ON THE CATEGORY OF LOCAL HOMEOMORPHISMS WITH UNIQUE PATH LIFTING PROPERTY

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ABSTRACT. In this talk, we discuss on the category of local homeomorphisms of topological spaces with unique path lifting property. We intend to find a classification of these local homeomorphisms similar to that of covering maps.

This is a joint work with Hamid Torabi and Behrooz Mashayekhy.

1. INTRODUCTION

Biss [2, Theorem 5.5] showed that for a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of open subgroups of its fundamental group \( \pi_1(X, x) \). There is a misstep in the proof of the above theorem. In fact, Biss assumed that every fibration with discrete fiber is a covering map which is not true in general.

Torabi et al.[7] pointed out the above misstep and gave the true classification of connected covering spaces of X according to open subgroups of the fundamental group \( \pi_1(X, x) \). In fact, for a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of subgroups of its fundamental group \( \pi_1(X, x) \), with an open normal subgroup in \( \pi_1^{qlop}(X, x) \). We know every covering map is a local homeomorphism. In this talk, we intend to study the category of local homeomorphisms \( p : \tilde{X} \to X \) for a fixed topological space X with unique path lifting property. Our main contribution is to find a classification for these local homeomorphisms.

2. NOTATIONS AND PRELIMINARIES

For a topological space X, by a path in X we mean a continuous map \( \alpha : [0, 1] \to X \). The points \( \alpha(0) \) and \( \alpha(1) \) are called the initial point and the terminal point of \( \alpha \), respectively. A loop \( \alpha \) is a path with \( \alpha(0) = \alpha(1) \). For a path \( \alpha : [0, 1] \to X \), \( \alpha^{-1} \) denotes a path such that \( \alpha^{-1}(t) = \alpha(1-t) \), for all \( t \in [0, 1] \). Denote \([0,1] \) by \( I \), two paths \( \alpha, \beta : I \to X \) with the same initial and terminal points are called

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homotopic relative to end points if there exists a continuous map \( F : I \times I \to X \) such that
\[
F(t, s) = \begin{cases} 
\alpha(t) & s = 0 \\
\beta(t) & s = 1 \\
\alpha(0) = \beta(0) & t = 0 \\
\alpha(1) = \beta(1) & t = 1.
\end{cases}
\]
Homotopy relative to end points is an equivalent relation and the homotopy class containing a path \( \alpha \) is denoted by \([\alpha]\). For paths \( \alpha, \beta : I \to X \) with \( \alpha(1) = \beta(0) \), \( \alpha \star \beta \) denotes the concatenation of \( \alpha \) and \( \beta \) which is a path from \( I \) to \( X \) such that
\[
(\alpha \star \beta)(t) = \begin{cases} 
\alpha(2t) & 0 \leq t \leq 1/2 \\
\beta(2t - 1) & 1/2 \leq t \leq 1.
\end{cases}
\]
The set of all homotopy classes of loops relative to the end point \( x \) in \( X \) under the binary operation \([\alpha][\beta] = [\alpha \star \beta]\) forms a group which is called the fundamental group of \( X \) and is denoted by \( \pi_1(X, x) \) (see [6]). The set of all loops with initial point \( x \) in \( X \) is called the loop space of \( X \) denoted by \( \Omega(X, x) \) (see [5]).

The quasitopological fundamental group \( \pi_1^{\text{qtop}}(X, x) \) is the quotient space of the loop space \( \Omega(X, x) \) equipped with the compact-open topology with respect to the function \( \Omega(X, x) \to \pi_1(X, x) \) identifying path components (see [2]). It should be mentioned that \( \pi_1^{\text{qtop}}(X, x) \) is a quasitopological group in the sense of [1] and it is not always a topological group (see [3],[4]).

**Definition 2.1.** [5] Assume that \( X \) and \( \tilde{X} \) are topological spaces. The continuous map \( p : \tilde{X} \to X \) is called a local homeomorphism if for every point \( \tilde{x} \in \tilde{X} \) there exists an open set \( \tilde{W} \) such that \( \tilde{x} \in \tilde{W} \) and \( p(\tilde{W}) \subset X \) is open and the restriction map \( p|_{\tilde{W}} : \tilde{W} \to p(\tilde{W}) \) is a homeomorphism.

**Definition 2.2.** Let \( p : \tilde{X} \to X \) be a local homeomorphism and let \( f : (Y, y_0) \to (X, x_0) \) be a continuous map with \( f(y_0) = x_0 \). Let \( \tilde{x}_0 \) be in the fiber over \( x_0 \). If there exist a continuous function \( \tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0) \) such that \( p \circ \tilde{f} = f \), then \( \tilde{f} \) is called a lifting for \( f \).

**Definition 2.3.** Assume that \( X \) and \( \tilde{X} \) are topological spaces and \( p : \tilde{X} \to X \) is a continuous map. Let \( \tilde{x}_0 \) be in the fiber over \( x_0 \). The map \( p \) has "unique path lifting property" if for every path \( f \) in \( X \), there exists a unique continuous function \( \tilde{f} : (I, 0) \to (\tilde{X}, \tilde{x}_0) \) with \( p \circ \tilde{f} = f \).

Let \( X \) be a fixed topological space. The set of all local homeomorphisms of \( X \) with unique path lifting property forms a category. In this category a morphism from \( p : \tilde{X} \to X \) to \( q : \tilde{Y} \to X \) is a continuous function \( h : \tilde{X} \to \tilde{Y} \) such that \( p = q \circ h \).

**Definition 2.4.** [6] Let \( \tilde{X} \) and \( X \) be topological spaces and let \( p : \tilde{X} \to X \) be continuous. An open set \( U \) in \( X \) is evenly covered by \( p \) if \( p^{-1}(U) \) is a disjoint union of open sets \( S_i \) in \( \tilde{X} \), called sheets, such that \( p|_{S_i} : S_i \to U \) is a homeomorphism for every \( i \).

**Definition 2.5.** [6] If \( X \) is a topological space, then an ordered pair \( (\tilde{X}, p) \) is a covering space of \( X \) if:
(1) \( \tilde{X} \) is a path connected topological space;
(2) \( p : \tilde{X} \to X \) is continuous;
(3) each \( x \in X \) has an open neighborhood \( U = U_x \) that is evenly covered by \( p \).

3. Main Results

**Theorem 3.1. (Local Homeomorphism Homotopy Theorem for Paths)** Let \((\tilde{X}, p)\) be a local homeomorphism of \( X \) with unique path lifting property. Consider the following diagram of continuous maps

\[
\begin{array}{ccc}
I & \xrightarrow{f} & (\tilde{X}, \tilde{x}_0) \\
\downarrow{\tilde{F}} & & \\
I \times I & \xrightarrow{F} & (X, x_0)
\end{array}
\]

where \( j(t) = (t, 0) \) for all \( t \in I \). Then there exists a unique continuous map \( \tilde{F} : I \times I \to \tilde{X} \) which makes the diagram commutative.

**Theorem 3.2. (Lifting Criterion)** If \( Y \) is connected and locally path connected, \( f : (Y, y_0) \to (X, x_0) \) is continuous and \( p : \tilde{X} \to X \) is a local homeomorphism with unique path lifting property, where \( \tilde{X} \) is path connected, then there exists a unique \( \tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0) \) such that \( p \circ \tilde{f} = f \) if and only if \( f_* (\pi_1(Y, y_0)) \subset p_* (\pi_1(\tilde{X}, \tilde{x}_0)) \).

**Corollary 3.3.** If \( Y \) is simply connected, locally path connected and \( p : \tilde{X} \to X \) is a local homeomorphism with unique path lifting property, where \( \tilde{X} \) is path connected, then any continuous map \( f : (Y, y_0) \to (X, x_0) \) has a lifting to \( \tilde{X} \).

**Corollary 3.4.** Suppose \( X \) is connected, locally path connected and \( p : \tilde{X} \to X \), \( q : \tilde{Y} \to X \) are local homeomorphisms with unique path lifting property where \( \tilde{X} \), \( \tilde{Y} \) are path connected. If \( p_* (\pi_1(\tilde{X}, \tilde{x}_0)) = q_* (\pi_1(\tilde{Y}, \tilde{y}_0)) \), then there exists a homeomorphism \( h : (\tilde{Y}, \tilde{y}_0) \to (\tilde{X}, \tilde{x}_0) \) such that \( p \circ h = q \).

**Theorem 3.5.** Let \( p : \tilde{X} \to X \) be a local homeomorphism with unique path lifting property and let \( x_0, x_1 \in X \) and \( f, g : I \to X \) be paths with \( f(0) = g(0) = x_0 \), \( f(1) = g(1) = x_1 \) and \( \tilde{x}_0 \in p^{-1}(x_0) \). If \( F : f \simeq g \) rel \( I \) and \( \tilde{f}, \tilde{g} \) are the lifting of \( f \) and \( g \) respectively with \( \tilde{f}(0) = \tilde{x}_0 = \tilde{g}(0) \), then \( \tilde{F} : \tilde{f} \simeq \tilde{g} \) rel \( I \).

**Theorem 3.6.** Let \( p : \tilde{X} \to X \) be a local homeomorphism with unique path lifting property where \( \tilde{X} \) is path connected. If \( x_0, x_1 \in X \), \( Y_0 = p^{-1}(x_0) \) and \( Y_1 = p^{-1}(x_1) \), then \( |Y_0| = |Y_1| \).

**Theorem 3.7.** If \( X \) is connected, locally path connected and \( H \) is a subgroup of \( \pi_1(X, x) \), then there exists a local homeomorphism \( p : \tilde{X} \to X \) with unique path lifting property such that \( p_* (\pi_1(\tilde{X}, \tilde{x})) = H \) if and only if \( H \) is an open subgroup of \( \pi_1^{top}(X, x) \). Moreover there is a 1-1 correspondence between equivalent classes of local homeomorphisms of \( X \) (in category of local homeomorphism with unique path lifting property) and the conjugacy classes of open subgroups of the quasitopological fundamental group \( \pi_1^{top}(X, x) \).
Definition 3.8. \( p : \tilde{X} \rightarrow X \) is called a regular local homeomorphism if \( p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \) is a normal subgroup of \( \pi_1(X, x_0) \).

Theorem 3.9. Every regular local homeomorphism is a cover map.

References


