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A Quotient OF Topological Fundamental Groups

Hamid Torabi

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, Mashhad, Iran
hamid_torabi86@yahoo.com

Abstract

In this talk, we discuss on the topological properties of a quotient of topological fundamental groups via a new subgroups of fundamental group, namely small generated subgroup, constructed by small loops which presence of them is equivalent to absence of homotopically Hausdorffness properties.

1 Introduction

In 2002, a work of Biss initiated the development of a theory in which the familiar fundamental group $\pi_1(X, x)$ of a topological space X becomes a topological space denoted by $\pi_1^{top}(X, x)$ by endowing it with the quotient topology inherited from the path components of based loops in X with the compact-open topology. Among other things, Biss claimed that $\pi_1^{top}(X, x)$ is a topological group. However, there is a gap in his proof. Brazas discovered some interesting counterexamples for continuity of multiplication in $\pi_1^{top}(X, x)$ (for more details, see [1]).

In fact, $\pi_1^{top}(X, x)$ was a quasitopological group, that is, a group with a topology such that inversion and all translations are continuous. Althogh, Brazas by removing some open subsets of $\pi_1^{qtop}(X, x)$ make it a topological group, but it is an interesting question that when these two topologies are equivalent. In the sequel, by introducing some spaces, we give a partial answer to this question.

If a space X is not homotopically Hausdorff, then there exist $x \in X$ and a nontrivial loop in X based at x which is homotopic to a loop in every neighborhood

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U of x. Z. Virk [4] called these loops as small loops and showed that for every $x \in X$ they form a subgroup of $\pi_1(X, x)$ which is named small loop group and denoted by $\pi_1^s(X, x)$. In general, various points of X have different small loop groups and hence in order to have a subgroup independent of the base point, Virk [4] introduced the SG (small generated) subgroup, denoted by $\pi_1^{sg}(X, x)$, as the subgroup generated by the following set

$$\{[\alpha * \beta * \alpha^{-1}] \mid [\beta] \in \pi_1^s(X, \alpha(1)),\ \alpha \in P(X, x)\},$$

where $P(X, x)$ is the space of all paths from I into X with initial point x ((see [3] for further details)).

Throughout this article, all the homotopies between two paths are relative to end points, X is a topological space with the base point $x \in X$.

2 Main results

Definition 2.1. ([4]) The *small loop group* $\pi_1^s(X, x)$ of (X, x) is the subgroup of the fundamental group $\pi_1(X, x)$ consisting of all homotopy classes of small loops. The SG subgroup of $\pi_1(X, x)$, denoted by $\pi_1^{sg}(X, x)$, is the subgroup generated by the following set

$$\{[\alpha * \beta * \alpha^{-1}] \mid [\beta] \in \pi_1^s(X, \alpha(1)),\ \alpha \in P(X, x)\},$$

where $P(X, x)$ is the space of all paths in X with initial point x.

Definition 2.2. We call a space X semi-locally small generated if and only if for each $x \in X$ there exists an open neighborhood U of x such that $i_*\pi_1(U, x) \leq \pi_1^{sg}(X, x)$, where $i : U \hookrightarrow X$ is the inclusion map.

Theorem 2.3. If (X, x) is a pointed topological space and U is an open neighborhood of the identity element $[e_x] \in \pi_1^{top}(X, x)$, then $\pi_1^{sg}(X, x) \subseteq U$.

Corollary 2.4. Every nonempty open or closed subset of $\pi_1^{qtop}(X, x)$ is a disjoint union of some cosets of $\pi_1^{sg}(X, x)$.

Proof. Since $\pi_1^{qtop}(X, x)$ is the disjoint union of all cosets of $\pi_1^{sg}(X, x)$, it suffices to prove the theorem for open subsets of $\pi_1^{qtop}(X, x)$. For this, let V be a nonempty open subset of $\pi_1^{qtop}(X, x)$ and $g \in V$. Then $g^{-1}V$ is an open subset of $\pi_1^{qtop}(X, x)$ containing $[e_x]$ and hence by Theorem 2.3 , $\pi_1^{sg}(X, x) \subseteq g^{-1}V$ which implies that $g\pi_1^{sg}(X, x) \subseteq V$. Hence $V = \bigcup_{g \in V} g\pi_1^{sg}(X, x)$. □

The natural quotient map $p : \pi_1^{qtop}(X, x) \longrightarrow \frac{\pi_1(X,x)}{\pi_1^{sg}(X,x)}$ induce the quotient topology on the algebraic quotient group $\frac{\pi_1(X,x)}{\pi_1^{sg}(X,x)}$ which we denote it by $(\frac{\pi_1(X,x)}{\pi_1^{sg}(X,x)})^{top}$. By the previous corollary we can prove that:

Theorem 2.5. For a topological space X, $\pi_1^{qtop}(X,x)$ is a topological group if an only if $(\frac{\pi_1(X,x)}{\pi_1^{sg}(X,x)})^{top}$ is topological group.

Theorem 2.6. For a topological space X, $\pi_1^{qtop}(X,x)$ is a indiscrete topological group if an only if $(\frac{\pi_1(X,x)}{\pi_1^{sg}(X,x)})^{top}$ is indiscrete topological group.

Theorem 2.7. For a topological space X, $(\frac{\pi_1(X,x)}{\pi_1^{sg}(X,x)})^{top}$ is discrete topological group if and only if X is semi-locally small generated.

Corollary 2.8. If X is semi-locally small generated, then $\pi_1^{qtop}(X,x)$ is topological group.

By the following example, we use Theorem 2.5 to find a non semi-locally small generated space with topological fundamental group as a topological group.

Example 2.9. Let HA be the Harmonic Archipelago space and let $X = [0,1] \cup_n (\{1/n\} \times HA_{n+1})$, where HA_n is scaled Harminic Archipelago by the scaler $1/n$. $\pi_1^{qtop}(X,x)$ is topological group since $(\frac{\pi_1(X,x)}{\pi_1^{sg}(X,x)})^{top}$ is topological group.

References

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