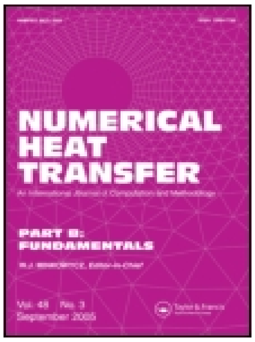

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MODIFICATION OF *k*-ε TURBULENT MODEL USING KINETIC ENERGY–PRESERVING METHOD

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In this article, the kinetic energy–preserving (KEP) scheme and, also, the way of applying this scheme to the* k–ε *turbulent model are introduced. This study aims to introduce a stable method in which a few artificial dissipation terms are added to the governing equations in a way that the intensity of the solution fluctuations is reduced and, therefore, the problem stability increases. In accord with the importance of the study of turbulent flows, the effects of the fluctuating velocity terms on the calculation of all fluxes in the governing equations are scrutinized as well. Also, the influence from applying the KEP scheme on the* k–ε *turbulent model is investigated. This article reveals that by using the KEP scheme and, afterwards, improving the discretization method of the velocity fluctuation terms in* k–ε *equations, the accuracy of the results obtained is enhanced without a need to add artificial dissipation terms (or by minimizing their values).

1. INTRODUCTION

This article pursues the idea, dating back to the early days of scientific computing, of the energy method for stability of numerical analysis. Very often, artificial dissipation terms are added to the equations to capture shock waves and increase stabilities. For example, in the scalar method proposed by Jameson, Smith, and Turkel [1, 2] (JST method), in regions where there is a strong pressure gradient, a considerable amount of artificial dissipation term is added.

If artificial dissipation terms are not completely controlled, damping can occur in all points of the computational domain. In such a situation, some flow properties such as turbulent vortexes may not be captured by the numerical simulation. In order to resolve the problems mentioned, Jameson [3, 4] has recently developed a new scheme in which the total kinetic energy is preserved. By preserving the total kinetic energy, this scheme provides the stability needed for the calculation of the viscous flow even in complicated problems including shock waves. Since no artificial dissipation terms are added to the equations (or their values are negligible), some noticeable properties are expected to be captured in this scheme that do not exist in the other similar schemes [3].

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NOMENCLATURE

A_{op}	area of the surface flux, m^2
C_D	drag coefficient
C_L	lift coefficient
C_p	pressure coefficient
C_p	specific heat capacity, J/kg. k
dS	surface element, m^2
dV	volume element, m^3
E	total energy, J/kg
f	flux vector
H	total enthalpy, J/kg
k	specific kinetic energy, m^2/s^2
k	fluctuating kinetic energy, m^2/s^2
k	thermal conductivity, W/m. k
M	Mach number
n^j	normal surface vector, m^2
p	pressure, Pa
Pr	Prandtl number
q	heat flux, w
Re	Reynolds number
S_{op}	surface between o and p, m^2
t	time, s
T	temperature, K
u	velocity, m/s
u, v, w	velocity components in x, y, z directions, respectively, m/s
vol	element volume, m^3
w	state vector
x	coordinate direction, m
δ	Kroneker delta
ε	dissipation rate, m^2/s^3
λ	dynamic viscosity, Pa· s
μ	dynamic viscosity, Pa· s
υ	kinematic viscosity, m^2/s
ρ	density, kg/m^3
σ	viscous stress ($kg/m.s^2$)

Subscripts

b	boundary node
m^k, m^j, m^i	previous node in i, j, and k directions
o	mean node
p	neighbor node
p^k, p^j, p^i	next node in i, j, and k directions

Superscripts

i, j, k	counting index (indicating the coordinate direction)
n	number of time step
–	mean value
′	fluctuating term

Previous works by the authors show that the kinetic energy–preserving (KEP) method leads to the same results as the other methods until the number of grids is low [5]. The challenge is to improve the KEP method to gain better results at low points of the grid. To do this, some corrections have to be made in the KEP method. After making such corrections, some inaccuracies may occur in the kinetic energy preservation. However, due to the improvement of the results for a low number of the grids, such inaccuracies can be neglected. Boundary conditions are of great importance in this method. Wang Li et al. [6] have studied a second-order additional source term method for handling boundary conditions. Corrected boundary conditions must be considered in this scheme.

There are several research reports about this method. The KEP scheme has been studied in one-dimensional viscous flow in a shock tube by Jameson [7], and in two-dimensional viscous flow in a shock tube and also two-dimensional viscous flow around a plunging airfoil by Allaneu and Jameson [8, 9]. Lehmkuhl et al. [10] used the KEP method for the flow around wind turbine blades. Trias et al. [11] presented a fully -conservative discretization of the Navier-Stokes equations for unstructured meshes. This method was tested for a buoyancy-driven turbulent flow. Baez Vidal et al. [12] compared the KEP and Godunov schemes on the flow around a NACA 0012 airfoil. Herbin and Latche [13] presented a kinetic energy–preserving operator for the MAC discretization of compressible Navier-Stokes equations. Edoh and Karagozian [14] applied KEP discretization schemes for high-Reynolds-number propulsive application. Moreover, Chandrashekar [15, 16], Kok [17], and Yan and Jin [18] scrutinized this method in their articles.

Also, Javadi and Pasandideh-Fard [5] have applied the KEP method in a one-dimensional inviscid convergent-divergent nuzzle flow, two-dimensional inviscid flow over a bump, and two-dimensional viscous flow over a NACA 0012 airfoil. It was shown that the KEP scheme is more accurate if the number of mesh points is increased; and, in contrast to other schemes, there is no limit in increasing the grid points.

Studying the kinetic energy of the flow is of great importance, especially in study of the turbulence existing in the flow. Most flows are naturally turbulent. Therefore, the analysis of the effect of the turbulence is very important. There are several methods to study the effects of the turbulence. For example, the RANS, LES, and DNS methods can be noted. Direct numerical simulation (DNS) of the turbulence is the most correct approach to solve turbulent flows. Large-eddy simulation (LES) is similar to DNS, because both offer three-dimensional and time-dependent solution for the Navier-Stokes equations [19]. So they both need a fine enough grid. At high Reynolds numbers, the DNS method can be replaced by the LES method. Although LES has been computationally developed more than DNS and therefore is a good alternative to DNS at higher Reynolds number, its application on engineering flows remains expensive, unless using wall functions. According to the existing problems in these methods, it is useful to apply Reynolds-averaged Navier-Stokes (RANS) models to reduce the computational costs [20].

Simple turbulent models (such as zero-equation models to two-equation models) create almost accurate results for simple flows in which only one component of the Reynolds stresses in the momentum equations is important [21]. One of the useful RANS models is the k–ε turbulent model. Several cases have been modeled by this turbulent model. For example, Maoa and Zhanga [22], Pulat et al. [23], and Choi et al. [24] used this model to study the behavior of the kinetic energy of a fluid.

The KEP method is a new scheme which has been less studied so far. Moreover, utilization of this method in different types of flows has not been performed yet. Jameson has derived the flux equations only for the main velocity u. The separation of the main velocity into averaged and fluctuated velocity components creates a new formula for calculating the amount of $\bar{u}_i\acute{u}_j$ or $\overline{\acute{u}_i\acute{u}_j}$. The influence of implementing these relations on different models and methods causes the calculation of the fluctuating terms to change. In fact, the most important innovations of this article are to implement the relations of the fluctuating terms and, also, to make changes in calculation of $\overline{\acute{u}_i\acute{u}_j}$ in the k–ε model.

In the following, the KEP method and its application on the k–ε turbulent model are introduced.

2. INTRODUCTION OF THE KINETIC ENERGY–PRESERVING (KEP) METHOD

This section introduces the kinetic energy-preserving scheme. It should be noted that the equations given in Section 2.1 are derived from [25]. Also, the relations presented in Sections 2.2 and 2.3 are obtained for the first time by the authors. The mathematical details of these two sets of equations are completely given in Appendixes A and B.

2.1. The KEP Method and Its Implementation in the Navier-Stokes Equations (to Calculate the Existing Flux)

The governing equations, including the Navier-Stokes (N-S) and energy equations, are as follows:

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x^i} f^i(w) = 0 \tag{1}$$

where

$$w = \begin{bmatrix} \rho \\ \rho u^1 \\ \rho u^2 \\ \rho u^3 \\ \rho E \end{bmatrix} \quad f^i = \begin{bmatrix} \rho v^i \\ \rho u^i u^1 - \sigma^{i1} + p\delta^{i1} \\ \rho u^i u^2 - \sigma^{i2} + p\delta^{i2} \\ \rho u^i u^3 - \sigma^{i3} + p\delta^{i3} \\ \rho u^i H - u^j \sigma^{ij} - q^j \end{bmatrix} \tag{2}$$

Also, the viscous stress tensor and the heat flux are obtained from the relations (3) and (4):

$$\sigma^{ij} = \mu\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) + \lambda\delta^{ij}\frac{\partial u^k}{\partial x^k} \tag{3}$$

$$q^j = -k\frac{\partial T}{\partial x^j} \tag{4}$$

In the above equations, μ and λ are the viscosity coefficients, in which λ is usually defined as $\lambda = -\frac{2}{3}\mu$. δ^{ij} is the Kronecker delta and k is the thermal conductivity. The k variable is defined as the kinetic energy, as can be seen in Eq. (5):

$$k = \rho\frac{u^{i^2}}{2} \quad \frac{\partial k}{\partial w} = \left[-\frac{u^{i^2}}{2}, u^1, u^2, u^3, 0\right] \tag{5}$$

By derivation of k with respect to time, Eq. (6) is obtained:

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^{i^2}\right) = u^i\frac{\partial}{\partial t}\left(\rho u^i\right) - \frac{u^{i^2}}{2}\frac{\partial \rho}{\partial t} \tag{6}$$

By using the equation of state and the flux vectors, if the spatial derivatives are replaced with time derivatives, Eq. (7) is obtained:

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x^j}\left[u^j\left(p + \rho\frac{v^{i^2}}{2}\right) - u^i\sigma^{ij}\right] = p\frac{\partial v^j}{\partial x^j} - \sigma^{ij}\frac{\partial u^i}{\partial x^j} \tag{7}$$

By integrating the above equation, Eq. (8) is obtained:

$$\frac{\partial}{\partial t}\int_{\Omega} k\, dV = -\int_{\partial\Omega}\left[u^j\left(p + \rho\frac{v^{i^2}}{2}\right) - u^i\sigma^{ij}\right]n^j\, dS + \int_{\Omega}\left(p\frac{\partial u^j}{\partial x^j} - \sigma^{ij}\frac{\partial u^i}{\partial x^j}\right) dV \tag{8}$$

where n^j is the normal surface vector and dS and dV are the surface and volume elements, respectively. Equation (8) shows the total kinetic energy–preserving equation for three-dimensional viscous flow.

Now the equations are discretized by the finite-volume method. For any o node, the neighbor nodes are specified by p. Also, the surface between these two nodes is defined as A_{op}. n^i_{op} is representative of the normal surface vector, in which i denotes the direction of this vector. The following equations can be written for a control volume:

$$n^i_{op} = -n^i_{po} \tag{9}$$

$$S^i_{op} = A_{op} n^i_{op} \tag{10}$$

$$S^i_o = -\sum_p S^i_{op} \tag{11}$$

The conservation equation comes in the form below:

$$\mathrm{vol}_o \frac{\partial w_o}{\partial t} + \sum\nolimits_{\mathrm{neighbor}} f^i_{op}.n^i_{op} A_{op} = 0 \tag{12}$$

or

$$\mathrm{vol}_o \frac{\partial w_o}{\partial t} + \sum_{\mathrm{neighbor}} f^i_{op}.S^i_{op} = 0 \tag{13}$$

where the state vector and the flux vector can be defined as

$$w_o = \begin{bmatrix} \rho_o \\ \rho_o u^1_o \\ \rho_o u^2_o \\ \rho_o u^3_o \\ \rho_o E_o \end{bmatrix} \quad f^i_{op} = \begin{bmatrix} (\rho v^i)_o \\ (\rho u^i u^1)_{op} + (p\delta^{i1} - \sigma^{i1})_{op} \\ (\rho u^i u^2)_{op} + (p\delta^{i2} - \sigma^{i2})_{op} \\ (\rho u^i u^3)_{op} + (p\delta^{i3} - \sigma^{i3})_{op} \\ (\rho u^i H)_{op} + (u^j \sigma^{ij} + q^j)_{op} \end{bmatrix} \tag{14}$$

It can be shown that if the relations (15) and (16) are true,

$$(\rho u^i u^j)_{op} = \frac{1}{2} (\rho u^i)_{op} \left(u^j_p + u^j_o \right) \tag{15}$$

$$(p\delta^{ij} - \sigma^{ij})_{op} = \frac{1}{2} (p\delta^{ij} - \sigma^{ij})_o + \frac{1}{2} (p\delta^{ij} - \sigma^{ij})_p \tag{16}$$

Equation (13) comes in the form

$$\begin{aligned} \frac{d}{dt} \sum\nolimits_o \mathrm{vol}_o\, k_o = & -\sum\nolimits_b S^j_b \left(u^j_b \left(p_b + \rho_b \frac{u^{i2}_b}{2} \right) - u^i_b \sigma^{ij}_b \right) \\ & + \sum\nolimits_o \left(p_o \sum\nolimits_p \frac{u^i_o + u^i_p}{2} S^i_{op} - \sigma^{ij}_o \sum_p \frac{u^i_o + u^i_p}{2} S^i_{op} \right) \end{aligned} \tag{17}$$

That is the discrete form of Eq. (8), where b represents the boundary points. For a variable q, the averaged value is defined as

$$\bar{q}_{op} = \frac{q_o + q_p}{2} \tag{18}$$

So

$$\left(\rho u^i u^j\right)_{op} = \left(\rho u^i\right)_{op} \bar{u}^j_{op} \tag{19}$$

Different types of calculation of the flux can be determined to ensure the preservation of total kinetic energy. For example,

$$\left(\rho u^i\right)_{op} = \bar{\rho}_{op} \bar{u}^i_{op} \tag{20}$$

$$\left(\rho u^i\right)_{op} = \overline{\rho u^i}_{op} \tag{21}$$

Similarly,

$$\left(\rho u^i u^j\right)_{op} = \bar{\rho}_{op} \bar{u}^i_{op} \bar{u}^j_{op} \tag{22}$$

$$\left(\rho u^i H\right)_{op} = \bar{\rho}_{op} \bar{u}^i_{op} \bar{H}_{op} \tag{23}$$

2.2. Applying the KEP Method on Fluxes Including Velocity Fluctuations

It can be shown that if the following relations are true for the fluctuating velocity terms, the total kinetic energy in the computational domain is preserved. These relations are fully proved in Appendix A.

$$\left(\rho \bar{u}^i \acute{u}^j\right)_{op} = \frac{1}{2} \left(\rho \bar{u}^i\right)_{op} \left(\acute{u}^j_o + \acute{u}^j_p\right) \tag{24}$$

$$\left(\rho \acute{u}^i \bar{u}^j\right)_{op} = \frac{1}{2} \left(\rho \acute{u}^i\right)_{op} \left(\bar{u}^j_o + \bar{u}^j_p\right) \tag{25}$$

$$\left(\rho \acute{u}^i \acute{u}^j\right)_{op} = \frac{1}{2} \left(\rho \acute{u}^i\right)_{op} \left(\acute{u}^j_o + \acute{u}^j_p\right) \tag{26}$$

$$\left(\rho \bar{u}^i \bar{u}^j\right)_{op} = \frac{1}{2} \left(\rho \bar{u}^i\right)_{op} \left(\bar{u}^j_o + \bar{u}^j_p\right) \tag{27}$$

By separating the average and fluctuating components of the velocity, Eqs. (28) and (29) are obtained to calculate the viscous fluxes of the momentum equation:

$$\left(\mu \frac{\partial \bar{u}^i}{\partial x^j}\right)_{op} = \frac{1}{2} \left(\mu \frac{\partial \bar{u}^i}{\partial x^j}\right)_o + \frac{1}{2} \left(\mu \frac{\partial \bar{u}^i}{\partial x^j}\right)_p \tag{28}$$

$$\left(\mu \frac{\partial \acute{u}^i}{\partial x^j}\right)_{op} = \frac{1}{2} \left(\mu \frac{\partial \acute{u}^i}{\partial x^j}\right)_o + \frac{1}{2} \left(\mu \frac{\partial \acute{u}^i}{\partial x^j}\right)_p \tag{29}$$

2.3. Applying the KEP Method on the Fluxes of the *k* and ε Equations

The k and ε equation are defined by Eqs. (30) and (31):

$$\rho\left(\frac{\partial k}{\partial t}+\frac{\partial k}{\partial x^j}\bar{u}^j\right)=-\frac{\partial}{\partial x^j}\left(\overline{\acute{p}\acute{u}^j}+\frac{\rho}{2}\overline{\left(\acute{u}^i\right)^2\acute{u}^j}\right)+\mu\frac{\partial}{\partial x^j}\left(\frac{\partial k}{\partial x^j}\right)+\left(-\rho\overline{\acute{u}^i\acute{u}^j}\frac{\partial\bar{u}^i}{\partial x^j}\right)-\rho\varepsilon \quad (30)$$

$$\begin{aligned}\frac{\partial}{\partial t}(\rho\varepsilon)+\rho\bar{u}^j\frac{\partial}{\partial x^j}(\varepsilon)-\mu\frac{\partial}{\partial x^j}\left(\frac{\partial\varepsilon}{\partial x^j}\right)+\rho\frac{\varepsilon^2}{k}=\overline{-2\nu\frac{\partial\acute{u}^i}{\partial x^k}\frac{\partial^2\acute{p}}{\partial x^k\partial x^i}}-\overline{\acute{u}^j\frac{\partial}{\partial x^j}\left[\mu\left(\frac{\partial\acute{u}^i}{\partial x^k}\right)^2\right]}\\ -2\mu\frac{\partial\bar{u}^i}{\partial x^j}\overline{\frac{\partial\acute{u}^i}{\partial x^k}\frac{\partial\acute{u}^j}{\partial x^k}}-2\mu\frac{\partial\bar{u}^j}{\partial x^k}\overline{\frac{\partial\acute{u}^i}{\partial x^k}\frac{\partial\acute{u}^i}{\partial x^j}}-2\mu\frac{\partial^2\bar{u}^i}{\partial x^k\partial x^j}\overline{\acute{u}^j\frac{\partial\acute{u}^i}{\partial x^k}}-2\mu\overline{\frac{\partial\acute{u}^i}{\partial x^k}\frac{\partial\acute{u}^j}{\partial x^k}\frac{\partial\acute{u}^i}{\partial x^j}}\end{aligned} \quad (31)$$

The first and the third terms of the right-hand side of Eq. (30) and all of the terms on the right-hand side of Eq. (31) are obtained by modeling.

Next, Eqs. (32)–(35) illustrate how to implement KEP method in the k–ε equation in order to calculate the flux. It is worth noting that, as can be seen in the following relations, the average value of the flux is not calculated. In other words, the significance of the main points in obtaining the results is more than the neighbor points. These relations are proved in Appendix B.

$$\left(\rho u^i\mathrm{k}\right)_{op}=\frac{1}{4}\left(\rho u^i\right)_{op}\left(3k_o+k_p\right) \quad (32)$$

$$\left(\mu\frac{\partial k}{\partial x^j}\right)_{op}=\frac{3}{4}\left(\mu\frac{\partial k}{\partial x^j}\right)_o+\frac{1}{4}\left(\mu\frac{\partial k}{\partial x^j}\right)_p \quad (33)$$

$$\left(\rho u^i\varepsilon\right)_{op}=\frac{1}{4}\left(\rho u^i\right)_{op}\left(3\varepsilon_o+\varepsilon_p\right) \quad (34)$$

$$\left(\mu\frac{\partial\varepsilon}{\partial x^j}\right)_{op}=\frac{3}{4}\left(\mu\frac{\partial\varepsilon}{\partial x^j}\right)_o+\frac{1}{4}\left(\mu\frac{\partial\varepsilon}{\partial x^j}\right)_p \quad (35)$$

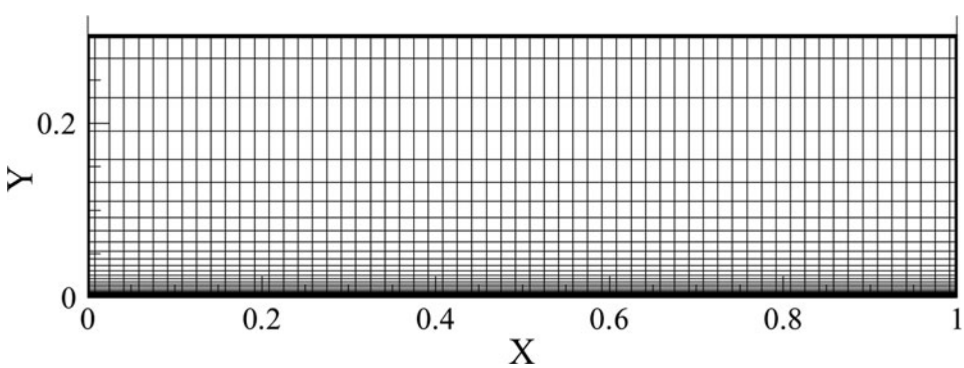


Figure 1. Geometry and mesh generation of the flat plate.

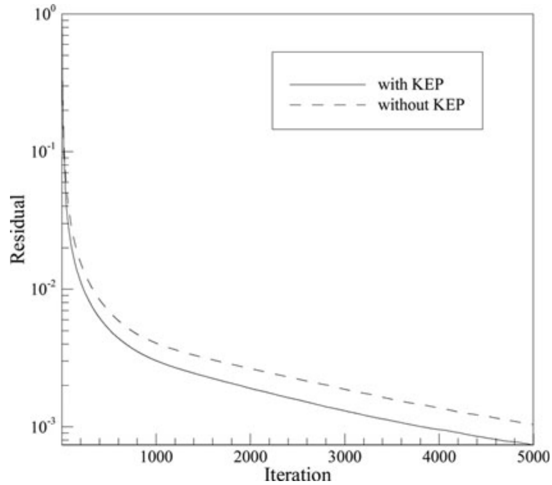


Figure 2. Convergence history, $Re = 2 \times 10^6$.

3. RESULTS AND DISCUSSION

In the following, two-dimensional viscous flow over a flat plate is first investigated, and then the results of the high-Reynolds transonic NACA 0012 airfoil achieved by using this method are presented.

3.1. Two-Dimensional Viscous Flow over a Flat Plate

In this section, the results obtained from applying the KEP method in the k–ε turbulent model are presented and compared with k–ε turbulent model results when the KEP method is not added to them.

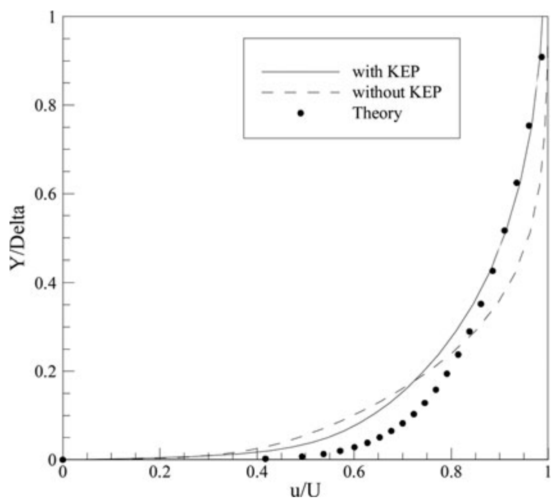


Figure 3. Velocity profile, $X = 0.8$, $Re = 2 \times 10^6$.

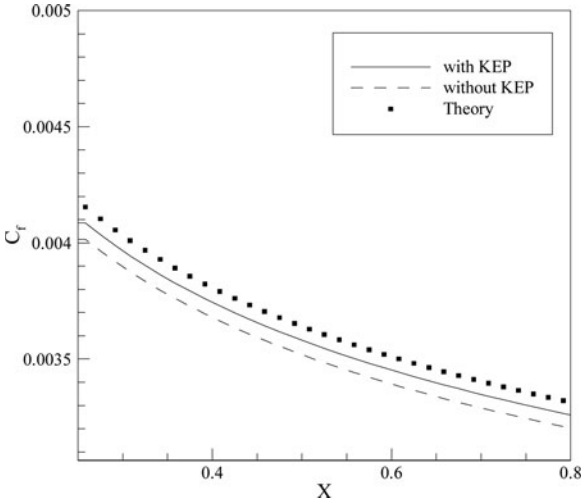


Figure 4. Distribution of friction coefficient, $\mathrm{Re} = 2 \times 10^6$.

The geometry considered in this study is a flat plate. In this section, the influence of the governing parameters on the results is investigated for the plate cross section at $x = 0.8$. All of the parameters are dimensionless. The Mach number and Reynolds number of the air flow are assumed to be 0.6 and 2×10^6, respectively. Moreover, the specific heat capacity is $C_P = 1{,}004\ \mathrm{J/kg.\,k}$, and the Prandtl number is $\mathrm{Pr} = 0.72$.

A schematic of the problem is shown in Figure 1 along with the mesh generation. At the inlet, the amounts of all parameters are assumed to be equal with those of the free stream. Values of velocity, density, k, and ε in the exit cross section are considered to be the same as their values at the points next to them in the domain,

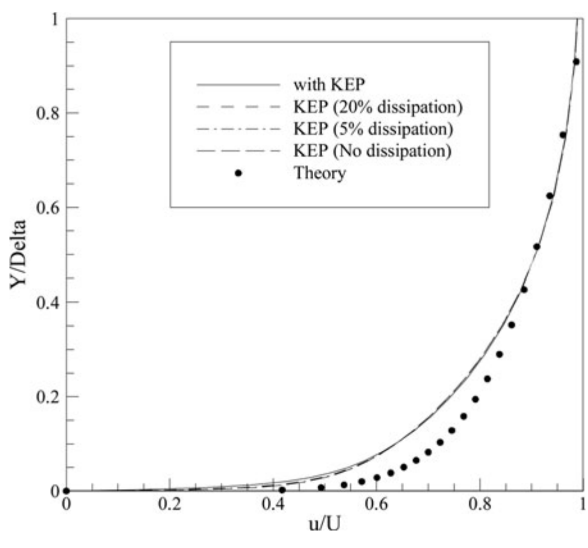


Figure 5. Velocity profile for KEP method with different artificial dissipation, $X = 0.8$, $\mathrm{Re} = 2 \times 10^6$.

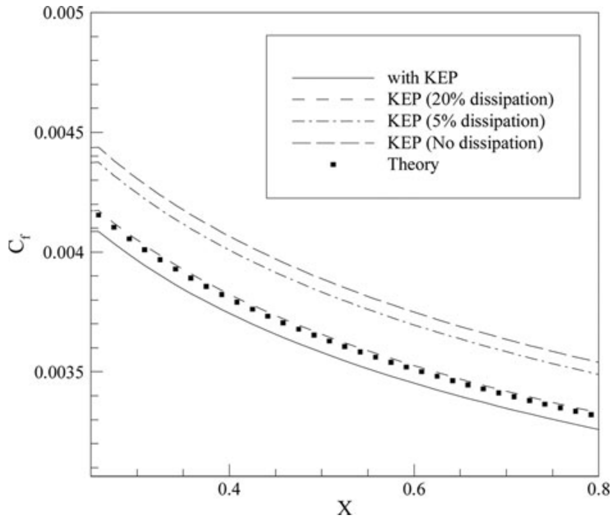


Figure 6. Distribution of friction coefficient for KEP method with different artificial dissipation, $Re = 2 \times 10^6$.

and pressure values in the exit cross section are equal to the pressure of the free stream. For the points at the bottom, values of density and pressure are set to be equal to their upper points. Velocity, k, and ε over the flat plate are considered to be zero. The generated mesh to study the results is (60×40) points. The grid generation is uniform in the x direction, but an expansion coefficient of 1.2 is applied in the y direction.

Figure 2 shows the convergence history for both methods. The convergence process occurs faster for the KEP method. Figure 3 shows the velocity profile in cross section $x = 0.8$. As can be seen, the results achieved by using the two methods

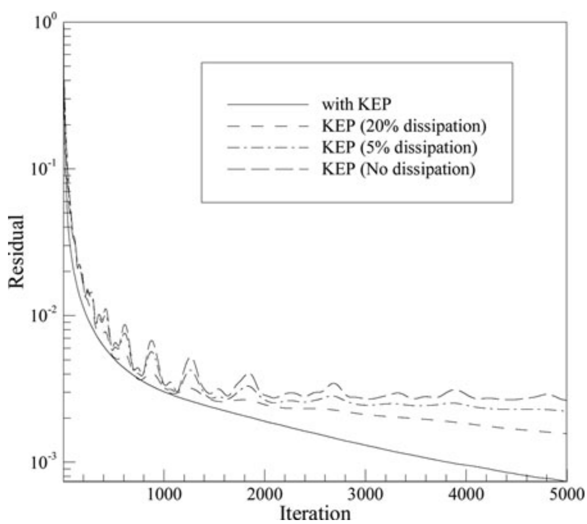


Figure 7. Convergence history for KEP method with different artificial dissipation, $Re = 2 \times 10^6$.

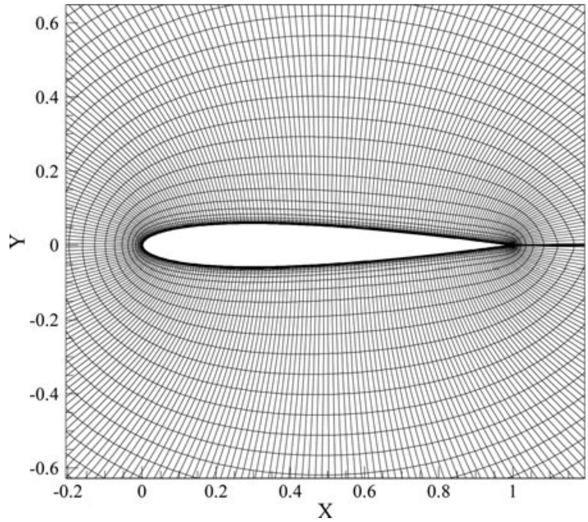


Figure 8. Mesh generation for NACA 0012 airfoil.

are close together, however, a slight improvement in the results obtained using the KEP can be observed. Figure 4 shows the values of local friction coefficient. By focusing on this figure, it can be noticed that with applying the KEP method in the *k*–ε model, the results are improved.

Using the KEP method in the *k*–ε model has another significant asset, which is expressed in this section. As previously mentioned, a few artificial dissipation terms are added to the governing equations without making the numerical solution diverge or reducing the accuracy of the results. This section presents the results of applying the KEP method, while in this case a little artificial dissipation term is added to

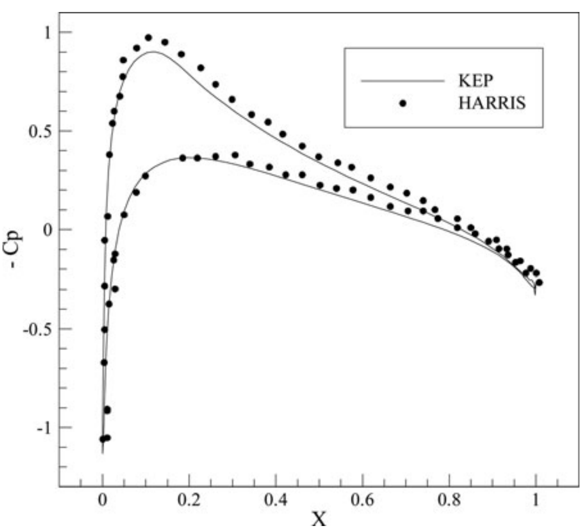


Figure 9. Coefficient pressure versus X ($M = 0.7$, $\alpha = 1.49$).

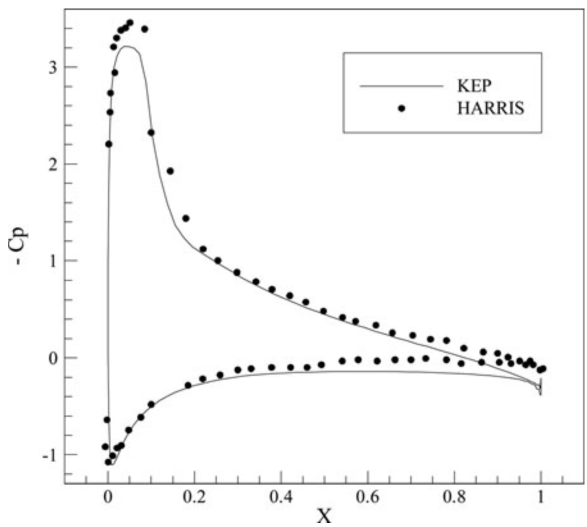


Figure 10. Coefficient pressure versus X ($M = 0.55$, $\alpha = 8.34$).

the equation. The adding procedure of the dissipation terms is accomplished in three steps. In the first step, the dissipation term, which is 20% of that of the SCDS method [1], is added to the governing equations. Next, 5% of the dissipation term of the SCDS method is added and, finally, the KEP method is used without any artificial dissipation term. Figures 5–7, illustrate the results achieved. As captured in these figures, it is possible to make the simulation converge, without producing large fluctuations, by adding only a few artificial dissipation terms. In such a phenomenon, the accuracy of the results is not reduced. On the other hand, without using the KEP method, when the artificial dissipation terms are eliminated or their values decrease

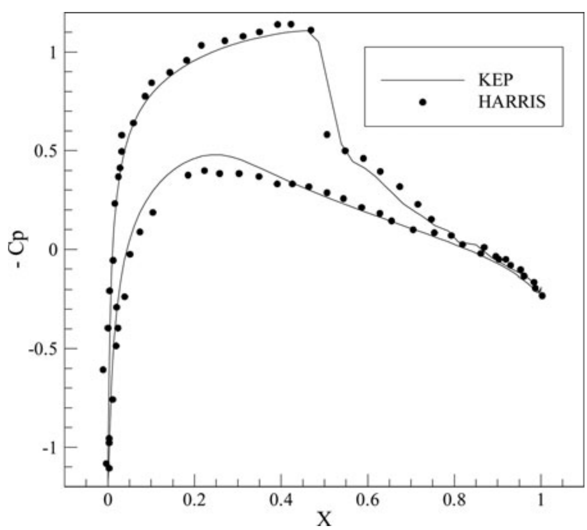
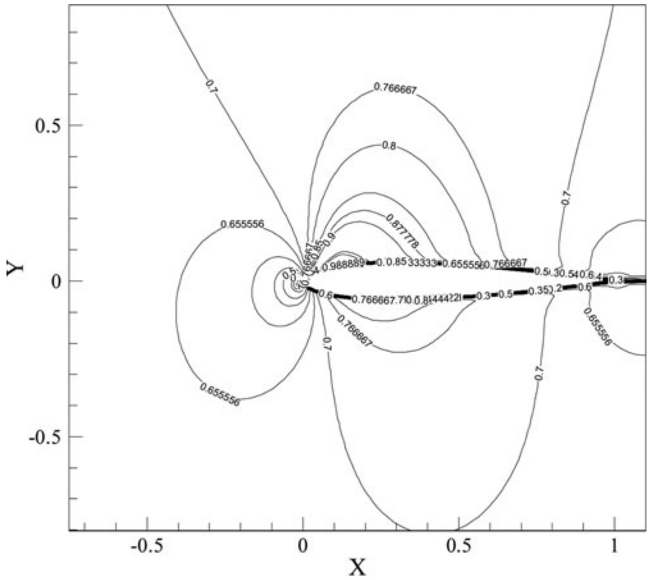
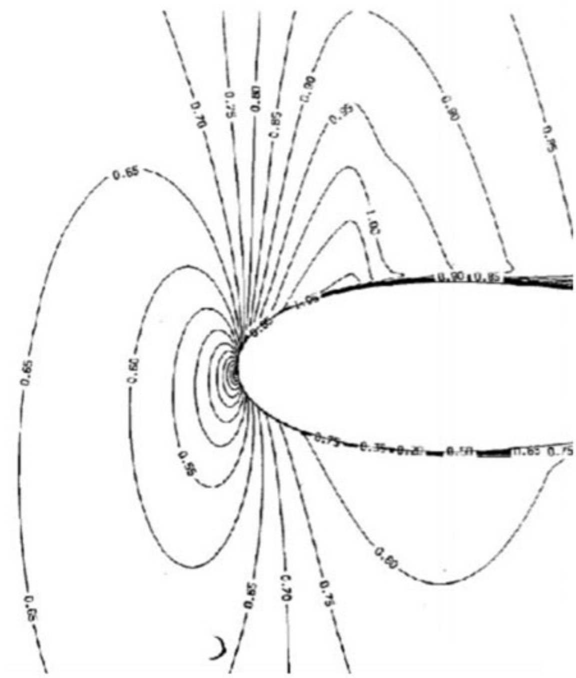


Figure 11. Coefficient pressure versus X ($M = 0.799$, $\alpha = 2.26$).



(*a*) KEP results



(*b*) Maksymiuk results

Figure 12. Mach contours ($M = 0.7$, $\alpha = 1.49$).

to 5% or 20% of those of the SCDS method, the numerical simulation diverge. Consequently, the results presented are related only to the situation when the KEP method is used.

3.2. Low-Dissipation, High-Reynolds Transonic NACA 0012 Airfoil

The second case is a NACA 0012 airfoil. A series of general C meshes generated with a hyperbolic solver are used for the Row field computations. The total number of grid points is (298 × 99). Figure 8 illustrates a view of the mesh which is used for the NACA 0012 cases.

For all these cases, the Reynolds number is 9 million. The KEP method with low dissipation (20% of the artificial dissipation of the SCDS method) is considered and also compared with the experimental data of Harris [26] and computational data of Maksymiuk [27].

First, computations were carried out for the NACA 0012 airfoil at three specified conditions. Mach numbers, angles of attack, and Reynolds numbers are specified, and the variations of C_p with respect to dimensionless displacement X are plotted. The KEP results are compared with the experimental data of Harris [26] in Figures 9–11. The results agree very well with experiment. Mach contours of the first case (M = 0.7, α = 1.49) are shown in Figure 12.

In Figures 13 and 14, the variation of lift coefficient with angle of attack and also with drag coefficient at a free-stream Mach number of 0.7 is illustrated. The results which were computed agree well with experimental data at geometric angles of attack up to about 5.0°. The drag coefficient (Figure 14) is also calculated reasonably well in comparison to experimental data.

Figure 15 shows the drag coefficient distribution with respect to Mach number. Results are compared with Maksymiuk's results [27]. However, the KEP results are in agreement with the computational data of Maksymiuk.

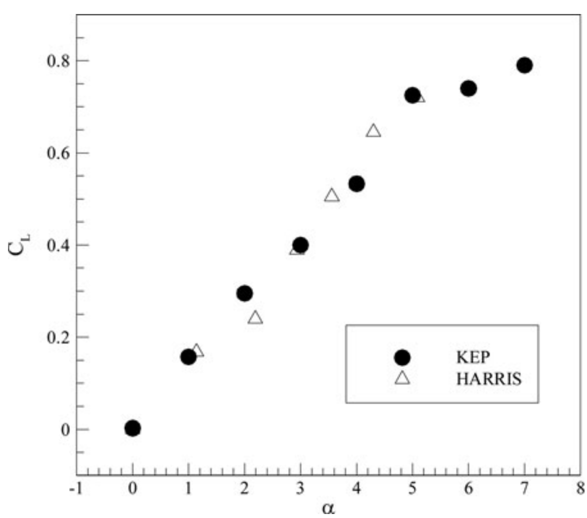


Figure 13. Lift coefficient versus angle of attack ($M = 0.7$).

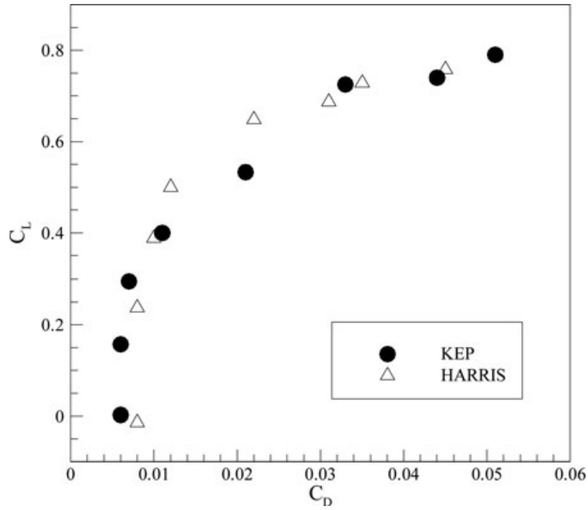


Figure 14. Lift coefficient versus drag coefficient ($M = 0.7$).

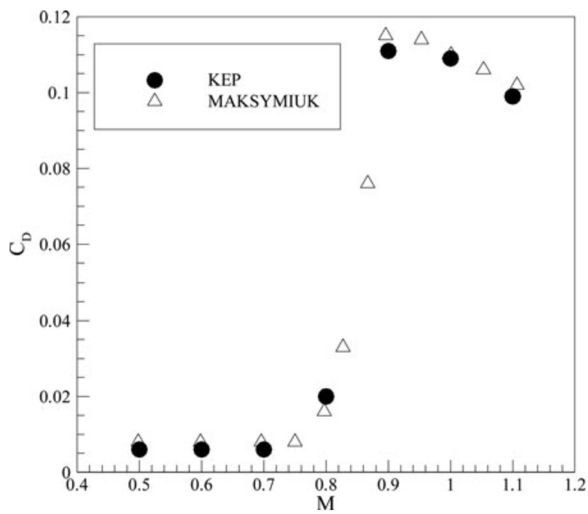


Figure 15. Drag coefficient versus Mach number ($\alpha = 0.0$).

4. CONCLUSION

In this article, the kinetic energy–preserving scheme, and the way of applying this scheme to the k–ε turbulent model, were introduced. The results obtained showed that by implementing the KEP method, the accuracy of the results is improved. Furthermore, by using this scheme, it is possible to attain a converged solution and, also, accurate results without a need to add any artificial dissipation terms (or by minimizing their values). Obtaining an accurate solution is also possible when only a few dissipation terms are added. If the KEP method is not implemented or the term of the artificial dissipation is eliminated or reduced, the numerical simulation diverges.

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APPENDIX A. EQUATIONS INCLUDING FLUCTUATING TERMS IN THE KEP METHOD FOR THREE-DIMENSIONAL VISCOUS FLOW

Necessary conditions for calculating the flux, in the KEP method for three-dimensional viscous flows, are expressed in Eqs. (15) and (16). The following equations can be written to calculate the flux:

$$(\rho uu)_{op} = \frac{1}{2}(\rho u)_{op}(u_o + u_p) \tag{36-A}$$

$$(\rho uv)_{op} = \frac{1}{2}(\rho u)_{op}(v_o + v_p) \tag{36-B}$$

$$(\rho uw)_{op} = \frac{1}{2}(\rho u)_{op}(w_o + w_p) \tag{36-C}$$

$$(\rho vu)_{op} = \frac{1}{2}(\rho v)_{op}(u_o + u_p) \tag{36-D}$$

$$(\rho vv)_{op} = \frac{1}{2}(\rho v)_{op}(v_o + v_p) \tag{36-E}$$

$$(\rho vw)_{op} = \frac{1}{2}(\rho v)_{op}(w_o + w_p) \tag{36-F}$$

$$(\rho wu)_{op} = \frac{1}{2}(\rho w)_{op}(u_o + u_p) \tag{36-G}$$

$$(\rho wv)_{op} = \frac{1}{2}(\rho w)_{op}(v_o + v_p) \tag{36-H}$$

$$(\rho ww)_{op} = \frac{1}{2}(\rho w)_{op}(w_o + w_p) \tag{36-I}$$

For example, Eq. (36-B) is checked. For other equations we can act similarly. By replacing mean and fluctuating velocity terms in Eq. (36-B), the following expression is obtained:

$$\begin{aligned}
&\frac{1}{2}\left[(\rho\bar{u})_{op}\bar{v}_p + (\rho\bar{u})_{op}\acute{v}_p + (\rho\bar{u})_{op}\bar{v}_o + (\rho\bar{u})_{op}\acute{v}_o\right] \\
&+\frac{1}{2}\left[(\rho\acute{u})_{op}\bar{v}_p + (\rho\acute{u})_{op}\acute{v}_p + (\rho\acute{u})_{op}\bar{v}_o + (\rho\acute{u})_{op}\acute{v}_o\right] \\
&-\left[(\rho\bar{u}\bar{v})_{op} + (\rho\acute{u}\acute{v})_{op} + (\rho\bar{u}\acute{v})_{op} + (\rho\acute{u}\bar{v})_{op}\right] = 0
\end{aligned} \tag{37}$$

Using the time average from Eq. (37),

$$\begin{aligned}
&\frac{1}{2}\left[(\rho\bar{u})_{op}\bar{v}_p + 0 + (\rho\bar{u})_{op}\bar{v}_o + 0\right] + \frac{1}{2}\left[0 + \overline{(\rho\acute{u})_{op}\acute{v}_p} + 0 + \overline{(\rho\acute{u})_{op}\acute{v}_o}\right] \\
&-\left[(\rho\bar{u}\bar{v})_{op} + \overline{(\rho\acute{u}\acute{v})_{op}} + 0 + 0\right] = 0
\end{aligned} \tag{38}$$

Equation (38) is subtracted from Eq. (37), and the result is:

$$\begin{aligned}
&\frac{1}{2}\left[\underbrace{(\rho\bar{u})_{op}\acute{v}_p}_{1} + \underbrace{(\rho\bar{u})_{op}\acute{v}_o}_{2} + \underbrace{(\rho\acute{u})_{op}\acute{v}_p}_{3} - \underbrace{\overline{(\rho\acute{u})_{op}\acute{v}_p}}_{4}\right] \\
&+\frac{1}{2}\left[\underbrace{(\rho\acute{u})_{op}\acute{v}_o}_{5} - \underbrace{\overline{(\rho\acute{u})_{op}\acute{v}_o}}_{6} + \underbrace{(\rho\acute{u})_{op}\bar{v}_p}_{7} + \underbrace{(\rho\acute{u})_{op}\bar{v}_o}_{8}\right] \\
&-\left[\underbrace{(\rho\acute{u}\acute{v})_{op}}_{9} - \underbrace{\overline{(\rho\acute{u}\acute{v})_{op}}}_{10} + \underbrace{(\rho\bar{u}\acute{v})_{op}}_{11} + \underbrace{(\rho\acute{u}\bar{v})_{op}}_{12}\right] = 0
\end{aligned} \tag{39}$$

There are different forms satisfying the above equation for the calculation of the fluctuating flux. For example, if the total value of sentences 3, 5, and 9 is equal to zero, the total amount of the average of these sentences, i.e., 4, 6, and 10, will be zero as well. As a result, in order for Eq. (39) to be zero, the summation of the remaining sentences must be zero. For instance, it is possible to equalize the summation of terms 1 and 2 with sentence 11 and, also, the summation of sentences 7 and 8 with term 12. After implementing these assumptions, Eqs. (40)–(42) are obtained:

$$(\rho\bar{u}\acute{v})_{op} = \frac{1}{2}(\rho\bar{u})_{op}(\acute{v}_o + \acute{v}_p) \tag{40}$$

$$\left(\rho\acute{u}\bar{v}\right)_{op}=\frac{1}{2}\left(\rho\acute{u}\right)_{op}\left(\bar{v}_o+\bar{v}_p\right) \tag{41}$$

$$\left(\rho\acute{u}\acute{v}\right)_{op}=\frac{1}{2}\left(\rho\acute{u}\right)_{op}\left(\acute{v}_o+\acute{v}_p\right) \tag{42}$$

After replacing Eqs. (40)–(42) in Eq. (37), the following equation is obtained:

$$\left(\rho\bar{u}\bar{v}\right)_{op}=\frac{1}{2}\left(\rho\bar{u}\right)_{op}\left(\bar{v}_o+\bar{v}_p\right) \tag{43}$$

In general, we can write:

$$\left(\rho\bar{u}^i\acute{u}^j\right)_{op}=\frac{1}{2}\left(\rho\bar{u}^i\right)_{op}\left(\acute{u}^j_o+\acute{u}^j_p\right) \tag{44}$$

$$\left(\rho\acute{u}^i\bar{u}^j\right)_{op}=\frac{1}{2}\left(\rho\acute{u}^i\right)_{op}\left(\bar{u}^j_o+\bar{u}^j_p\right) \tag{45}$$

$$\left(\rho\acute{u}^i\acute{u}^j\right)_{op}=\frac{1}{2}\left(\rho\acute{u}^i\right)_{op}\left(\acute{u}^j_o+\acute{u}^j_p\right) \tag{46}$$

$$\left(\rho\bar{u}^i\bar{u}^j\right)_{op}=\frac{1}{2}\left(\rho\bar{u}^i\right)_{op}\left(\bar{u}^j_o+\bar{u}^j_p\right) \tag{47}$$

In the above equations, i and j vary from 1 to 3 and represent the x, y, and z directions, respectively. Equations (44)–(47) involve fluctuating terms in the KEP method for three-dimensional viscous flow. Equation (16) can be deduced:

$$\left(\mu\frac{\partial u^i}{\partial x^j}\right)_{op}=\frac{1}{2}\left(\mu\frac{\partial u^i}{\partial x^j}\right)_o+\frac{1}{2}\left(\mu\frac{\partial u^i}{\partial x^j}\right)_p \tag{48}$$

By separating the mean term and velocity fluctuations, similar to the process described above, the following equations are attained to calculate the viscous flux of the momentum equation:

$$\left(\mu\frac{\partial \bar{u}^i}{\partial x^j}\right)_{op}=\frac{1}{2}\left(\mu\frac{\partial \bar{u}^i}{\partial x^j}\right)_o+\frac{1}{2}\left(\mu\frac{\partial \bar{u}^i}{\partial x^j}\right)_p \tag{49}$$

$$\left(\mu\frac{\partial \acute{u}^i}{\partial x^j}\right)_{op}=\frac{1}{2}\left(\mu\frac{\partial \acute{u}^i}{\partial x^j}\right)_o+\frac{1}{2}\left(\mu\frac{\partial \acute{u}^i}{\partial x^j}\right)_p \tag{50}$$

APPENDIX B. APPLICATION OF THE KEP METHOD IN THE k AND ε EQUATIONS OF THE k–ε MODEL

The subscripts used in this section are defined as follows. The subscript o refers to the main point, subscripts p^i and p^j refer to the following points in the i and j directions, respectively. Moreover, the subscripts m^i and m^j are representative of the earlier points in the i and j directions, respectively. In order to clarify the matter, Figure 16 is presented.

In this section, the discretization method of the k equation is presented.

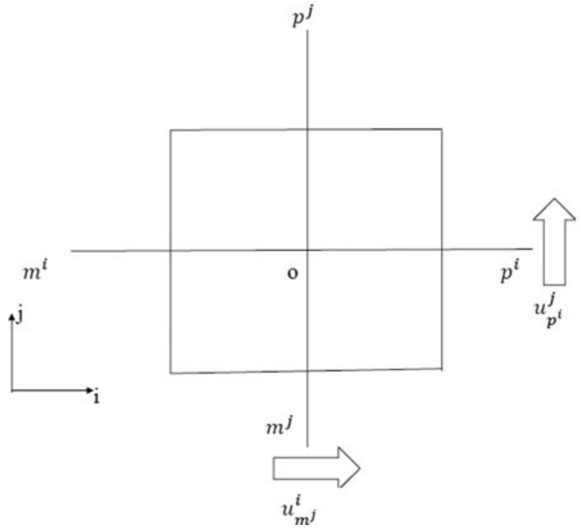

Figure 16. Index notation for meshes.

Discretization of the *k* Equation

The conservative form of the Navier-Stokes equation is given in (51):

$$\rho\left(\frac{\partial u^i}{\partial t}+\frac{\partial(u^iu^j)}{\partial x^j}\right)=-\frac{\partial p}{\partial x^i}+\mu\frac{\partial}{\partial x^j}\left(\frac{\partial u^i}{\partial x^j}\right) \tag{51}$$

The discretization of the above equation is obtained as

$$\begin{aligned}&\rho\left(\frac{u_o^{i,n+1}-u_o^{i,n}}{\Delta t}\right)+\rho\left[\frac{\left(u^i_{p^j}+u^i_o\right)\left(u^j_{p^j}+u^j_o\right)}{4\Delta x^j}-\frac{\left(u^i_{m^j}+u^i_o\right)\left(u^j_{m^j}+u^j_o\right)}{4\Delta x^j}\right]\\&=-\frac{p_{p^i}-p_{m^i}}{2\Delta x^i}+\mu\left[\frac{(\partial u^i/\partial x^j)_{p^j}+(\partial u^i/\partial x^j)_o}{2\Delta x^j}-\frac{(\partial u^i/\partial x^j)_{m^j}+(\partial u^i/\partial x^j)_o}{2\Delta x^j}\right]\end{aligned} \tag{52}$$

The averaged form of the Navier-Stokes equation is given in (53):

$$\rho\frac{\partial\bar{u}^i}{\partial t}+\left(\rho\bar{u}^j\frac{\partial\bar{u}^i}{\partial x^j}+\overline{\rho\acute{u}^j\frac{\partial\acute{u}^i}{\partial x^j}}\right)=-\frac{\partial\bar{p}}{\partial x^i}+\mu\frac{\partial}{\partial x^j}\frac{\partial\bar{u}^i}{\partial x^j} \tag{53}$$

The Navier-Stokes equations are multiplied by u^i and, afterwards, their average is written as

$$\overline{\rho u^i\frac{\partial u^i}{\partial t}}+\overline{\rho u^iu^j\frac{\partial u^i}{\partial x^j}}=-\overline{u^i\frac{\partial p}{\partial x^i}}+\overline{\mu u^i\frac{\partial}{\partial x^j}\left(\frac{\partial u^i}{\partial x^j}\right)} \tag{54}$$

Equation (53) is written as a conservative form:

$$\rho\frac{\partial\bar{u}^i}{\partial t}+\rho\frac{\partial(\bar{u}^i\bar{u}^j)}{\partial x^j}+\rho\frac{\partial\overline{\left(\acute{u}^i\acute{u}^j\right)}}{\partial x^j}=-\frac{\partial\bar{p}}{\partial x^i}+\mu\frac{\partial}{\partial x^j}\left(\frac{\partial\bar{u}^i}{\partial x^j}\right) \tag{55}$$

The above equation can be discretized as follows:

$$\begin{aligned}&\rho\left(\frac{\overline{u_o^{i,n+1}-u_o^{i,n}}}{\Delta t}\right)+\rho\left[\frac{\left(\overline{u_{p^j}^i}+\overline{u_o^i}\right)\left(\overline{u_{p^j}^j}+\overline{u_o^j}\right)}{4\Delta x^j}-\frac{\left(\overline{u_{m^j}^i}+\overline{u_o^i}\right)\left(\overline{u_{m^j}^j}+\overline{u_o^j}\right)}{4\Delta x^j}\right]\\&+\rho\left[\frac{\overline{\left(\acute{u}_{p^j}^i+\acute{u}_o^i\right)\left(\acute{u}_{p^j}^j+\acute{u}_o^j\right)}}{4\Delta x^j}-\frac{\overline{\left(\acute{u}_{m^j}^i+\acute{u}_o^i\right)\left(\acute{u}_{m^j}^j+\acute{u}_o^j\right)}}{4\Delta x^j}\right]\\&=-\frac{\overline{p_{p^i}}-\overline{p_{m^i}}}{2\Delta x^i}+\mu\left[\frac{\overline{(\partial u^i/\partial x^j)_{p^j}}+\overline{(\partial u^i/\partial x^j)_o}}{2\Delta x^j}-\frac{\overline{(\partial u^i/\partial x^j)_{m^j}}+\overline{(\partial u^i/\partial x^j)_o}}{2\Delta x^j}\right]\end{aligned} \tag{56}$$

Equation (54) is written as a conservative form:

$$\overline{\rho u^i\frac{\partial u^i}{\partial t}}+\overline{\rho u^i\frac{\partial(u^iu^j)}{\partial x^j}}=-\overline{u^i\frac{\partial p}{\partial x^i}}+\overline{\mu u^i\frac{\partial}{\partial x^j}\left(\frac{\partial u^i}{\partial x^j}\right)} \tag{57}$$

The above equation can be discretized as follows:

$$\begin{aligned}&\overline{\rho\left(\frac{u_o^{i,n+1}-u_o^{i,n}}{\Delta t}\right)\left(\frac{u_o^{i,n+1}+u_o^{i,n}}{2}\right)}\\&+\rho\left[\frac{\overline{\left(u_{p^j}^i+u_o^i\right)\left(u_{p^j}^j+u_o^j\right)}}{4\Delta x^j}u_o^i-\frac{\overline{\left(u_{m^j}^i+u_o^i\right)\left(u_{m^j}^j+u_o^j\right)}}{4\Delta x^j}u_o^i\right]\\&=-\frac{\overline{p_{p^i}-p_{m^i}}}{2\Delta x^i}u_o^i+\mu\overline{\left[\frac{(\partial u^i/\partial x^j)_{p^j}+(\partial u^i/\partial x^j)_o}{2\Delta x^j}-\frac{(\partial u^i/\partial x^j)_{m^j}+(\partial u^i/\partial x^j)_o}{2\Delta x^j}\right]u_o^i}\end{aligned} \tag{58}$$

Equation (56) is multiplied by $\overline{u_o^i}$:

$$\begin{aligned}&\rho\left(\frac{\overline{u_o^{i,n+1}-u_o^{i,n}}}{\Delta t}\right)\overline{\left(\frac{u_o^{i,n+1}+u_o^{i,n}}{2}\right)}\\&+\rho\left[\frac{\left(\overline{u_{p^j}^i}+\overline{u_o^i}\right)\left(\overline{u_{p^j}^j}+\overline{u_o^j}\right)}{4\Delta x^j}\overline{u_o^i}-\frac{\left(\overline{u_{m^j}^i}+\overline{u_o^i}\right)\left(\overline{u_{m^j}^j}+\overline{u_o^j}\right)}{4\Delta x^j}\overline{u_o^i}\right]\\&+\rho\left[\frac{\overline{\left(\acute{u}_{p^j}^i+\acute{u}_o^i\right)\left(\acute{u}_{p^j}^j+\acute{u}_o^j\right)}}{4\Delta x^j}\overline{u_o^i}-\frac{\overline{\left(\acute{u}_{m^j}^i+\acute{u}_o^i\right)\left(\acute{u}_{m^j}^j+\acute{u}_o^j\right)}}{4\Delta x^j}\overline{u_o^i}\right]\\&=-\frac{\overline{p_{p^i}}-\overline{p_{m^i}}}{2\Delta x^i}\overline{u_o^i}+\mu\left[\frac{\overline{(\partial u^i/\partial x^j)_{p^j}}+\overline{(\partial u^i/\partial x^j)_o}}{2\Delta x^j}\overline{u_o^i}-\frac{\overline{(\partial u^i/\partial x^j)_{m^j}}+\overline{(\partial u^i/\partial x^j)_o}}{2\Delta x^j}\overline{u_o^i}\right]\end{aligned} \tag{59}$$

Equation (59) is subtracted from Eq. (58):

$$
\begin{aligned}
&\overline{\rho\left(\frac{\acute{u}_o^{i,n+1}-\acute{u}_o^{i,n}}{\Delta t}\right)\left(\frac{\acute{u}_o^{i,n+1}+\acute{u}_o^{i,n}}{2}\right)} \\
&+\rho\left[\frac{\overline{\left(\acute{u}_{p^j}^i+\acute{u}_o^i\right)}}{4\Delta x^j}\acute{u}_o^i\left(\overline{u_{p^j}^j}+\overline{u_o^j}\right)-\frac{\overline{\left(\acute{u}_{m^j}^i+\acute{u}_o^i\right)}}{4\Delta x^j}\acute{u}_o^i\left(\overline{u_{m^j}^j}+\overline{u_o^j}\right)\right] \\
&+\rho\left[\frac{\overline{\left(\acute{u}_{p^j}^j+\acute{u}_o^j\right)}}{4\Delta x^j}\acute{u}_o^i\left(\overline{u_{p^j}^i}+\overline{u_o^i}\right)-\frac{\overline{\left(\acute{u}_{m^j}^j+\acute{u}_o^j\right)}}{4\Delta x^j}\acute{u}_o^i\left(\overline{u_{m^j}^i}+\overline{u_o^i}\right)\right] \\
&+\rho\left[\overline{\frac{\left(\acute{u}_{p^j}^i+\acute{u}_o^i\right)\left(\acute{u}_{p^j}^j+\acute{u}_o^j\right)}{4\Delta x^j}\acute{u}_o^i}-\overline{\frac{\left(\acute{u}_{m^j}^i+\acute{u}_o^i\right)\left(\acute{u}_{m^j}^j+\acute{u}_o^j\right)}{4\Delta x^j}\acute{u}_o^i}\right] \\
&=-\overline{\frac{\acute{p}_{p^i}-\acute{p}_{m^i}}{2\Delta x_i}\acute{u}_o^i}+\mu\overline{\left[\frac{\left(\partial\acute{u}^i/\partial x^j\right)_{p^j}+\left(\partial\acute{u}^i/\partial x^j\right)_o}{2\Delta x^j}\acute{u}_o^i-\frac{\left(\partial\acute{u}^i/\partial x^j\right)_{m^j}+\left(\partial\acute{u}^i/\partial x^j\right)_o}{2\Delta x^j}\acute{u}_o^i\right]}
\end{aligned}
\tag{60}
$$

The second term on the left-hand side of Eq. (60) is the convective term of the k equation that is the discretized form of $\rho\overline{u^j}\,\overline{\acute{u}^i\left(\frac{\partial\acute{u}^i}{\partial x^j}\right)}=\frac{1}{2}\rho\overline{u^j}\left[\overline{\frac{\partial(\acute{u}^i)^2}{\partial x^j}}\right]$. Considering the amount of 1/2 entered in the above equation, and considering the following approximations,

$$\acute{u}_{p^j}^i\acute{u}_o^i=\frac{\acute{u}_{p^j}^{i^2}+\acute{u}_o^{i^2}}{2} \tag{61}$$

$$\acute{u}_{m^j}^i\acute{u}_o^i=\frac{\acute{u}_{m^j}^{i^2}+\acute{u}_o^{i^2}}{2} \tag{62}$$

And, also, using the relation (63) to define the parameter k,

$$k=\frac{1}{2}\overline{\left(\acute{u}^i\right)^2} \tag{63}$$

The convective term of the k equation is attained as

$$\rho\left[\frac{k_{p^j}+3k_o}{8\Delta x_j}\left(\overline{u_{p^j}^j}+\overline{u_o^j}\right)-\frac{3k_o+k_{m^j}}{8\Delta x_j}\left(\overline{u_{m^j}^j}+\overline{u_o^j}\right)\right] \tag{64}$$

In order to calculate the convective flux term of the k equation passing through the interface of nodes o and p, the following equation can be written:

$$\left(\rho u^i\mathrm{k}\right)_{op}=\frac{1}{4}\left(\rho u^i\right)_{op}\left(3k_o+k_p\right) \tag{65}$$

The last term of Eq. (60) is obtained as

$$\begin{aligned}&\mu\overline{\left[\frac{\left(\partial\acute{u}^i/\partial x^j\right)_{p^j}+\left(\partial\acute{u}^i/\partial x^j\right)_o}{2\Delta x^j}\acute{u}^i_o-\frac{\left(\partial\acute{u}^i/\partial x^j\right)_{m^j}+\left(\partial\acute{u}^i/\partial x^j\right)_o}{2\Delta x^j}\acute{u}^i_o\right]}\\&=\mu\overline{\left[\frac{\left(\partial\acute{u}^i_o\acute{u}^i/\partial x^j\right)_{p^j}+\left(\partial\acute{u}^i_o\acute{u}^i/\partial x^j\right)_o}{2\Delta x^j}-\frac{\left(\partial\acute{u}^i_o\acute{u}^i/\partial x^j\right)_{m^j}+\left(\partial\acute{u}^i_o\acute{u}^i/\partial x^j\right)_o}{2\Delta x^j}\right]}\\&-\mu\overline{\left[\frac{\left(\partial\acute{u}^i/\partial x^j\right)_{p^j}+\left(\partial\acute{u}^i/\partial x^j\right)_o}{2\Delta x^j}\left(\frac{\partial\acute{u}^i}{\partial x^j}\right)_o-\frac{\left(\partial\acute{u}^i/\partial x^j\right)_{m^j}+\left(\partial\acute{u}^i/\partial x^j\right)_o}{2\Delta x^j}\left(\frac{\partial\acute{u}^i}{\partial x^j}\right)_o\right]}\end{aligned}\tag{66}$$

The second part of the right-hand side of the above equation is the source term of the k equation. To applying the approximations of (61) and (62), the diffusion term of the k equation is gained in the form

$$\mu\left(\frac{(\partial k/\partial x^j)_{p^j}+3(\partial k/\partial x^j)_o}{4\Delta x^j}-\frac{(\partial k/\partial x^j)_{m^j}+3(\partial k/\partial x^j)_o}{4\Delta x^j}\right)\tag{67}$$

In order to calculate the diffusion flux term of the k equation passing through the interface of nodes o and p, Eq. (68) is written:

$$\left(\mu\frac{\partial k}{\partial x^j}\right)_{op}=\frac{3}{4}\left(\mu\frac{\partial k}{\partial x^j}\right)_o+\frac{1}{4}\left(\mu\frac{\partial k}{\partial x^j}\right)_p\tag{68}$$

Implementing the KEP Method in the ε Equation of *k*–ε Model

In this section, the discretization method of the ε equation is presented. The ε equation is defined by Eqs. (31). The convective term of the ε equation is discretized in the following form:

$$\begin{aligned}&\rho\nu\left(\frac{\overline{u^j_{p^j}}+\overline{u^j_o}}{4\Delta x^j}\right)\overline{\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o\left[\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o+\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_{p^j}\right]}\\&-\rho\nu\left(\frac{\overline{u^j_{m^j}}+\overline{u^j_o}}{4\Delta x^j}\right)\overline{\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o\left[\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o+\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_{m^j}\right]}\end{aligned}\tag{69}$$

By defining ε as follows,

$$\varepsilon=\nu\overline{\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)^2}\tag{70}$$

And considering the approximations (71) and (72),

$$\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_{p^j}=\frac{\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)^2_o+\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)^2_{p^j}}{2} \tag{71}$$

$$\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_{m^j}=\frac{\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)^2_o+\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)^2_{m^j}}{2} \tag{72}$$

Eq. (69) is as obtained as

$$\rho\left(\frac{\varepsilon_{p^j}+3\varepsilon_o}{8\Delta x_j}\left(\overline{u^j_{p^j}}+\overline{u^j_o}\right)-\frac{3\varepsilon_o+\varepsilon_{m^j}}{8\Delta x_j}\left(\overline{u^j_{m^j}}+\overline{u^j_o}\right)\right) \tag{73}$$

In order to calculate the convective flux term of the ε equation passing through the interface of nodes o and p, the following equation can be written:

$$\left(\rho u^i\varepsilon\right)_{op}=\frac{1}{4}\left(\rho u^i\right)_{op}\left(3\varepsilon_o+\varepsilon_p\right) \tag{74}$$

The diffusion term of the ε equation is discretized in the form

$$\begin{aligned}&\left(\frac{\mu\nu}{2\Delta x^j}\right)\frac{\partial}{\partial x^j}\overline{\left[\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o+\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_{p^j}\right]}\\&-\left(\frac{\mu\nu}{2\Delta x^j}\right)\frac{\partial}{\partial x^j}\overline{\left[\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o+\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_o\left(\frac{\acute{u}^i_{p^k}-\acute{u}^i_{m^k}}{2\Delta x^k}\right)_{m^j}\right]}\end{aligned} \tag{75}$$

Considering the approximations (71) and (72), Eq. (75) is obtained as

$$\mu\left(\frac{(\partial\varepsilon/\partial x^j)_{p^j}+3(\partial\varepsilon/\partial x^j)_o}{4\Delta x^j}-\frac{(\partial\varepsilon/\partial x^j)_{m^j}+3(\partial\varepsilon/\partial x^j)_o}{4\Delta x^j}\right) \tag{76}$$

In order to calculate the diffusion flux term of the ε equation passing through the interface of nodes o and p, Eq. (77) is written:

$$\left(\mu\frac{\partial\varepsilon}{\partial x^j}\right)_{op}=\frac{3}{4}\left(\mu\frac{\partial\varepsilon}{\partial x^j}\right)_o+\frac{1}{4}\left(\mu\frac{\partial\varepsilon}{\partial x^j}\right)_p \tag{77}$$