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Changing Productivity Index Using a Special DMU in DEA

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Abstract
The Malmquist Index is the prominent index for measuring the productivity change in Decision Making Units (DMUs) in multiple time periods that use Data Envelopment Analysis (DEA) models with Variable Return to Scale (VRS) and Constant Return to Scale (CRS) technology. In this paper, we compute the Malmquist Index based on common weights evaluation, and using this method we can rank DMUs according to logical criteria.

Keywords: Data Envelopment Analysis, Common Weights, Malmquist Index, Anti Ideal.

Introduction

Data Envelopment Analysis (DEA) is a mathematical programming technique that measures the relative efficiency of Decision Making Units (DMUs) with multiple inputs and outputs. Charnes et al. (1978) first proposed DEA as an evaluation tool to measure and compare the relative efficiency of DMUs. Their model assumed Constant Returns to Scale (CRS, the CCR model).The model with Variable Return to Scale (VRS, the BCC model) was developed by Banker et al. (1984). The Malmquist Index is the most important index for measuring the relative productivity change in DMUs in multiple time periods. This index was introduced by Caves et al. (1982).Later, Fare, Gross Kopf, Lindgren, and Ross (FGLR, Fare et al, 1992), and … (FGNZ, Fare et al., 1994) used DEA model (CRS) and VRS for measuring the Malmquist Index. The rest of the paper is organized as follows: Sections 2 describes DEA. In Section 3, we explain computation of common weight. Section 4 deals with computing of efficiency using common weight in different periods. In Section 5, we compute Malmquist Index based on common weight through defining Anti Ideal DMU. The last section summarizes the main points and draws conclusions.

Data Envelopment Analysis (DEA)

Assuming that there are n DMUs, each with m inputs and s outputs, the relative efficiency of a particular DMUo $\left(o \in \{1,2,\ldots,n\}\right)$ is obtained by solving the following fractional programming problem:

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$$\theta_o = \max \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

$$\begin{aligned} & subject\ to: \\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1 \qquad j = 1,2,\ldots,n \qquad\qquad (1) \\ & u_r \ge 0 \qquad r = 1,2,\ldots.,s \\ & v_i \ge 0 \qquad i = 1,2,\ldots.,m \end{aligned}$$

where j is the DMU index, $j = 1,2,\ldots,n, r$ is the output index, $r = 1,2,\ldots,s$ and i the input index $i = 1,2,\ldots,m$, y_{ri} the value of the rth output for the $\mathbf{j}$th DMU, x_{ij} the value of the $\mathbf{i}$ input for the $\mathbf{j}^{th}$ DMU, u_r the weight given to the rth output, v_i the weight given to the $\mathbf{i}$ input. DMU_O is efficient if and only if $w_o = 1$.

DMU_O selects weights that maximize its output-to-input ratio, subject to the constraints. A relative efficiency score of 1 indicates that the DMU under consideration is efficient, whereas a score less than 1 implies that it is inefficient. This fractional program can be converted into a linear programming problem where the optimal value of the objective function indicates the relative efficiency of DMUO. The reformulated linear programming problem, also known as the Linear CCR model, is as follows:

$$\theta_o^* = w_o = \max \sum_{r=1}^{s} u_r y_{ro}$$

$$\begin{aligned} & subject\ to: \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad j = 1,2,\ldots,n \qquad\qquad (2) \\ & u_r \ge 0 \qquad r = 1,2,\ldots.,s \\ & v_i \ge 0 \qquad i = 1,2,\ldots.,m \end{aligned}$$

Common Weight

(For more information about this subject see Jahanshahloo et. al., 2010.)

Definition 1

The virtual positive anti ideal DMU is a DMU with maximize inputs of all of DMUs as its input and minimize outputs of all DMUs as its output. That is, if we show positive ideal DMU with $\overline{DMU} = (\bar{X}, \bar{Y})$, then $\bar{x}_i = \max\{x_{ij} \mid j = 1,2,\ldots,n\}, (i = 1,2,\ldots,m)$ and $\bar{y}_r = \min\{y_{rj} \mid j = 1,2,\ldots,n\}, (r = 1,2,\ldots,s)$.

Definition 2

An ideal level is one straight line that passes through the origin and positive ideal DMU with slope 1.0. In Fig.1, the vertical and horizontal axes are set to be the virtual output (weighed sum of s outputs) and the virtual input (weighed sum of m inputs), respectively. Also, ox is an ideal line, and $\overline{DMU} = (\sum_{i=1}^{m} \bar{x}_i v_i', \sum_{r=1}^{s} \bar{y}_r u_r')$ is an ideal DMU. The notation of a decision variable with superscript symbols","represents an arbitrarily assigned value. For any DMU_N, DMU_M, if given one set of weights

u'_r $(r = 1,2,\dots,s)$ and v'_i $(i = 1,2,\dots,m)$, then the coordinate of points M', N' and N0 in Fig. 1 are $(\sum_{i=1}^{m} x_{iM} v'_i, \sum_{r=1}^{s} y_{rM} u'_r)$ and $(\sum_{i=1}^{m} x_{iN} v'_i, \sum_{r=1}^{s} y_{rN} u'_r)$. The virtual gaps between points M' and M'^p on the horizontal axes and vertical axes are denoted as $\Delta_{M'}^{I}$ and $\Delta_{M'}^{O}$, respectively. Similarly, for points N' and N'^p, the gaps are $\Delta_{N'}^{I}$ and $\Delta_{N'}^{O}$. We observe that there exists a total virtual gap $\Delta_{M'}^{I} + \Delta_{M'}^{O} + \Delta_{N'}^{O} + \Delta_{N'}^{O}$to the ideal line. Let the notation of a decision variable with superscript", " represent the optimal value of the variable. We want to determine an optimal set of weights u_r^* $(r = 1,2,\dots,s)$ and v_i^* $(i = 1,2,\dots,m)$such that both points M^* and N^* below the ideal line could be as close to their projection points, M^{*p} and N^{*p} on the ideal line, as possible. In other words, by adopting the optimal weights, the total virtual gaps $\Delta_{M^*}^{I} + \Delta_{M^*}^{O} + \Delta_{N^*}^{O} + \Delta_{N^*}^{O}$to the ideal line is the shortest to both DMUs. As for the constraint, the numerator is the weighted sum of outputs plus the vertical gap Δ_j^O, and the denominator is the weighted sum of inputs minus the horizontal virtual gap Δ_j^I. The constraint implies that the direction closest to the ideal line is upward and leftward at the same time. The ratio of the numerator to the denominator equals 1.0, which means that the projection point on the ideal line is reached. Therefore, we have the following model:

$$\Delta^* = \min \sum_{j=1}^{n} \Delta_j^I + \Delta_j^O$$

$$\text{S.t} \quad \frac{\sum_{r=1}^{s} u_r \bar{y}_r}{\sum_{i=1}^{m} v_i \bar{x}_i} = 1$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \Delta_j^O}{\sum_{i=1}^{m} v_i x_{ij} - \Delta_j^I} = 1, \qquad j = 1,2,\dots,n \qquad (3)$$

$$\Delta_j^I, \Delta_j^O \geq 0, \qquad j = 1,2,\dots,n$$

$$u_r \geq \epsilon > 0, \qquad r = 1,2,\dots,s$$

$$v_i \geq \epsilon > 0, \qquad i = 1,2,\dots,m$$

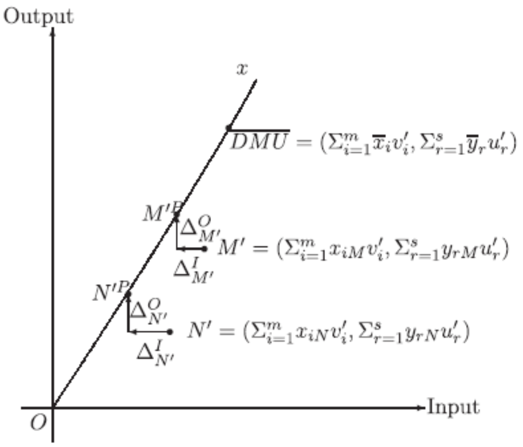

Fig. 1. Gap analysis showing *DMU* below the virtual ideal line.

$\epsilon$is positive Archimedean infinitesimal constant. The ratio form of constrains (1) can be rewritten in a linear form, so we have the following model:

$$\Delta^* = \min \sum_{j=1}^{n} \Delta_j^I + \Delta_j^O$$

$$\text{S.t} \quad \sum_{r=1}^{s} u_r \bar{y}_r - \sum_{i=1}^{m} v_i \bar{x}_i = 0$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \Delta_j^I + \Delta_j^O = 0 \qquad j = 1,2,\dots,n \quad (4)$$
$$\Delta_j^I, \Delta_j^O \geq 0, \qquad j = 1,2,\dots,n$$
$$u_r \geq \epsilon > 0, \qquad r = 1,2,\dots,s$$
$$v_i \geq \epsilon > 0, \qquad i = 1,2,\dots,m$$

Then, if we let $\Delta_j^I + \Delta_j^O$, be Δ_j (4), then simplified to the following linear programming (5).

$$\Delta^* = \min \sum_{j=1}^{n} \Delta_j$$
$$\text{S.t} \quad \sum_{r=1}^{s} u_r \bar{y}_r - \sum_{i=1}^{m} v_i \bar{x}_i = 0 \qquad (*)$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \Delta_j = 0 \qquad j = 1,2,\dots,n \quad (5)$$
$$\Delta_j \geq 0, \qquad j = 1,2,\dots,n$$
$$u_r \geq \epsilon > 0, \qquad r = 1,2,\dots,s$$
$$v_i \geq \epsilon > 0, \qquad i = 1,2,\dots,m$$

If a DMUj wason positive ideal, then we use the definition of the CWA efficiency score of DMUj that Liu and Peng (2006) defined as the following equation:

$$\theta_j^* = \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}} \qquad j = 1,2,\dots,n \qquad (6)$$

Therefore, the CWA efficiency score of it is 1.0. So that constrain (*) in (5) becomes redundant and this model becomes the same as the CWA model in Liu and Peng (2006). On the other hand, the ideal line is the benchmark line. We conclude that the CWA model is a special case of (5) in this paper. Therefore, DMUj is CWA efficient if $\Delta_j^* = 0$ or $\theta_j^* = 1$.Otherwise, DMUj is CWA inefficient.

Definition 3

The performance of DMUj is better than that of DMUi if $\Delta_j < \Delta_i$(For more information on this subject, see Jahanshahloo et al., 2010).

Computing Efficiency using common weights in different periods and different models of DEA

We can compute $\theta_{k(t)}^{*t(CRS)}, \theta_{k(t)}^{*t(VRS)}$(ideal DMU and DMUs in period t, frontier period = t).As in the previous section, wherex_{ij}^t, $y_{rj}^t$are substitutedx_{ij}, y_{rj}. ($\theta_{k(t+1)}^{*t+1(CRS)}, \theta_{k(t+1)}^{*t+1(VRS)}$(ideal DMU and DMUs in period t+1, frontier period = t+1)).

DEA model of CRS technology in input orientation, ideal DMU and DMUs in period t, frontier period = t+1.

Phase 1

$$\Delta^{*(t)} = \min \sum_{j=1}^{n} \Delta_j^{\,t}$$
$$\text{S.t} \quad \sum_{r=1}^{s} u_r^{t+1} \bar{y}_r^t - \sum_{i=1}^{m} v_i^{t+1} \bar{x}_i^t = 0$$

$$\sum_{r=1}^{s} u_r^{t+1} y_{rj}^{t+1} - \sum_{i=1}^{m} v_i^{t+1} x_{ij}^{t+1} + \Delta_j{}^t = 0 \qquad j = 1,2,\dots,n \quad (7)$$

$$\Delta_j{}^t \geq 0, \qquad j = 1,2,\dots,n$$

$$u_r^{t+1} \geq \epsilon > 0, \qquad r = 1,2,\dots,s$$

$$v_i^{t+1} \geq \epsilon > 0, \qquad i = 1,2,\dots,m$$

Phase 2

Therefore, by solving Model (7), we obtain $v_i^{*(t+1)}$, $u_r^{*(t+1)}$. So, using common weight, efficiency is:

$$\theta_{j(t)}^{*t+1(CRS)} = \frac{\sum_{r=1}^{s} u_r^{*(t+1)} y_{rj}^{t}}{\sum_{i=1}^{m} v_i^{*(t+1)} x_{ij}^{t}} \qquad j = 1,2,\dots,n \quad (8)$$

DEA model of CRS technology in input orientation, ideal DMU and DMUs in period t+1, frontier period = t.

Phase 1:

$$\Delta^{*(t+1)} = \min \sum_{j=1}^{n} \Delta_j{}^{t+1}$$

$$\text{S.t} \quad \sum_{r=1}^{s} u_r^t \bar{y}_r^{t+1} - \sum_{i=1}^{m} v_i^t \bar{x}_i^{t+1} = 0$$

$$\sum_{r=1}^{s} u_r^t y_{rj}^t - \sum_{i=1}^{m} v_i^t x_{ij}^t + \Delta_j{}^{t+1} = 0 \qquad j = 1,2,\dots,n \quad (9)$$

$$\Delta_j{}^{t+1} \geq 0, \qquad j = 1,2,\dots,n$$

$$u_r^t \geq \epsilon > 0, \qquad r = 1,2,\dots,s$$

$$v_i^t \geq \epsilon > 0, \qquad i = 1,2,\dots,m$$

Phase 2: Therefore, solving Model (9), we obtain $v_i^{*(t)}$, $u_r^{*(t)}$. So, using common weight, efficiency is:

$$\theta_{j(t+1)}^{*t(CRS)} = \frac{\sum_{r=1}^{s} u_r^{*(t)} y_{rj}^{t+1}}{\sum_{i=1}^{m} v_i^{*(t)} x_{ij}^{t+1}} \qquad j = 1,2,\dots,n \quad (10)$$

Likewise, we can compute $\theta_{j(t+1)}^{*t(VRS)}$ *and* $\theta_{j(t)}^{*t+1(VRS)}$.

New Method for computing Malmquist Index based on Common Weights in different models of DEA

According to the computation of $\theta_{j(t)}^{*t(VRS)}$, $\theta_{j(t)}^{*t(CRS)}$, ... in the previous section, consider the following equations:

$$EC_{\theta^*} = \frac{\theta_{(t+1)}^{*t+1(CRS)}}{\theta_{(t)}^{*t(CRS)}} \quad (11) \qquad PEC_{\theta^*} = \frac{\bar{\theta}_{(t+1)}^{t+1(VRS)}}{\bar{\theta}_{(t)}^{t(VRS)}} \quad (12)$$

$$TC_{\theta^*} = \left[\frac{\theta_{(t)}^{*t(CRS)}}{\theta_{(t)}^{*t+1(CRS)}} \times \frac{\theta_{(t+1)}^{*t(CRS)}}{\theta_{(t+1)}^{*t+1(CRS)}}\right]^{\frac{1}{2}} \quad (13)$$

$$SEC_{\theta^*} = \left[\frac{\theta_{(t)}^{*t(VRS)}}{\theta_{(t)}^{*t(CRS)}} \times \frac{\theta_{(t+1)}^{*t+1(CRS)}}{\theta_{(t+1)}^{*t+1(VRS)}}\right] \quad (14)$$

Where $EC_{\theta^*}$is efficiency change based on θ^*, $PEC_{\theta^*}$is pure efficiency change based on θ^*, $TC_{\theta^*}$is technology change based on θ^*, and $SEC_{\theta^*}$is scale efficiency change based on θ^*. The Malmquist Index and its FGLR and FGNZ decompositions are as follows (For more details, see Fare et al., 1992, 1994.). Similarly, we can compute Malmquist Index.

Malmquist Index based on θ^* $(MI_{\theta^*}) = EC_{\theta^*} \times TC_{\theta^*}$(15)

Malmquist Index based on θ^* $(MI_{\theta^*}) = PEC_{\theta^*} \times SEC_{\theta^*} \times TC_{\theta^*}$(16)

We define Malmquist Index Disparity and Expanded Malmquist Index Disparity:

$$MID = \frac{MI_{\theta} - MI_{\theta^*}}{MI_{\theta}} \times 100 \qquad (17)$$

Conclusion

To obtain the relative efficiency of DMUs, we use means of weights, and through this method we compute Malmquist Index. The result seems be quite satisfactory. Using a new method (common weights), we can rank DMUs according to logical criteria. The result of the performance of this method can be seen in a numerical example.

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