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On Relationship Between Generalized Covering Subgroups of Fundamental Groups

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Abstract

In this talk we are interested to focus on subgroups of topologized fundamental groups and some relationships between generalized covering subgroups and some famous subgroups of the fundamental group equipped with the compact-open topology.

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1 Introduction and Motivation

We recall that a continuous map $p : \tilde{X} \to X$ is a covering map if every point of X has an open neighborhood which is evenly covered by p. It is well-known that the induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is a monomorphism and so $\pi_1(\tilde{X}, \tilde{x}) \cong p_*\pi_1(\tilde{X}, \tilde{x})$ is a subgroup of $\pi_1(X, x)$.

Some people extended the notion of covering maps and introduced semicoverings [3] and generalized coverings [1,2,4]. These generalizations focus on keeping some properties of covering maps and eliminating the evenly covered property. Brazas [3] introduced semicoverings by removing evenly covered property and keeping local homeomorphism and unique path lifting properties. For generalized coverings, the local homeomorphism is replaced with unique lifting property [2,4]. A subgroup H of the fundamental group $\pi_1(X, x)$ is called covering, semicovering, generalized covering subgroup if there is a covering, semicovering, generalized covering map $p : \left(\tilde{X}, \tilde{x}\right) \to (X, x)$ such that $H = p_*\pi_1(\tilde{X}, \tilde{x})$, respectively.

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Brazas [2, Theorem 15] showed that the intersection of any collection of generalized covering subgroups of $\pi_1(X,x)$ is also a generalized covering subgroup. We denoted the intersection of all generalized covering subgroups of $\pi_1(X,x)$ by $\pi_1^{gc}(X,x_0)$. Based on some recent works of [5,6,8,9] there is a chain of some effective subgroups of the fundamental group as follows:

$$\{e\} \leq \pi_1^{s}(X,x) \leq \pi_1^{sg}(X,x) \leq \widetilde{\pi}_1^{sp}(X,x) \leq \pi_1^{sp}(X,x) \leq \pi_1(X,x). \qquad (*)$$

In continue, we find the location of the subgroup $\pi_1^{gc}(X,x_0)$ in the chain $(*)$.

2 Notations and Preliminaries

The definition of generalized covering maps based on unique lifting property, is as follows.

Definition 2.1. A pointed continuous map $p:\tilde{X}\to X$ has **UL (unique lifting)** property if for every connected, locally path connected space (Y,y_0) and every continuous map $f:(Y,y_0)\to(X,x_0)$ with $f_*\pi_1(Y,y_0)\subseteq p_*\pi_1(\tilde{X},\tilde{x}_0)$ for chosen $\tilde{x}_0\in p^{-1}(x_0)$, there exists a unique continuous lifting $\tilde{f}$ with $p\circ\tilde{f}=f$ and $\tilde{f}(y_0)=\tilde{x}_0$. If $\tilde{X}$ be a connected, locally path connected space and $p:\tilde{X}\to X$ is surjective with UL property, then $\tilde{X}$ is called a **generalized covering space** for X. A subgroup $H\leq\pi_1(X,x_0)$ is called a generalized covering subgroup of $\pi_1(X,x_0)$ if there is a generalized covering $p:\left(\tilde{X},\tilde{x}_0\right)\to(X,x_0)$ such that $H=p_*\pi_1(\tilde{X},\tilde{x}_0)$.

Recently, the following lemma stated and proved using pullbacks by Brazas [2, Theorem 15]. However, the authors [1, Corollary 2.8] gave another simple proof.

Lemma 2.2. *If $\{H_j \mid j\in J\}$ is any set of generalized covering subgroup of $\pi_1(X,x_0)$, then $H=\cap H_j$ is a generalized covering subgroup.*

Definition 2.3. For a pointed space (X,x_0) we define $\pi_1^{\mathbf{gc}}(\mathbf{X},\mathbf{x_0})=\cap\{H\leq\pi_1(X,x_0)\mid H$ *is a generalized covering subgroup*$\}$.

Corollary 2.4. *For a connected, locally path connected space (X,x_0), $\pi_1^{gc}(X,x_0)$ is a generalized covering subgroup of $\pi_1(X,x_0)$.*

A loop $\alpha:\left(I,\dot{I}\right)\to(X,x)$ is called *small* if and only if there exists a representative of the homotopy class $[\alpha]\in\pi_1(X,x)$ in every open neighborhood U of x. Z. Virk [10] introduced two interesting subgroups of the fundamental group based on small loops, $\pi_1^{s}(X,x_0)$, the collection of all small loops at $x_0\in X$, and $\pi_1^{sg}(X,x_0)$ the collection of all small generated loops i.e. the loops that generated by the following set

$$\{\left[\alpha*\beta*\alpha^{-1}\right] \ / \ [\beta]\in \pi_1^{s}(X,\alpha(1)),\ \alpha\in P(X,\ x_0)\},$$

that is independent of choice of the base point [9]. Torabi et al. [8, Definition 1.2] named $\pi_1^{sp}(X,x_0)$ the intersection of all Spanier subgroups $\pi(\mathfrak{u},x_0)$ where $\mathfrak{u}$ is an open cover of X and $\widetilde{\pi}_1^{sp}(X,x_0)$ the intersection of all path Spanier subgroups $\widetilde{\pi}(\mathfrak{v},x_0)$ where $\mathfrak{v}$ is a path open cover of X [8, Section 3]. Then they presented [8, Theorem 2.1] a relationship order between some subgroups of $\pi_1(X, x_0)$ as the following chain of subgroups:

$$\{e\} \leq \pi_1^s(X, x_0) \leq \pi_1^{sg}(X, x_0) \leq \widetilde{\pi}_1^{sp}(X, x_0) \leq \pi_1^{sp}(X, x_0) \leq \pi_1(X, x_0).$$

They also showed that [8, Theorem 2.2] the closure of the trivial subgroup contains $\pi_1^{sg}(X, x_0)$ and implied that $\overline{\pi_1^{sg}(X, x_0)} = \overline{\{e\}}$, where the closure is based on the topology on fundamental group that inherited from the compact-open topology on the loop space by the natural quotient map.

3 Main Results

We are going to find the location of $\pi_1^{gc}(X, x_0)$ in the chain $(\star)$, because this chain gives us a viewpoint to find out which subgroups of the fundamental group are nice candidates to be covering, semicovering or generalized covering subgroups and which of them can not be a candidate. For instance, it is well-known that every covering subgroup contains $\pi_1^{sp}(X, x_0)$ [7]. This fact gives us useful tool to distinguish covering subgroups, i.e. for $H \leq \pi_1(X, x_0)$ if $\pi_1^{sp}(X, x_0) \cap H \neq \pi_1^{sp}(X, x_0)$, then H cannot be a covering subgroup. If $\pi_1^{sp}(X, x_0) \cap H = \pi_1^{sp}(X, x_0)$, then H is a covering subgroup [5, Corollary 3.7] if and only if it contains a normal open subgroup of $\pi_1^{qtop}(X, x_0)$. There is a similar result for semicoverings, i.e. if $\widetilde{\pi}_1^{sp}(X, x_0) \cap H \neq \widetilde{\pi}_1^{sp}(X, x_0)$, then H cannot be a semicovering subgroup. If $\widetilde{\pi}_1^{sp}(X, x_0) \cap H = \widetilde{\pi}_1^{sp}(X, x_0)$ for locally path connected spaces, then H is a semicovering subgroup if and only if it is an open subgroup of $\pi_1^{qtop}(X, x_0)$. These results allow us to introduce and name ***incoverable area*** and ***insemicoverable area*** in the chain subgroups $(\star)$. For $H \leq \pi_1(X, x_0)$, H is called ***incoverable*** subgroup if $\pi_1^{sp}(X, x_0) \cap H \neq \pi_1^{sp}(X, x_0)$ and is called ***insemicoverable*** subgroup if $\widetilde{\pi}_1^{sp}(X, x_0) \cap H \neq \widetilde{\pi}_1^{sp}(X, x_0)$. After locating the place of $\pi_1^{gc}(X, x_0)$ in the chain $(\star)$, it will be easy to express a similar result for generalized coverings. In addition, the chain $(\star)$ allows us to obtain some new results on the categorical relationship between coverings, semicoverings and generalized coverings.

Theorem 3.1. *For a locally path connected space* (X, x_0) , $\pi_1^{sg}(X, x_0) \leq \pi_1^{gc}(X, x_0) \leq \overline{\pi_1^{sg}(X, x_0)}$.

Similar to coverings and semicoverings, we define ***ingeneralized covering area*** in the chain $(\star)$ that don't contain $\pi_1^{gc}(X, x_0)$, or ***ingeneralized covering subgroup*** for every subgroup H with $\pi_1^{gc}(X, x_0) \cap H \neq \pi_1^{gc}(X, x_0)$. The first part of the following corollary assert this fact, but the second part needs more details that proved in [1, Theorem 2.7].

Corollary 3.2. *Let* $H \leq \pi_1(X, x_0)$*, If* $\pi_1^{gc}(X, x_0) \cap H \neq \pi_1^{gc}(X, x_0)$*, then* H *is an ingeneralized covering subgroup. If* $\pi_1^{gc}(X, x_0) \cap H = \pi_1^{gc}(X, x_0)$*, then* H *is a generalized covering subgroup if and only if* $H = (p_H)_*\pi_1(\tilde{X}_H, \tilde{e}_H)$*, where* $p_H : \tilde{X}_H \to X$ *is the end point projection.*

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