On the Whisker Topology on Fundamental Group

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Abstract

In this talk, after reviewing concepts of compact-open topology, Whisker topology and Lasso topology on fundamental groups, we present some topological properties for the Whisker topology on a fundamental group.

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1 Introduction

The concept of a natural topology on the fundamental group appears to have originated with Hurewicz [8] in 1935. The topology inherited from the loop space by quotient map, where equipped with compact-open topology, on fundamental group is denoted by \( \pi_1^{top}(X, x_0) \). Spanier [10, Theorem 13 on page 82] introduced a different topology that Dydak et al. [4] called it the Whisker topology and denoted by \( \pi_1^{wh}(X, x_0) \). They also introduced a new topology on \( \pi_1(X, x_0) \) and called it the Lasso topology to characterize the unique path lifting property which is denoted by \( \pi_1^{lh}(X, x_0) \) and showed that this topology makes the fundamental group a topological group [3]. However Biss [2] claimed that \( \pi_1^{top}(X, x_0) \) is a topological group, but it is shown that the multiplication map is not continuous, in general, hence \( \pi_1^{top}(X, x_0) \) is a quasitopological group (see [6]). In this talk, we show that \( \pi_1^{wh}(X, x_0) \) is not a topological group, in general. In addition, it is not even a semitopological group, but it has some properties similar to topological groups. For instance, every open subgroup of \( \pi_1^{wh}(X, x_0) \) is also a closed subgroup of \( \pi_1^{wh}(X, x_0) \) and \( \pi_1^{wh}(X, x_0) \) is \( T_0 \) if and only if it is \( T_2 \). Moreover, \( \pi_1^{wh}(X, x_0) \) is a homogenous and regular space, and it is totally seperated if and only if is \( T_0 \).

2 Notation and Preliminaries

Definition 2.1. Let \( H \) be a subgroup of \( \pi_1(X, x_0) \) and \( P(X, x_0) = \{ \alpha : (I, 0) \to (X, x_0) \mid \alpha \) is a path \} be a path space. Then \( \alpha_1 \sim \alpha_2 \mod H \) if \( \alpha_1(1) = \alpha_2(1) \) and \([\alpha_1 * \alpha_2^{-1}] \in H\). It is easy to check that this is an equivalence relation on \( P(X, x_0) \). The equivalence class of \( \alpha \) is denoted by \( \langle \alpha \rangle_H \). Now one can define the quotient space \( \tilde{X}_H = \frac{P(X, x_0)}{\sim} \) and the

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map \( p_H : (\tilde{X}_H, e_H) \to (X, x_0) \) by \( p_H(\langle \alpha \rangle_H) = \alpha(1) \) where \( e_H \) is the class of constant path at \( x_0 \).

For \( \alpha \in P(X, x_0) \) and an open neighborhood \( U \) of \( \alpha(1) \), a continuation of \( \alpha \) in \( U \) is a path \( \beta \in P(X, x_0) \) of the form \( \beta = \alpha \ast \gamma \), where \( \gamma(0) = \alpha(1) \) and \( \gamma(I) \subseteq U \). Thus we can define a set \( \langle U, \langle \alpha \rangle_H \rangle = \{ \langle \beta \rangle_H \in X_H | \beta \text{ is a continuation of } \alpha \text{ in } U \} \) where \( U \) is an open neighborhood of \( \alpha(1) \) in \( X \). It is shown that the subsets \( \langle U, \langle \alpha \rangle_H \rangle \) as defined above form a basis for a topology on \( \tilde{X}_H \) for which the function \( p_H : \tilde{X}_H \to X \) is continuous [9, Theorem 10.31]. Moreover, if \( X \) is path connected, then \( p_H \) is surjective. This topology on \( \tilde{X}_H \) is called the Whisker topology [4].

**Definition 2.2.** Let \( p_e : \tilde{X}_e \to X \) be the defined end point projection map for \( \{e\} \leq \pi_1(X, x_0) \) and put \( p_e^{-1}(x_0) \) as a subspace of \( (\tilde{X}_e, \tilde{x}_0) \) with its default Whisker topology. One can transfer this topology by the bijection \( f : \pi_1(X, x_0) \to p_e^{-1}(x_0) \) into \( \pi_1(X, x_0) \) with \( [\alpha] \mapsto \langle \alpha \rangle_H \). The fundamental group with Whisker topology is denoted by \( \pi_1^{wh}(X, x_0) \).

Fishcer and Zastrow [7, Lemma 2.1.] have shown that the Whisker topology is finer than the inherited topology from loop space with compact-open topology on \( \pi_1(X, x_0) \) which is denoted by \( \pi_1^{top}(X, x_0) \).

3 Main results

In this section we are going to present some interesting properties of \( \pi_1^{wh}(X, x_0) \). At first, it seems necessary to characterize the open subsets and subgroups of \( \pi_1^{wh}(X, x_0) \). Let \( [\alpha] \in \pi_1(X, x_0) \), for every open subset \( U \) of \( x_0 \) there is a bijection \( \varphi_\alpha : \pi_1(U, x_0) \to \langle U, [\alpha] \rangle \cap p_e^{-1}(x_0) \) defined by \( \varphi_\alpha([\gamma]) = [\alpha \ast \gamma] \). It is easy to check that \( \varphi_\alpha \) is a well defined bijection.

The collection \( \{[\alpha]\ast \pi_1(U, x_0) | [\alpha] \in \pi_1(X, x_0) \text{ and } U \text{ open subset of } x_0 \} \) form a basis for the Whisker topology on \( \pi_1(X, x_0) \). Moreover, these basis elements are closed and hence they are clopen subsets.

The left (right) topological group is a group equipped with a topology that makes all of the left (right) translations continuous. A semitopological group is a left topological group which is also a right topological group [1, Section 1.2.]. \( \pi_1^{wh}(X, x_0) \) is not a right topological group in general, hence it is not a semitopological group. For example see the Hawaiian earring is not a topological group since the inverse map in \( \pi_1^{wh}(HE, *) \) is not continuous [4]. Recall that a non-empty topological space \( X \) is called a \( G^- \) space, for a group \( G \), if it is equipped with an action of \( G \) on \( X \). A homogeneous space is a \( G^- \) space on \( X \) which \( G \) acts transitively.

**Proposition 3.1.** \( \pi_1^{wh}(X, x_0) \) is a homogenous space.

**Proof.** Clearly \( \pi_1^{wh}(X, x_0) \) acts on itself. To show that this action is transitive, it is enough to prove that left translation map in \( \pi_1^{wh}(X, x_0) \) is homeomorphism. It is known that every left topological group is a homogenous space. Hence \( \pi_1^{wh}(X, x_0) \) is a homogenous space.

**Corollary 3.2.** Every open subgroup of \( \pi_1^{wh}(X, x_0) \) is a closed subgroup.

Recall that a topological space is called totally separated if for every pair of disjoint points there exists a clopen subset which contains one of points and does not contain another. The following proposition state some separation axioms for \( \pi_1^{wh}(X, x_0) \).
Proposition 3.3. For a connected and locally path connected space $X$, the following statement are equivalent:

1. $\pi_1^{wh}(X,x_0)$ is $T_0$.
2. $\pi_1^{wh}(X,x_0)$ is $T_1$.
3. $\pi_1^{wh}(X,x_0)$ is $T_2$.
4. $\pi_1^{wh}(X,x_0)$ is $T_3$ ($T_3 = \text{regular} + T_1$).
5. $\pi_1^{wh}(X,x_0) = 1$, where $\pi_1^{wh}(X,x_0)$ is the collection of small loops at $x_0$.
6. $\pi_1^{wh}(X,x_0)$ is totally separated.

Moreover, $\pi_1^{wh}(X,x_0)$ is regular.

Corollary 3.4. If the right translation in $\pi_1^{wh}(X,x_0)$ are continuous, then $\pi_1^{wh}(X,x_0)$ is a topological group.

It seems interesting to know that when $\pi_1^{wh}(X,x_0)$ has the countable axiom properties.

Proposition 3.5. If $X$ is a first countable space, then $\pi_1^{wh}(X,x_0)$ is also first countable.

Proof. Let $\beta_{x_0}$ be a countable neighborhood basis at $x_0$ and let $[f] \in \pi_1^{wh}(X,x_0)$. Then the collection $\beta_f = \{ [f]_i(V,x_0) | V \in \beta_{x_0} \}$ form a countable neighborhood basis at $[f]$.

Proposition 3.6. The closure of trivial element in $\pi_1^{wh}(X,x_0)$ is equals to $\pi_1^1(X,x_0)$.

References


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