}|z_{jk} > t_{j:m:n}, \mu, \sigma) = \exp(\zeta) - \ln\sigma + \Big(\frac{z_{jk}-\mu}{\sigma}\Big) \\ - \exp\Big(\frac{z_{jk}-\mu}{\sigma}\Big), \quad z_{jk} > t_{j:m:n}. \end{aligned} \tag{53}$$

Let $\zeta = \frac{t_j-\mu}{\sigma}$; $j = 1, 2, \ldots, m$, thus

$$Z_{11} = \frac{\partial^2 \ln f_{\underline{Z}|\underline{T}}(z_{jk}|z_{jk} > t_{j:m:n}, \mu, \sigma)}{\partial \mu^2} = -\frac{1}{\sigma^2}\left[\exp(\zeta) - \exp\left(\frac{z_{jk}-\mu}{\sigma}\right)\right],$$

$$Z_{22} = \frac{\partial^2 \ln f_{\underline{Z}|\underline{T}}(z_{jk}|z_{jk} > t_{j:m:n}, \mu, \sigma)}{\partial \sigma^2} = \frac{1}{\sigma^2}\Bigg[1 + 2\left(\frac{z_{jk}-\mu}{\sigma}\right) + \zeta^2 \exp(\zeta) + 2\zeta \exp(\zeta)$$
$$- \left(\frac{z_{jk}-\mu}{\sigma}\right)^2 \exp\left(\frac{z_{jk}-\mu}{\sigma}\right) - 2\left(\frac{z_{jk}-\mu}{\sigma}\right)\exp\left(\frac{z_{jk}-\mu}{\sigma}\right)\Bigg],$$

$$Z_{12} = \frac{\partial^2 \ln f_{\underline{Z}|\underline{T}}(z_{jk}|z_{jk} > t_{j:m:n}, \mu, \sigma)}{\partial \mu \partial \sigma} = -\frac{1}{\sigma^2}\Bigg[1 + \exp(\zeta) + \zeta \exp(\zeta) - \exp\left(\frac{z_{jk}-\mu}{\sigma}\right)$$
$$- \left(\frac{z_{jk}-\mu}{\sigma}\right)\exp\left(\frac{z_{jk}-\mu}{\sigma}\right)\Bigg], \tag{54}$$

and finally, $Z_{13} = Z_{23} = Z_{33} = 0$.

Again, the missing information can be given by taking expectation (with respect to the conditional distribution of $\underline{Z}$ given $\underline{T}$ in (16)) from the second partial derivatives

$$I^{(j)}_{\underline{Z}|\underline{T}}(\underline{\theta}) = -E\left[\frac{\partial^2 \ln f_{\underline{Z}|\underline{T}}(z_{jk}|t_{j:m:n}, \underline{\theta})}{\partial \underline{\theta}^2}\right], \quad \text{where} \quad \underline{\theta} = (\mu, \sigma, u_s).$$

Finally, the observed information $I_{\underline{T}}(\underline{\theta})$ could easily be found. As mentioned before this subsection, by the central limit theorem for large samples, we can obtain Monte Carlo simulated CPs, the AMLPI of U_s, the asymptotic predictive ML intervals of unknown parameters (μ and σ) in the EM algorithm and the corresponding AMLPI of Y_s and asymptotic predictive ML intervals of α and β for the Weibull distribution.

4.2. Interval prediction on the basis of the approximate ML

With respect to (42) and similar to expectations in (45)–(48), we have

$$E\left(\exp\left\{2(\frac{T_j-\mu}{\sigma})\right\}\right) = 2C'_{j-1}\sum_{k=1}^{j}\frac{a'_k}{(\gamma'_k)^3}, \tag{55}$$

and similarly,

$$E\left(\left(\frac{T_j-\mu}{\sigma}\right)\exp\left\{2\left(\frac{T_j-\mu}{\sigma}\right)\right\}\right) = C'_{j-1}\sum_{k=1}^{j}\frac{a'_k}{(\gamma'_k)^3}\left[2\psi(3) - 2\ln\gamma'_k\right], \tag{56}$$

$$E\left(\left(\frac{T_j-\mu}{\sigma}\right)^2\exp\left\{2\left(\frac{T_j-\mu}{\sigma}\right)\right\}\right)$$
$$= C'_{j-1}\sum_{k=1}^{j}\frac{a'_k}{(\gamma'_k)^3}\left[2\psi^2(3) + 2\psi'(3) + 2\ln^2\gamma'_k - 4\psi(3)\ln\gamma'_k\right], \tag{57}$$
$$E\left(\left(\frac{T_j-\mu}{\sigma}\right)^2\right) = C'_{j-1}\sum_{k=1}^{j}\frac{a'_k}{\gamma'_k}\left[\psi^2(1) + \psi'(1) + \ln^2\gamma'_k - 2\psi(1)\ln\gamma'_k\right]. \tag{58}$$

After taking expectation from the second partial derivatives, simplifying them and replacing (44)–(47) and (55)–(58) in the expectation (similar to Subsection 4.1), we can derive the Fisher information matrix of the approximate ML prediction method and therefore, the AMLPI of U_s and the asymptotic predictive ML intervals of unknown parameters μ and σ, or

Table 1. Various censoring scheme R_i, $i = 1, 2, \ldots, m$ and S_i, $i = 1, 2, \ldots, M$ for $n = 50, N = 26, m = 35$, and $M = 13$.

Case	$R_i, i = 1, 2, \ldots, m$	$S_i, i = 1, 2, \ldots, M$
1	$15, 0, \ldots, 0$	$13, 0, \ldots, 0$
2	$\overbrace{0, \ldots, 0}^{17}, 15, \overbrace{0 \ldots, 0}^{17}$	$\overbrace{0, \ldots, 0}^{6}, 13, \overbrace{0, \ldots, 0}^{6}$
3	$0, \ldots, 0, 15$	$0, \ldots, 0, 13$
4	$\overbrace{2, \ldots, 2}^{7}, 1, \overbrace{0 \ldots, 0}^{27}$	$\overbrace{0, \ldots, 0}^{6}, 13, \overbrace{0, \ldots, 0}^{6}$

corresponding AMLPI of Y_s and asymptotic predictive ML intervals of α and β based on the Weibull distribution under the approximate ML prediction. For more details, see Balakrishnan et al. (2004) and Balakrishnan and Hossain (2007).

5. Numerical results

In this section, the performance of the proposed procedures are investigated by a simulation study and two illustrative examples.

5.1. Simulation Study

This subsection is devoted to test the performance of the obtained MLP of Y_s, MLPEs of α and β, AMLPI of Y_s and the asymptotic predictive ML intervals of α and β in a future random sample. For the three introduced prediction methods, we can compare the MLPs of the Y_s $(1 \leq s \leq M)$ on the basis of their biases as well as MSPEs and MLPEs of α and β in terms of their biases and MSEs. Also, the AMLPIs of Y_s and the asymptotic predictive ML intervals of α and β can be compared on the basis of their CPs and ALs in different methods of prediction described in Sections 3 and 4.

For given values of the parameters $(\mu = 0,\ \sigma = 1)$ and for different $s\,(1 \leq s \leq M)$, according to an algorithm proposed by Balakrishnan and Sandhu (1995), a progressively Type-II censored extreme value sample is generated for given values of the censoring scheme R_i, $i = 1, 2, \ldots, m$. For the three mentioned methods for prediction, the corresponding MLPs of Y_s, MLPEs of α and β, 95% AMLPIs of Y_s and asymptotic 95% predictive ML intervals of α and β are derived based on the results in Sections 3 and 4 in the large samples (sufficiently large m and n). On the basis of real values for $\mu_0 = 0,\ \sigma_0 = 1$ and $U_{s0} = E(U_s) = \mu C_{s-1} \sum_{i=1}^{s} \frac{a_i}{\gamma_i} + \sigma C_{s-1} \sum_{i=1}^{s} \frac{a_i}{\gamma_i} [\psi(1) - \ln \gamma_i]$ (or equivalently, $\alpha = \exp(-\mu/\sigma),\ \ \beta = 1/\sigma$ and $Y_s = \exp(U_s)$), we replicate the above process 10,000 times and report biases, MSPEs of the MLPs, and MSEs of the MLPEs, as well as CPs and ALs of the 95% AMLPIs and asymptotic 95% predictive ML intervals. The results were obtained by the statistical package R (2013) and are shown in Tables 2–4 for different methods of prediction. Also, Table 1 displays three different cases of R_i and S_i's for $n = 50$, $N = 26$, $m = 35$, and $M = 13$.

From Tables 2–4, we observed that three methods of point prediction for Y_s in terms of bias and MSPE of the MLP of Y_s do not have any significant difference. In addition, we can see almost identical ALs and CPs of the AMLPI of Y_s for Cases 1, 2, and 4, whereas we have shorter ALs of the AMLPI of Y_s under EM algorithm in the Case 3 (Type-II censored data). Also, for point prediction of the parameters, using the EM algorithm method $(\tilde{\alpha}\ , \tilde{\beta})$ yields smaller biases for the MLPEs of parameters. On the other hand, in Cases 1, 2, and 4, the ALs

Table 2. Normality simulation results for the numerical method approach.

		Bias (MSPE)			AL (CP)		
Case	s	y_s	α	β	y_s	α	β
1	3	0.051(0.007)	0.018(0.037)	0.053(0.022)	1.106(0.999)	0.691(0.940)	0.578(0.912)
	8	0.064(0.028)	0.015(0.036)	0.051(0.022)	1.620(0.999)	0.688(0.939)	0.556(0.909)
	13	− 0.076(0.304)	0.019(0.038)	0.054(0.023)	5.653(0.998)	0.689(0.931)	0.564(0.906)
2	3	0.032(0.002)	0.037(0.038)	0.054(0.022)	0.561(0.998)	0.884(0.983)	0.407(0.821)
	8	0.024(0.009)	0.032(0.037)	0.053(0.021)	1.002(0.999)	0.883(0.983)	0.395(0.812)
	13	−0.051(0.242)	0.034(0.037)	0.054(0.021)	6.054(0.999)	0.884(0.982)	0.396(0.812)
3	3	0.035(0.003)	0.047(0.044)	0.073(0.032)	1.712(0.999)	12.795(0.999)	1.212(0.975)
	8	0.038(0.007)	0.047(0.043)	0.073(0.033)	6.217(0.999)	10.563(0.999)	1.008(0.956)
	13	0.031(0.015)	0.046(0.044)	0.071(0.033)	14.107(0.999)	10.250(0.999)	0.978(0.954)
4	3	0.034(0.003)	0.022(0.036)	0.051(0.020)	0.628(0.998)	0.695(0.948)	0.536(0.909)
	8	0.028(0.010)	0.025(0.037)	0.052(0.020)	0.998(0.999)	0.692(0.941)	0.512(0.895)
	13	−0.034(0.218)	0.025(0.036)	0.051(0.020)	5.727(0.999)	0.693(0.943)	0.512(0.903)

and CPs of the asymptotic predictive ML intervals of unknown parameters are not different. But for the Case 3, the EM method for interval prediction of the parameters has the smaller ALs, too.

In addition, according the viewpoint of reviewer we have added the bootstrap (boot-p) method to our simulation study. We have obtained the bias (MSPE) for MLP of Y_s, the biases (MSEs) of the MLPEs, the ALs (CPs) of the 95% approximate boot-p prediction interval

Table 3. Normality simulation results for the EM algorithm.

		Bias (MSPE)	Bias (MSE)		AL (CP)		
Case	s	y_s	α	β	y_s	α	β
1	3	0.053(0.007)	0.000(0.037)	− 0.000(0.020)	1.107(0.999)	0.685(0.938)	0.589(0.911)
	8	0.064(0.028)	− 0.000(0.037)	0.000(0.020)	1.618(0.999)	0.682(0.936)	0.564(0.913)
	13	− 0.10(0.307)	0.000(0.041)	0.000(0.021)	5.629(0.996)	0.685(0.936)	0.574(0.905)
2	3	0.033(0.002)	− 0.000(0.036)	− 0.000(0.018)	0.613(0.999)	0.660(0.939)	0.585(0.903)
	8	0.022(0.008)	0.000(0.037)	0.000(0.019)	0.957(0.999)	0.661(0.919)	0.555(0.900)
	13	− 0.033(0.250)	0.000(0.037)	0.000(0.018)	5.933(0.999)	0.659(0.920)	0.552(0.922)
3	3	0.036(0.003)	− 0.000(0.043)	− 0.000(0.027)	0.659(0.999)	0.665(0.925)	0.776(0.929)
	8	0.033(0.007)	− 0.000(0.040)	0.000(0.026)	0.684(0.998)	0.665(0.925)	0.705(0.917)
	13	0.036(0.014)	0.000(0.045)	− 0.000(0.029)	0.923(0.999)	0.670(0.928)	0.708(0.878)
4	3	0.034(0.003)	0.000(0.034)	− 0.000(0.017)	0.633(0.997)	0.675(0.943)	0.575(0.924)
	8	0.028(0.011)	0.000(0.037)	0.000(0.017)	1.000(0.999)	0.671(0.929)	0.540(0.919)
	13	− 0.035(0.020)	0.000(0.033)	0.000(0.018)	5.821(0.999)	0.674(0.946)	0.544(0.909)

Table 4. Normality simulation results for the approximate ML prediction.

		Bias (MSPE)	Bias (MSE)		AL (CP)		
Case	s	y_s	α	β	y_s	α	β
1	3	0.042(0.006)	0.042(0.039)	0.041(0.021)	0.720(0.996)	0.693(0.938)	0.550(0.917)
	8	0.039(0.025)	0.047(0.040)	0.037(0.020)	1.481(0.999)	0.694(0.935)	0.549(0.917)
	13	− 0.108(0.299)	0.042(0.040)	0.042(0.021)	5.169(0.997)	0.693(0.935)	0.551(0.909)
2	3	0.031(0.002)	0.052(0.041)	0.057(0.022)	0.400(0.990)	0.893(0.978)	0.395(0.799)
	8	0.015(0.008)	0.057(0.042)	0.056(0.022)	0.894(0.999)	0.895(0.979)	0.395(0.805)
	13	− 0.093(0.231)	0.052(0.041)	0.056(0.022)	5.291(0.999)	0.893(0.977)	0.395(0.808)
3	3	0.034(0.003)	0.051(0.045)	0.070(0.033)	1.663(0.999)	10.237(0.999)	0.973(0.954)
	8	0.037(0.007)	0.047(0.043)	0.067(0.032)	6.163(0.999)	10.208(0.999)	0.971(0.957)
	13	0.028(0.014)	0.050(0.044)	0.068(0.033)	14.190(0.999)	10.230(0.999)	0.971(0.955)
4	3	0.029(0.002)	0.047(0.040)	0.044(0.020)	0.417(0.993)	0.697(0.939)	0.505(0.901)
	8	0.018(0.009)	0.042(0.038)	0.046(0.019)	0.889(0.999)	0.696(0.943)	0.506(0.900)
	13	− 0.062(0.021)	0.045(0.039)	0.042(0.019)	5.055(0.999)	0.697(0.940)	0.504(0.908)

Table 5. Boot-p simulation results for the numerical method approach.

		Bias (MSPE)	Bias (MSE)		AL (CP)		
Case	s	y_s	α	β	y_s	α	β
1	3	0.062(0.008)	0.026(0.039)	0.093(0.029)	0.259(0.800)	0.688(0.932)	0.564(0.876)
	8	0.057(0.028)	0.036(0.046)	0.093(0.030)	0.579(0.896)	0.692(0.912)	0.565(0.875)
	13	− 0.206(0.310)	0.019(0.040)	0.095(0.029)	1.999(0.887)	0.689(0.915)	0.566(0.874)
2	3	0.042(0.003)	0.058(0.045)	0.102(0.031)	0.168(0.795)	0.674(0.921)	0.563(0.844)
	8	0.024(0.009)	0.068(0.047)	0.091(0.027)	0.352(0.917)	0.676(0.907)	0.554(0.866)
	13	− 0.195(0.272)	0.066(0.057)	0.104(0.033)	1.753(0.858)	0.674(0.885)	0.563(0.838)
3	3	0.044(0.004)	0.069(0.054)	0.117(0.043)	0.177(0.782)	0.676(0.898)	0.685(0.868)
	8	0.053(0.012)	0.082(0.060)	0.128(0.045)	0.605(0.854)	0.676(0.895)	0.690(0.856)
	13	0.030(0.026)	0.081(0.055)	0.129(0.047)	3.628(0.996)	0.675(0.899)	0.689(0.855)
4	3	0.040(0.003)	0.038(0.040)	0.082(0.026)	0.176(0.822)	0.680(0.933)	0.540(0.878)
	8	0.033(0.011)	0.037(0.038)	0.084(0.026)	0.372(0.929)	0.679(0.936)	0.537(0.874)
	13	− 0.128(0.202)	0.036(0.036)	0.083(0.026)	1.693(0.905)	0.686(0.935)	0.537(0.868)

of Y_s and the approximate boot-p 95% predictive ML intervals of α and β for all Cases of three methods. The results with 1000 replications and 200 bootstrap samples are reported in Tables 5–7.

In order to compare the simulation results especially in terms of ALs of prediction intervals based on asymptotic results with boot-p method, it is observed that in more Cases, the ALs of prediction intervals are more shorter in comparison to the same method based on

Table 6. Boot-p simulation results for the EM algorithm.

		Bias (MSPE)	Bias (MSE)		AL (CP)		
Case	s	y_s	α	β	y_s	α	β
1	3	0.090(0.011)	0.000(0.024)	0.000(0.014)	0.279(0.800)	0.644(0.900)	0.544(0.900)
	8	0.049(0.021)	0.000(0.045)	0.000(0.011)	0.554(0.900)	0.683(0.900)	0.589(0.900)
	13	− 0.442(0.333)	− 0.000(0.055)	0.000(0.009)	1.794(0.900)	0.686(0.800)	0.549(0.999)
2	3	0.047(0.003)	0.000(0.033)	− 0.000(0.008)	0.176(0.800)	0.655(0.900)	0.563(0.900)
	8	0.042(0.010)	− 0.000(0.025)	0.000(0.009)	0.368(0.900)	0.658(0.999)	0.524(0.999)
	13	− 0.023(0.423)	0.000(0.053)	− 0.000(0.023)	1.891(0.900)	0.691(0.900)	0.550(0.900)
3	3	0.036(0.001)	0.000(0.026)	− 0.000(0.006)	0.176(0.999)	0.669(0.900)	0.665(0.999)
	8	0.053(0.010)	− 0.000(0.031)	0.000(0.022)	0.292(0.700)	0.667(0.999)	0.716(0.900)
	13	0.036(0.015)	0.000(0.027)	− 0.000(0.041)	0.449(0.900)	0.664(0.999)	0.680(0.900)
4	3	0.047(0.004)	− 0.000(0.022)	0.000(0.028)	0.184(0.600)	0.686(0.999)	0.566(0.800)
	8	0.018(0.010)	− 0.000(0.045)	0.000(0.016)	0.344(0.999)	0.718(0.900)	0.581(0.800)
	13	− 0.018(0.241)	− 0.000(0.039)	− 0.000(0.035)	1.618(0.800)	0.708(0.999)	0.559(0.700)

Table 7. Boot-p simulation results for the approximate ML prediction.

		Bias (MSPE)	Bias (MSE)		AL (CP)		
Case	s	y_s	α	β	y_s	α	β
1	3	0.043(0.006)	0.072(0.047)	0.062(0.023)	0.249(0.880)	0.689(0.914)	0.555(0.914)
	8	0.026(0.024)	0.070(0.046)	0.062(0.022)	0.569(0.924)	0.688(0.915)	0.556(0.914)
	13	− 0.025(0.317)	0.084(0.047)	0.056(0.023)	1.968(0.873)	0.693(0.912)	0.554(0.906)
2	3	0.039(0.003)	0.088(0.056)	0.103(0.033)	0.165(0.810)	0.681(0.889)	0.568(0.843)
	8	0.013(0.008)	0.100(0.059)	0.093(0.028)	0.336(0.918)	0.683(0.894)	0.556(0.862)
	13	− 0.212(0.233)	0.077(0.048)	0.091(0.027)	1.742(0.879)	0.681(0.910)	0.554(0.875)
3	3	0.044(0.004)	0.076(0.056)	0.119(0.044)	0.175(0.796)	0.672(0.900)	0.684(0.869)
	8	0.040(0.008)	0.084(0.053)	0.113(0.044)	0.293(0.893)	0.671(0.895)	0.679(0.851)
	13	0.020(0.014)	0.078(0.050)	0.106(0.042)	0.431(0.915)	0.673(0.903)	0.677(0.874)
4	3	0.034(0.003)	0.073(0.045)	0.075(0.025)	0.170(0.849)	0.686(0.920)	0.535(0.887)
	8	0.014(0.009)	0.078(0.050)	0.076(0.024)	0.363(0.926)	0.686(0.917)	0.536(0.883)
	13	− 0.222(0.227)	0.084(0.051)	0.086(0.028)	1.607(0.856)	0.686(0.904)	0.539(0.870)

the approximate normality. But in some Cases the CPs are lower than the nominal level 95% based on boot-p method.

In according to comparing three methods based on boot-p (Tables 5–7), we found that the results are similar to the comparing Tables 2–4 (asymptotic normality).

5.2. Illustrative examples

In this subsection, two datasets are used to illustrate the proposed prediction and estimation in the preceding sections.

Example 1 (Air conditioning data): Consider the following dataset of failure times of the air conditioning system of an airplane (due to Gupta and Kundu, 2001): 1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62 71, 71, 87, 90, 95, 120, 120, 225, 246, 261. For $m = 20, M = 13$, and $N = 25$, we let $\underline{R} = (0, \ldots, 0, 10)$, and $\underline{S} = (0, \ldots, 0, 12)$, Type-II censored data.

According to the simulation study's results, in the EM algorithm based on normality, the MLP of Y_{10} was given by 24.186 and the MLPEs of α and β were computed as 0.026 and 0.919, respectively.

Also, in the EM algorithm, the 95% AMLPI of Y_{10} and the asymptotic 95% predictive ML intervals of α and β were (9.846, 59.414), (0.000, 0.106) and (0.648, 1.582), respectively. Whereas, based on boot-p method in the EM algorithm, the MLP of Y_{10} was given by 25.107, and the MLPEs of α and β were obtained as 0.019 and 1.010, respectively.

In addition, the 95% approximate boot-p prediction interval of Y_{10} and the approximate boot-p 95% predictive ML intervals of α and β in the EM algorithm were reported as (14.953, 39.034), (0.000, 0.082) and (0.726, 1.641), respectively.

Example 2 (Waiting time data): According to Al-Mutairi et al. (2013), "we present the analysis of real data, partially considered in Ghitany et al. (2008), for illustrative purposes. The data represent the waiting times (in minutes) before customer service in two different banks (A and B)," it is clear that two datasets are independent. We found that based on the first sample's ML estimators of Weibull parameters (0.093, 1.235, respectively) the corresponding Kolmogorov–Smirnov (K-S) p-value sample is 0.978. On the other hand, the corresponding K-S p-value to test whether the Weibull model with 0.093 and 1.235 parameters fitted to the second dataset divided by 1.69 (sample Y) is 0.439. Namely, the first and second samples are independent with the same parameters of Weibull.

We know that $n = 60$ and $N = 100$. Now, we suppose that $m = 30$, $M = 80$. The aim is point and interval prediction of Y_{10} of the second sample based on the first sample's information. Let us $\underline{R} = (30, 0, \ldots, 0)$, and $\underline{S} = (20, 0, \ldots, 0)$, on the other hand we know the real $Y_{10} = 2/1.69$, (Al-Mutairi et al., 2013).

In the EM algorithm method, according to the simulation study based on normality, the MLP of Y_{10} was computed as 1.074 and the MLPEs of α and β were 0.117 and 1.388, respectively. And the 95% AMLPI of Y_{10} and the asymptotic 95% predictive ML intervals of α and β were found to be (0.501, 2.303), (0.030, 0.252), and (1.083, 1.933), respectively.

On the other hand, by boot-p method in the EM algorithm, the MLP of Y_{10} was 1.289 and the MLPEs of α and β were reported as 0.090 and 1.446, respectively. Also, the 95% approximate boot-p prediction interval of Y_{10} and the approximate boot-p 95% predictive ML intervals of α and β in the EM algorithm were given by (0.718, 2.106), (0.025, 0.205), and (1.138, 1.902), respectively.

6. Concluding remarks

In this article, we have obtained the MLP of Y_s, the MLPEs of unknown parameters for the Weibull distribution as well as the AMLPI of Y_s $(1 \leq s \leq M)$ and the asymptotic predictive ML intervals of unknown parameters using the numerical ML prediction, the EM algorithm and the approximate ML prediction. The Monte Carlo simulation results based on approximate normality and percentile boot-p methods in three methods of ML prediction were compared in terms of biases and MSEs (MSPEs) of point predictors as well as ALs and CPs of the prediction intervals of α, β, and Y_s.

Generally, based on the simulated results and for both of the normality and boot-p, the EM algorithm for point and interval prediction of Y_s and the parameters yields better results with respect to biases and ALs, especially under Type-II censoring scheme.

The proposed procedures for the prediction problem may be considered for other censoring schemes such as Type-II progressively hybrid censoring and for some other lifetime distributions such as the generalized exponential distribution (GE).

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