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A new method based on generalized Taylor expansion for computing a series solution of the linear systems

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ABSTRACT

In this paper, based on the generalized Taylor expansion and using the iteration matrix G of the iterative methods, we introduce a new method for computing a series solution of the linear systems. This method can be used to accelerate the convergence of the basic iterative methods. In addition, we show that, by applying the new method to a divergent iterative scheme, it is possible to construct a convergent series solution and to find the convergence intervals of control parameter for special cases. Numerical experiments are given to show the efficiency of the new method.

1. Introduction

Consider the linear system of equations,

$$Au = d, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is given, $d \in \mathbb{R}^n$ is known, and $u \in \mathbb{R}^n$ is unknown. One class of iterative methods is based on a splitting (M, N) of the matrix A, i.e.,

$$A = M - N, \tag{1.2}$$

where M is taken to be invertible and cheap to invert, meaning that a linear system with matrix coefficient M is much more economical to solve than (1.1). Based on (1.2), (1.1) can be written in the fixed-point form

$$u = Gu + c, \quad G = M^{-1}N, \quad c = M^{-1}d; \tag{1.3}$$

which yields the following iterative scheme for the solution of (1.1):

$$u^{(k+1)} = Gu^{(k)} + c, k = 0, 1, 2, \ldots, \quad u^{(0)} \in \mathbb{R}^n \text{ is arbitrary}. \tag{1.4}$$

There have been many studies about the convergence of the splitting iteration method (1.4), or in other words, the matrix splitting (1.2), when the coefficient matrix A and the iteration matrix G have particular properties (see, e.g. [1,3,4,13–15]). Also, for improving the rate of the convergence, many authors introduced the pereconditioned methods, see [5,7,11,12,16,17]. A sufficient and necessary condition for (1.4) to converge to the solution of (1.1), is that $\rho(G) < 1$, where $\rho(G)$ denotes the spectral radius of the iteration matrix G. In this paper based on the generalized Taylor expansion and using

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the iteration matrix G of the iterative method, we introduce a new method for computing a series solution of the linear system (1.1). We show that, this method can be used to accelerate the convergence of the convergent basic iterative methods. In addition, we prove that, under certain assumptions, this method can be applied to a divergent iterative scheme and to construct a convergent series solution.

This paper is organized as follows. In Section 2, by using generalized Taylor expansion we introduce the new method. In Section 3, we derive the conditions for improving the rate of convergence of the basic iterative methods. In Section 4, we apply the new method to the divergent iterative methods to construct a convergent series solution. We derive the convergence intervals and obtain the optimal value for the control parameter. In Section 5, some numerical examples are presented to show the efficiency of the method. Finally, we make some concluding remarks in Section 6.

2. New method based on generalized Taylor expansion

In the book [8], Liao introduces the generalized Taylor expansion to the nonlinear equation as follow,

$$f(t)=\lim_{m\to\infty}\sum_{l=0}^{m}\mu_0^{m,l}(\hbar)\frac{f^{(l)}(t_0)}{l!}(t-t_0)^l, \tag{2.1}$$

where $\lim_{m\to\infty}\mu_0^{m,l}(\hbar)=1$, for $l>1$. He controlled the convergence region of the generalized Taylor expansion (2.1) by the auxiliary parameter $\hbar\neq 0$. As [9], we call it the convergence control parameter $\hbar$.

In [9], through detailed analysis of some examples, Liu showed that the generalized Taylor series is only the usual Taylor expansion at point t_0. Here, by using this idea, we consider the Taylor expansion of $f(t)=\frac{1}{1-t}$ at point t_0

$$f(t)=\frac{1}{1-t_0}\left[1+\frac{t-t_0}{1-t_0}+\left(\frac{t-t_0}{1-t_0}\right)^2+\cdots+\left(\frac{t-t_0}{1-t_0}\right)^l+\cdots\right],$$

with the convergence region $|t-t_0|<|1-t_0|$. If we take $t_0=\alpha(1+\frac{1}{\hbar})$ where $\alpha=a+ib$, so the above expression becomes

$$f(t)=\frac{\hbar}{\hbar-\alpha(\hbar+1)}\left[1+\left(\frac{\hbar t-\alpha(\hbar+1)}{\hbar-\alpha(\hbar+1)}\right)+\left(\frac{\hbar t-\alpha(\hbar+1)}{\hbar-\alpha(\hbar+1)}\right)^2+\cdots+\left(\frac{\hbar t-\alpha(\hbar+1)}{\hbar-\alpha(\hbar+1)}\right)^l+\cdots\right]. \tag{2.2}$$

By assuming $\lambda_i\neq 1$, for $i=1,2,\ldots,n$, where λ_i is the eigenvalue of matrix G, and $\rho\left(\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}\right)<1$, if we apply (2.2) to the equation $f(G)=(I-G)^{-1}$, then one may obtain the following equation

$$f(G)=\frac{\hbar}{\hbar-\alpha(\hbar+1)}\left[I+\left(\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}\right)+\left(\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}\right)^2+\cdots+\left(\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}\right)^l+\cdots\right], \tag{2.3}$$

Taking

$$G_{\alpha,\hbar}=\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}, \tag{2.4}$$

yields

$$f(G)=\frac{\hbar}{\hbar-\alpha(\hbar+1)}[I+G_{\alpha,\hbar}+G_{\alpha,\hbar}^2+\cdots+G_{\alpha,\hbar}^l+\cdots]. \tag{2.5}$$

Let u_0 be an initial approximation to the exact solution u of the original system (1.1) and define the vectors

$$\begin{aligned} u_1 &= -\tfrac{\hbar}{\hbar-\alpha(\hbar+1)}[(I-G)u_0-c],\\ u_i &= G_{\alpha,\hbar}u_{i-1},\quad i=2,3,\ldots, \end{aligned} \tag{2.6}$$

and

$$v=\sum_{i=0}^{\infty}u_i=u_0+\sum_{i=1}^{\infty}G_{\alpha,\hbar}^{i-1}u_1 \tag{2.7}$$

It is obvious that if $\rho(G_{\alpha,\hbar})<1$, then the series $\sum_{i=1}^{\infty}G_{\alpha,\hbar}^{i-1}u_1$ converges and we have

$$v=u_0+(I-G_{\alpha,\hbar})^{-1}u_1=u_0+\left(\frac{\hbar}{\hbar-\alpha(\hbar+1)}\right)^{-1}(I-G)^{-1}u_1=(I-G)^{-1}c,$$

which is the exact solution of (1.3). So, when $\rho(G_{\alpha,\hbar})<1$ and $\lambda_i\neq 1$, for $i=1,2,\ldots,n$, a series of vectors can be computed by (2.6) and the convergence series (2.7) provides the following approximations to the exact solution of (1.1):

$$v_l=\sum_{i=0}^{l}u_i,\quad l=1,2,\ldots. \tag{2.8}$$

In this paper, our aim is to choose the convergence control parameter $\hbar\neq 0$ and appropriate $a+ib=\alpha\in\mathbb{C}$, so that $\rho(G_{\alpha,\hbar})<1$.

3. Convergent iterative methods

Here we suppose that the iterative method (1.4) converges to the exact solution of (1.1). For improving the rate of convergence of iterative methods, we try to find suitable parameters $\hbar$ and α such that $\rho(G_{\alpha,\hbar}) < \rho(G)$. We suppose that $|\alpha| = 1$ and $\lambda_i = Re(\lambda_i) + iIm(\lambda_i)$ and $\mu_i, i = 1, 2, \dots, n$ be the eigenvalues of G and $G_{\alpha,\hbar}$, respectively. In order to investigate the spectrum radius of $G_{\alpha,\hbar}$, we define the function $g_i(\hbar)$ as follows:

$$\begin{aligned} g_i(\hbar) &= |\hbar - \alpha(1+\hbar)|^2(|\mu_i|^2 - \rho(G)^2) = \alpha_i\hbar^2 + \beta_i\hbar + 1 - |\hbar - \alpha(1+\hbar)|^2\rho(G)^2 \\ &= \delta_i\hbar^2 + (\beta_i + \delta_i - \alpha_i)\hbar + (1 - \rho(G)^2), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \alpha_i &= (a - Re(\lambda_i))^2 + (b - Im(\lambda_i))^2, \\ \beta_i &= 2(1 - aRe(\lambda_i) - bIm(\lambda_i)), \\ \delta_i &= \alpha_i - 2(1-a)\rho(G)^2. \end{aligned} \tag{3.2}$$

We observe that, for fixed α, the relation $\rho(G_{\alpha,\hbar}) < \rho(G)$ satisfies if $g_i(\hbar) < 0$, for $i = 1, 2, \dots, n$. For having these relations, we present the following theorems.

Theorem 3.1. *Suppose that $\rho(G) < 1$, and $\lambda_i = Re(\lambda_i) + iIm(\lambda_i)$ and $\mu_i, i = 1, 2, \dots, n$ be the eigenvalues of G and $G_{\alpha,\hbar}$, respectively. If $\delta_i \neq 0$ and $\beta_i - \alpha_i + \delta_i \neq 0, i = 1, 2, \dots, n$, then $g_i(\hbar)$ has simple real roots $\gamma_1^{(i)} < \gamma_2^{(i)}$ with $\gamma_1^{(i)} < 0$.*

Proof. We begin by defining four index sets

$$\begin{aligned} N_1 &= \{i \,|\, |\lambda_i| = \rho(G) \text{and} \delta_i > 0\}, \\ N_2 &= \{i \,|\, |\lambda_i| = \rho(G) \text{and} \delta_i < 0\}, \\ N_3 &= \{i \,|\, |\lambda_i| \neq \rho(G) \text{and} \delta_i > 0\}, \\ N_4 &= \{i \,|\, |\lambda_i| \neq \rho(G) \text{and} \delta_i < 0\}. \end{aligned}$$

It is easy to see that, if $i \in N_3 \cup N_4$ then

$$g_i(-1) < 0 \text{ and } g_i(0) > 0.$$

Thus, g_i has a real root between -1 and 0.

For $i \in N_3$, since $\delta_i > 0$, we have

$$\text{sign}\, g_i(-\infty) = 1.$$

So, for $i \in N_3, g_i(\hbar)$ has real simple roots

$$\gamma_1^{(i)} < -1 < \gamma_2^{(i)} < 0,$$

and for all $\hbar \in [\gamma_1^{(i)}, \gamma_2^{(i)}]$, we have $g_i(\hbar) \leq 0$.

For $i \in N_4$, since $\delta_i < 0$, we have

$$\text{sign}\, g_i(\infty) = -1.$$

So, for $i \in N_4, g_i(\hbar)$ has real simple roots

$$-1 < \gamma_1^{(i)} < 0 < \gamma_2^{(i)},$$

and for all $\hbar \in (-\infty, \gamma_1^{(i)}] \cup [\gamma_2^{(i)}, \infty)$, we have $g_i(\hbar) \leqslant 0$.

For $i \in N_1 \cup N_2$, we have $\alpha_i - \beta_i = \rho(G)^2 - 1$, so

$$g_i(-1) = 0, \; g_i\left(\frac{\alpha_i - \beta_i}{\delta_i}\right) = 0,$$

and $\alpha_i - \beta_i < 0$. So, for $i \in N_1$, the assumption $\delta_i > 0$ implies that g_i has two real simple roots

$$\gamma_1^{(i)} < \gamma_2^{(i)} < 0,$$

where one of them is equal to -1. Therefore, for all $\hbar \in [\gamma_1^{(i)}, \gamma_2^{(i)}]$, we have $g_i(\hbar) \leq 0$.

For $i \in N_2$, from the assumption $\delta_i < 0$, we have

$$\gamma_1^{(i)} = -1 \text{ and } \quad \gamma_2^{(i)} > 0.$$

So, for all $\hbar \in (-\infty, -1] \cup [\gamma_2^{(i)}, \infty)$, we have $g_i(\hbar) \leqslant 0$. □

Theorem 3.2. *Suppose that $\rho(G) < 1$, and $\lambda_i = Re(\lambda_i) + iIm(\lambda_i)$ and $\mu_i, i = 1, 2, \ldots, n$, be the eigenvalues of G and $G_{\alpha,\hbar}$, respectively. If $\delta_i \neq 0$ and $\beta_i - \alpha_i + \delta_i \neq 0, i = 1, 2, \ldots, n$, then:*

(i) *$\hbar = -1$ belongs to the interval $[\phi, \psi]$, where*

$$\phi = \max_{i \in N_1 \cup N_3} \gamma_1^{(i)}, \psi = \min\{\min_{i \in N_2 \cup N_4} \gamma_1^{(i)}, \min_{i \in N_1 \cup N_3} \gamma_2^{(i)}\},$$

and $\rho(G_{\alpha,-1}) = \rho(G) < 1$.

(ii) *for each $\hbar \in [\phi, \psi]$ and $\hbar \neq -1$, the relation $\rho(G_{\alpha,\hbar}) < \rho(G) < 1$ holds.*

Proof

(i) From Theorem 3.1, it is clear that $-1 \in [\phi, \psi]$. For $\hbar = -1$, we have $G_{\alpha,-1} = G$. So, $\rho(G_{\alpha,-1}) = \rho(G) < 1$.

(ii) From Theorem 3.1, we observe that for each $\hbar \in [\phi, \psi]$ and $\hbar \neq -1$, we have $g_i(\hbar) < 0$, so the relation $\rho(G_{\alpha,\hbar}) < \rho(G) < 1$ holds. □

4. Divergent iterative methods

In this section, we try to find two parameters $\alpha = a + ib$ and $\hbar$, such that $\rho(G_{\alpha,\hbar}) < 1$, when the iterative method (1.4) is divergent.

As Section 3, let $\lambda_i = Re(\lambda_i) + iIm(\lambda_i)$ and $\mu_i, i = 1, 2, \ldots, n$ denote the eigenvalues of G and $G_{\alpha,\hbar}$, respectively. We define the function $g_i(\hbar)$ as follows:

$$g_i(\hbar) = |\hbar - \alpha(1 + \hbar)|^2(|\mu_i|^2 - 1) = \delta_i\hbar^2 + \xi_i\hbar, \tag{4.1}$$

where

$$\begin{aligned} \alpha_i &= (a - Re(\lambda_i))^2 + (b - Im(\lambda_i))^2, \\ \delta_i &= \alpha_i - ((1 - a)^2 + b^2), \\ \xi_i &= 2(a - aRe(\lambda_i) - bIm(\lambda_i)). \end{aligned} \tag{4.2}$$

We observe that $g_i(\hbar) < 0$ implies that $|\mu_i| < 1$. So, we can construct a convergent series solution if we use a control parameter $\hbar$ which satisfies $g_i(\hbar) < 0$, for $i = 1, 2, \ldots, n$. By the following theorems, we furnishes the convergence intervals and optimal values for control parameter $\hbar$.

Theorem 4.1. *Suppose that G has real eigenvalues $\lambda_i, i = 1, 2, \ldots, n$ and*

$$\lambda_{min} = \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n = \lambda_{max}.$$

Let $\alpha = a + ib, a \neq 0$, and $a \neq \frac{\lambda_{min} + \lambda_{max}}{2}$, then

(i) *$\rho(G_{\alpha,\hbar}) < 1$, if the eigenvalues $\lambda_i, i = 1, 2, \ldots, n$ and the parameters a and $\hbar$ take any values from their domains, as these are defined and given in the Table 1.*

Table 1
The possible domains of the parameters λ_i, a and $\hbar$.

Case	λ_i-domain	a-domain	$\hbar$-domain
1	$\lambda_{min} > 1$	$a > \frac{\lambda_{max}+1}{2}$	$\left(-\infty, \frac{-2a}{2a-\lambda_{max}-1}\right) \cup (0, +\infty)$
2	$\lambda_{min} > 1$	$0 < a < \frac{\lambda_{min}+1}{2}$	$\left(0, \frac{-2a}{2a-\lambda_{max}-1}\right)$
3	$\lambda_{min} > 1$	$0 > a$	$(\frac{-2a}{2a-\lambda_{max}-1}, 0)$
4	$-1 < \lambda_{max} < 1$	$a > \frac{\lambda_{max}+1}{2}$	$\left(\frac{-2a}{2a-\lambda_{min}-1}, 0\right)$
5	$\lambda_{max} < -1$	$a > 0$	$\left(\frac{-2a}{2a-\lambda_{min}-1}, 0\right)$
6	$\lambda_{max} < -1$	$0 > a > \frac{\lambda_{max}+1}{2}$	$\left(0, \frac{-2a}{2a-\lambda_{min}-1}\right)$
7	$\lambda_{min} < -1$ and $\lambda_{max} < 1$	$a < \frac{\lambda_{min}+1}{2}$	$\left(-\infty, \frac{-2a}{2a-\lambda_{min}-1}\right) \cup (0, +\infty)$
8	$\lambda_{min} > -1$ and $\lambda_{max} < 1$	$0 < a < \frac{\lambda_{min}+1}{2}$	$(-\infty, 0) \cup \left(\frac{-2a}{2a-\lambda_{min}-1}, +\infty\right)$
9	$\lambda_{min} > -1$ and $\lambda_{max} < 1$	$0 > a$	$\left(-\infty, \frac{-2a}{2a-\lambda_{min}-1}\right) \cup (0, +\infty)$

(ii) *If $\alpha = a \in \mathbb{R}$ and the eigenvalues $\lambda_i, i = 1, 2, \ldots, n$ and the parameters a, take any values from their domains in the Table 1, then*

$$\hbar_{opt} = \frac{2a}{\lambda_{max} + \lambda_{min} - 2a}$$

and

$$\rho_{opt}(G_{a,\hbar_{opt}}) = \frac{\lambda_{max} - \lambda_{min}}{|\lambda_{max} + \lambda_{min} - 2|}.$$

Proof

(i) Since $Im(\lambda_i) = 0$, from (4.2), for all cases we have

$$\begin{aligned} \delta_i &= (a - \lambda_i)^2 - (1 - a)^2 = (2a - \lambda_i - 1)(1 - \lambda_i) \\ \xi_i &= 2a(1 - \lambda_i), \end{aligned}$$

and $g_i(\hbar), i = 1, 2, \ldots, n$, has real simple roots

$$\hbar_1^{(i)} = 0 \quad \text{and} \quad \hbar_2^{(i)} = \frac{-\xi_i}{\delta_i} = \frac{-2a}{2a - \lambda_i - 1}.$$

Case (1): For $\lambda_{min} > 1$ and $a > \frac{\lambda_{max}+1}{2}$, we have $\delta_i < 0$ and $\hbar_2^{(i)} = \frac{-2a}{2a-\lambda_i-1} < 0$, for $i = 1, 2, \ldots, n$. So, for each value $\hbar \in (-\infty, \frac{-2a}{2a-\lambda_{max}-1}) \cup (0, +\infty)$, we have $g_i(\hbar) < 0, i = 1, 2, \ldots, n$, which imply that $\rho(G_{\alpha,\hbar}) < 1$.

For other cases, in a similar way, the convergence intervals for parameter $\hbar$ can be obtained.

(ii) Let, for fixed a and $\hbar$, $\mu_\hbar^{(i)}(a), i = 1, 2, \ldots, n$, represent the eigenvalues of $G_{\alpha,\hbar}$. Then, from (2.4), we have

$$\mu_\hbar^{(i)}(a) = \frac{\hbar\lambda_i - a(\hbar + 1)}{\hbar - a(\hbar + 1)}, \quad i = 1, 2, \ldots, n,$$

and

$$\max_{1 \leqslant i \leqslant n} |\mu_\hbar^{(i)}(a)| = \max\{|\mu_\hbar^{(1)}(a)|, |\mu_\hbar^{(n)}(a)|\}.$$

For fixed a, it is easy to see that, for

$$\hbar = \hbar_{opt} = \frac{2a}{\lambda_{max} + \lambda_{min} - 2a},$$

we have $|\mu_\hbar^{(1)}(a)| = |\mu_\hbar^{(n)}(a)|$ and

$$\rho_{opt}(G_{a,\hbar_{opt}}) = \frac{\lambda_{max} - \lambda_{min}}{|\lambda_{max} + \lambda_{min} - 2|}.$$

For each Case, by simple computations, we can show that the parameter $\hbar_{opt}$ lies in the corresponding convergence interval. □

Remark 4.2. The cases 7 and 8 of Table 1 show that, when the basic iterative method converges, then the new method converges too if the parameters a and $\hbar$ are chosen from the corresponding convergence intervals.

Theorem 4.3. *Let $\lambda_i = Re(\lambda_i) + iIm(\lambda_i), i = 1, 2, \ldots, n$, be the eigenvalues of G and*

$$Re_{min}(\lambda_i) = \min_{i=1}^{n}(Re(\lambda_i)) \quad \text{and} \quad Re_{max}(\lambda_i) = \max_{i=1}^{n}(Re(\lambda_i)).$$

Let $\theta_i = \frac{|\lambda_i|^2 - 1}{Re(\lambda_i) - 1}, i = 1, 2, \ldots, n$, and

$$\theta_{min} = \min_{i=1}^{n}(\theta_i) \quad \text{and} \quad \theta_{max} = \max_{i=1}^{n}(\theta_i).$$

Suppose that $\alpha = a \neq 0$ and $a \neq \frac{Re_{min}(\lambda_i) + Re_{max}(\lambda_i)}{2}$. Then $\rho(G_{\alpha,\hbar}) < 1$ if the values $Re(\lambda_i)$, $i = 1, 2, \ldots, n$, and the parameters a and $\hbar$ take any values from their domains, as these are defined and given in the Table 2.

Table 2
The possible domains of the parameters $Re(\lambda_i)$, a and $\hbar$.

Case	$Re(\lambda_i)$-domain	a-domain	$\hbar$-domain
1	$Re_{min}(\lambda_i) > 1$	$\frac{\theta_{max}}{2} < a$	$\left(-\infty, \frac{2a}{\theta_{max}-2a}\right) \cup (0, +\infty)$
2	$Re_{min}(\lambda_i) > 1$	$0 < a < \frac{\theta_{min}}{2}$	$\left(0, \frac{2a}{\theta_{max}-2a}\right)$
3	$Re_{min}(\lambda_i) > 1$	$a < 0$	$\left(\frac{2a}{\theta_{max}-2a}, 0\right)$
4	$Re_{max}(\lambda_i) < 1$	$0 < a$ and $\frac{\theta_{max}}{2} < a$	$\left(\frac{2a}{\theta_{min}-2a}, 0\right)$
5	$Re_{max}(\lambda_i) < 1$	$\frac{\theta_{max}}{2} < a < 0$	$\left(0, \frac{2a}{\theta_{min}-2a}\right)$
6	$Re_{max}(\lambda_i) < 1$	$0 < a < \frac{\theta_{min}}{2}$	$(-\infty, 0) \cup \left(\frac{2a}{\theta_{min}-2a}, +\infty\right)$
7	$Re_{max}(\lambda_i) < 1$	$a < \frac{\theta_{min}}{2}$ and $a < 0$	$\left(-\infty, \frac{2a}{\theta_{min}-2a}\right) \cup (0, +\infty)$

Proof. Since $b = 0$, from (4.2), we have

$$\begin{aligned} \delta_i &= Im(\lambda_i)^2 + (a - Re(\lambda_i))^2 - (1-a)^2 = (\theta_i - 2a)(Re(\lambda_i) - 1), \\ \xi_i &= 2a(1 - Re(\lambda_i)), \end{aligned} \tag{4.3}$$

and $g_i(\hbar), i = 1, 2, \ldots, n$, has real simple roots

$$\hbar_1^{(i)} = 0 \quad \text{and} \quad \hbar_2^{(i)} = \frac{-\xi_i}{\delta_i} = \frac{2a}{\theta_i - 2a}.$$

Case (1): For $Re_{min}(\lambda_i) > 1$ and $a > \frac{\theta_{max}}{2}$, we have $\theta_i > 0, \theta_i - 2a < 0$, and $\delta_i < 0$. So, for each $\hbar \in (-\infty, \frac{2a}{\theta_{max}-2a}) \cup (0, +\infty)$, we have $g_i(\hbar) < 0, i = 1, 2, \ldots, n$, which imply that $\rho(G_{\alpha,\hbar}) < 1$.

In a similar way, we can prove the other cases. □

Remark 4.4. For spacial value $\alpha = a = 1$, the new iterative method (presented in Section 2 and discussed in Sections 3 and 4) was introduced by the present authors in [10]. For other uses of the Taylor expansion in the field of linear and nonlinear analysis see Iliev and Kyurkchiev [6].

5. Numerical examples

In this section, we provide numerical experiments to illustrate the theoretical results obtained in Section 3 and 4. All numerical experiments are carried out using Matlab 7.12. We consider three matrices from the University of Florida Sparse Matrix Collection [2]. The test matrices and their properties are shown in Table 3.

Firstly, we used the Gauss–Seidel method (as the basic method) and the new method for the matrices pde225 and pde900. The two parameters $\hbar$ and $\alpha = a + ib$ are chosen according to the conditions imposed in Theorem 3.2. In Table 4 we report two parameters $\hbar, a$ (from the assumption $|a| = 1$, we have $b = \sqrt{1-a^2}$), and the spectral radii $\rho(G), \rho(G_{\alpha,\hbar})$. This table shows that $\rho(G_{\alpha,\hbar}) < \rho(G)$. From this numerical experiment, we get that the results are in concord with Theorem 3.2.

We considered the matrix mesh1e1 and the Richardson method as the basic iterative method to illustrate the results obtained from Theorem 4.1. The eigenvalues of the corresponding iteration matrix are real and $\lambda_{min} = \lambda_1 = -8.1342$, $\lambda_{max} = \lambda_n = -0.7401$, and $\rho(G) = 8.1342$. In Table 5, we report a and b $(\alpha = a + ib)$, $\hbar$, $\hbar$-domain, and $\rho(G_{\alpha,\hbar})$. For $b = 0$, the values of $\hbar_{opt}$ and ρ_{opt} are also presented in the last line of this table. We observe that by choosing the parameters α and $\hbar$ according to the conditions imposed in Theorem 4.1, we can construct a convergent series solution.

Finally, we considered the matrices pde225, pde900, and the AOR method as the basic iterative method with $\gamma = 1$ and $\omega = 3$. For the matrices pde225 and pde900, we have

$$\rho(L_{\gamma,\omega}) = 3.5670, Re_{min}(\lambda_i) = -2.9352, Re_{max}(\lambda_i) = 0.0981, \theta_{min} = -3.8687, \theta_{max} = 1.0981,$$

and

$$\rho(L_{\gamma,\omega}) = 2.4455, Re_{min}(\lambda_i) = -2.4455, Re_{max}(\lambda_i) = 0.7457, \theta_{min} = -1.4455, \theta_{max} = 1.7457,$$

Table 3
Test problem information.

Matrix	Order	nnz	Symmetric	Positive definite	Condition number
pde225	225	1065	no	no	39.0638
pde900	900	4380	no	no	152.562
mesh1e1	48	306	yes	yes	5.24933

Table 4
The basic method is convergent (Theorem 3.2).

pde225 $\rho(GS) = 0.9776$			pde900 $\rho(GS) = 0.9152$		
a	$\hbar$	$\rho(G_{\alpha,\hbar})$	a	$\hbar$	$\rho(G_{\alpha,\hbar})$
1	−0.75	0.7745	−1	−0.98	0.8870
0.9	−0.52	0.8565	−0.9	−0.8	0.8982
0.8	−0.5	0.9299	−0.8	−0.99	0.9075
0.7	−0.6	0.9599	−0.6	−0.9	0.9100

Table 5
The basic method is divergent (Theorem 4.1).

$a = 1$,	$\hbar$-domain$= (-0.2190, 0)$		$a = -4$,	$\hbar$-domain$= (-\infty, -9.2396) \cup (0, +\infty)$	
b	$\hbar$	$\rho(G_{\alpha,\hbar})$	b	$\hbar$	$\rho(G_{\alpha,\hbar})$
1	−0.1790	0.8282	1	9.1510	0.6956
1	−0.2200	1.0059	3	9.1510	0.7803
1	0.1000	1.4846	−5	9.1510	0.8582
0	−0.1790	0.6886	0	−30	0.8769
0	0.2200	1.0095	0	−9.2300	1.0002
0	0.1000	1.9134	0	1	0.8067
0	$\hbar_{opt} = -0.1839$	$\rho_{opt} = 0.68$	0	$\hbar_{opt} = 9.1510$	$\rho_{opt} = 0.68$

Table 6
The basic method is divergent (Theorem 4.3).

pde225				pde900			
a	$\hbar$-domain	$\hbar$	$\rho(G_{\alpha,\hbar})$	a	$\hbar$-domain	$\hbar$	$\rho(G_{\alpha,\hbar})$
3	$(-0.6080, 0)$	−0.23	0.9183	2	$(-0.7345, 0)$	−0.75	1.0678
		−0.6	0.9776			−0.7	0.8993
		0.5	1.4919			0.5	1.6903
		−0.7	1.3238			−0.63	0.8830
−2	$(-\infty. -30.4685) \cup (0, +\infty)$	0.5	0.8712	−1	$(-\infty, -3.6069) \cup (0, +\infty)$	0.25	0.9576
		−29	1.0012			−1	2.4389
		−31	0.9996			−3.8	0.9730
		−0.75	11.9529			−3.5	1.0036

respectively. The values of a and b ($\alpha = a + ib$), $\hbar$, the convergence domain of $\hbar$($\hbar$-domain), and $\rho(G_{\alpha,\hbar})$ are listed in Table 6. From this table, we see that these results accord with Theorem 4.3 and we conclude that the series solution converge if the parameters α and $\hbar$ are chosen according to the conditions imposed in Theorem 4.3.

6. Conclusion

In this paper, based on the generalized Taylor expansion, we proposed a new method which can be applied to the classical iterative methods for computing a convergent series solution for the linear system of equations. The theoretical results show that the new method can be used to accelerate the convergence of the basic iterative methods. In addition, we show that by applying the new method to a divergent iterative scheme, it is possible to construct a convergent series solution when the iteration matrix G of the iterative scheme has particular properties. The numerical experiments confirm the theoretical results and show the efficiency of the new method.

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References

[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in Mathematical Sciences, Academic Press, SIAM, New York, Philadelphia, PA, 1979, 1994.
[2] T.A. Davis, Y. Hu. The University of Florida Sparse Matrix Collection. ACM Transactions on Mathematical Software (to appear), <http://www.cise.ufl.edu/research/sparse/matrices>.

[3] A. Hadjidimos, A. Yeyios, Symmetric accelerated overrelaxation (SAOR) method, Math. Comput. Simul. XXlV (1982) 72–76.
[4] L.A. Hageman, D.M. Young, Applied Iterative Methods, Academic Press, New York, 1981.
[5] T.Z. Huang, X.Z. Wang, Y.D. Fu, Improving Jacobi methods for negative H-matrices linear systems, Appl. Math. Comput. 186 (2007) 1542–1550.
[6] A. Iliev, N. Kyurkchiev, Nontrivial Methods in Numerical Analysis: Selected Topics in Numerical Analysis, LAP LAMBERT Academic Publishing (2010).
[7] Y. Li, C. Li, S. Wu, Improvements of preconditioned AOR iterative methods for L-matrices, J. Comput. Appl. Math. 206 (2007) 656–665.
[8] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton, 2003.
[9] C.S. Liu, The essence of the homotopy analysis method, Appl. Math. Comput 216 (2010) 1299–1303.
[10] H. Nasabzadeh, F. Toutounian, Convergent Homotopy Analysis for solving linear systems, Advanc. Num. Ana., 2013, doi: http://dx.doi.org/10.1155/2013/732032.
[11] H. Wang, Y. -t. Li, A new preconditioned AOR iterative method for L-matrices, J. Comput. Appl. Math. 229 (2009) 47–53.
[12] L. Wang, Y. Song, Preconditioned AOR iterative methods for M-matrices, Comput. Appl. Math 226 (2009) 114–124.
[13] Y. Saad, Iterative Methods for Sparse Linear Systems, PWS Press, New York, 1995.
[14] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood CliGs, NJ, 1962;
R.S. Varga, Springer Series in Computational Mathematics, 27, Springer-Verlag, Berlin, 2000.
[15] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press. New York, London, 1971.
[16] J.H. Yun, A note on preconditioned AOR method for L-matrices, J. Comput. Appl. Math. 220 (2008) 13–16.
[17] Y. Zhang, T.-Z. Hung, Modified iterative methods for nonnegative matrices and M-matrices linear systems, Comput. Math. Appl. 50 (2005) 1587–1602.