A Quotient OF Topological Fundamental Groups

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Abstract

In this talk, we discuss on the topological properties of a quotient of topological fundamental groups via a new subgroups of fundamental group, namely small generated subgroup, constructed by small loops which presence of them is equivalent to absence of homotopically Hausdorffness properties.

1 Introduction

In 2002, a work of Biss initiated the development of a theory in which the familiar fundamental group $\pi_1(X, x)$ of a topological space $X$ becomes a topological space denoted by $\pi_1^{top}(X, x)$ by endowing it with the quotient topology inherited from the path components of based loops in $X$ with the compact-open topology. Among other things, Biss claimed that $\pi_1^{top}(X, x)$ is a topological group. However, there is a gap in his proof. Brazas discovered some interesting counterexamples for continuity of multiplication in $\pi_1^{top}(X, x)$ (for more details, see [1]).

In fact, $\pi_1^{top}(X, x)$ was a quasitopological group, that is, a group with a topology such that inversion and all translations are continuous. Although, Brazas by removing some open subsets of $\pi_1^{top}(X, x)$ make it a topological group, but it is an interesting question that when these two topologies are equivalent. In the sequel, by introducing some spaces, we give a partial answer to this question.

If a space $X$ is not homotopically Hausdorff, then there exist $x \in X$ and a nontrivial loop in $X$ based at $x$ which is homotopic to a loop in every neighborhood

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U of x. Z. Virk [4] called these loops as small loops and showed that for every \( x \in X \) they form a subgroup of \( \pi_1(X, x) \) which is named small loop group and denoted by \( \pi_1^s(X, x) \). In general, various points of X have different small loop groups and hence in order to have a subgroup independent of the base point, Virk [4] introduced the SG (small generated) subgroup, denoted by \( \pi_1^s(X, x) \), as the subgroup generated by the following set

\[
\{ [\alpha * \beta * \alpha^{-1}] | [\beta] \in \pi_1^s(X, x), \alpha \in P(X, x) \},
\]

where \( P(X, x) \) is the space of all paths from \( I \) into \( X \) with initial point \( x \) (see [3] for further details).

Throughout this article, all the homotopies between two paths are relative to end points, \( X \) is a topological space with the base point \( x \in X \).

2 Main results

Definition 2.1. ([4]) The small loop group \( \pi_1^s(X, x) \) of \( (X, x) \) is the subgroup of the fundamental group \( \pi_1(X, x) \) consisting of all homotopy classes of small loops. The SG subgroup of \( \pi_1(X, x) \), denoted by \( \pi_1^s(X, x) \), is the subgroup generated by the following set

\[
\{ [\alpha * \beta * \alpha^{-1}] | [\beta] \in \pi_1^s(X, x), \alpha \in P(X, x) \},
\]

where \( P(X, x) \) is the space of all paths in \( X \) with initial point \( x \).

Definition 2.2. We call a space \( X \) semi-locally small generated if and only if for each \( x \in X \) there exists an open neighborhood \( U \) of \( x \) such that \( i_*\pi_1(U, x) \leq \pi_1^s(X, x) \), where \( i : U \hookrightarrow X \) is the inclusion map.

Theorem 2.3. If \( (X, x) \) is a pointed topological space and \( U \) is an open neighborhood of the identity element \( [e_x] \in \pi_1^{top}(X, x) \), then \( \pi_1^s(X, x) \subseteq U \).

Corollary 2.4. Every nonempty open or closed subset of \( \pi_1^{top}(X, x) \) is a disjoint union of some cosets of \( \pi_1^s(X, x) \).

Proof. Since \( \pi_1^{top}(X, x) \) is the disjoint union of all cosets of \( \pi_1^s(X, x) \), it suffices to prove the theorem for open subsets of \( \pi_1^{top}(X, x) \). For this, let \( V \) be a nonempty open subset of \( \pi_1^{top}(X, x) \) and \( g \in V \). Then \( g^{-1}V \) is an open subset of \( \pi_1^{top}(X, x) \) containing \( [e_x] \) and hence by Theorem 2.3, \( \pi_1^s(X, x) \subseteq g^{-1}V \) which implies that \( g\pi_1^s(X, x) \subseteq V \). Hence \( V = \bigcup_{g \in V} g\pi_1^s(X, x) \).

The natural quotient map \( p : \pi_1^{top}(X, x) \rightarrow \pi_1^s(X, x) \) induce the quotient topology on the algebraic quotient group \( \pi_1^s(X, x) / \pi_1^s(X, x) \) which we denote it by \( (\pi_1^s(X, x))^{top} \). By the previous corollary we can prove that:
Theorem 2.5. For a topological space $X$, $\pi_1^{q\text{top}}(X, x)$ is a topological group if and only if $\left( \frac{\pi_1(X, x)}{\pi_1^y(X, x)} \right)^{\text{top}}$ is a topological group.

Theorem 2.6. For a topological space $X$, $\pi_1^{q\text{top}}(X, x)$ is an indiscrete topological group if and only if $\left( \frac{\pi_1(X, x)}{\pi_1^y(X, x)} \right)^{\text{top}}$ is an indiscrete topological group.

Theorem 2.7. For a topological space $X$, $\left( \frac{\pi_1(X, x)}{\pi_1^y(X, x)} \right)^{\text{top}}$ is a discrete topological group if and only if $X$ is semi-locally small generated.

Corollary 2.8. If $X$ is semi-locally small generated, then $\pi_1^{q\text{top}}(X, x)$ is a topological group.

By the following example, we use Theorem 2.5 to find a non semi-locally small generated space with topological fundamental group as a topological group.

Example 2.9. Let $HA$ be the Harmonic Archipelago space and let $X = [0, 1] \cup \{1/n\} \times HA_{n+1}$, where $HA_n$ is scaled Harmonic Archipelago by the scaler $1/n$. $\pi_1^{q\text{top}}(X, x)$ is topological group since $\left( \frac{\pi_1(X, x)}{\pi_1^y(X, x)} \right)^{\text{top}}$ is a topological group.

References


