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Optimal control of a class of non-linear time-delay systems via hybrid functions

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Optimal control problems for a class of time-delay non-linear systems with quadratic performance index are studied. The properties of the hybrid functions which consist of block-pulse functions and orthonormal Taylor series are presented. By expanding various time functions in the systems as their truncated hybrid functions, we attain to algebraic equations and by means of operational matrices of integration, delay and product we reduce the solution of optimal control problem to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: time-delay system; orthonormalization; operational matrix; hybrid functions.

1. Introduction

The dynamics of many control systems may be expressed by time-delay equations. Delays occur frequently in incubation periods, mechanics, viscoelasticity, physiology, population dynamics, communication, information technologies, maturation times, microbial processes, age structure, biological, chemical, electronic and transportation systems (Hoffman & Kunze, 1971; Jamshidi & Wang, 1984; Jiang & Schaufelberger, 1992; Bernatz, 2010; Farahi & Dadkhah, 2014; Liu, 2015). The delay may appear in the system state, control input and/or output. Therefore, the control of time-delay systems has been interested by many engineers and scientists, due to its variety presence in the realistic models of phenomena. Analytical methods, especially in optimal control of time-delay systems, have less implementation ability, and the application of Pontryagin's maximum principle to the optimization of control systems with time-delays as outlined by Kharatishvili (1961) results in a system of coupled two-point boundary value problem involving both delay and advance terms whose exact solution, except in very special cases, is very difficult (Marzban & Razzaghi, 2004). So different numerical methods have been devised to overcome the problems arising from the applications of analytical methods while the main object of all computational aspects of optimal time-delay systems have been to produce a new method to avoid the solution of the mentioned two-point boundary value problem.

However, it should be noted that most of these approaches are not much developed for optimal control of non-linear time-delay systems. One may assume that for overcoming the complexity of the non-linear time-delay systems in optimal control problems, dynamic programming approaches are one of the most effective techniques (e.g. Dadebo & Luus, 1992), but applications of dynamic programming methods have some difficulties due to the need to provide an appropriate level model and define recursive relationships for each case problem (Borzabadi & Asadi, 2013). Note that the computational algorithm considering a linear approximation of the original system which is defined about a nominal trajectory as in Jamshidi & Wang (1984) is not reliable and may lead to large errors. An approach based on discretization techniques with necessary conditions to obtain approximate optimal control and the state for optimal control problems (with non-linear delay systems) was proposed in Gollman *et al.* (2009), but achieving the necessary conditions in some problems and the implementation of the approach may be faced with difficulties. Koshkouei *et al.* (2012) proposed a method based on measure theory, functional analysis and linear programming and extended it in order to optimize a definite objective function, and to design an appropriate optimal control for the non-linear time-delay systems. Recently, Marzban & Hoseini (2015) considered numerical treatment of non-linear optimal control problems.

Orthogonal functions (OFs) and polynomial series have received considerable attentions in dealing with various problems of dynamic systems (Marcellan & Assche, 2006). For such kind of problem, the approach is that of converting the underlaying differential equation governing the dynamical system to an algebraic form through the use of an operational matrix of integration which can be uniquely determined based on the particular OFs. Special attentions have been given to applications of orthogonal functions in solving control or optimal control problems. For example, in Lo *et al.* (2009), Palanisamy & Rao (1983), Walsh functions are used to solve linear delays control problems, the applications of block-pulse functions in control systems are considered in Deb *et al.* (2006), Jiang & Schaufelberger (1992), Mohammadzadeh & Lakestani (2015), in Kung & Lee (1983) one can see the use of Laguerre polynomials in solving time invariant linear delay systems, Legendre polynomials are also used to solve optimal control of non-linear systems (Marzban & Razzaghi, 2004; Khellat & Yousefi, 2006; Mohan & Kar, 2012). Chebyshev series (Marzban & Shahsiah, 2011; Shaban *et al.*, 2013), Taylor series (Marzban & Razzaghi, 2006), Bernoulli polynomials (Nazemi & Shabani, 2014) and Fourier series (Bernatz, 2010; Farahi & Dadkhah, 2014) are used to solve control or optimal control problems.

The aim of this paper is to introduce a new numerical method to solve the optimal control of non-linear systems. This method consists of reducing the optimal control problem to a set of algebraic equations by expanding the state and control vectors as hybrid functions with unknown coefficients. These hybrid functions, which consist of block-pulse functions and orthonormal Taylor series, are given. The operational matrices of integration and delay are introduced. The necessary conditions of optimality are derived as a system of algebraic equations in the unknown coefficients of state and control vectors, as well as Lagrange multipliers. These coefficients are determined in such a way that the necessary conditions for extremization are imposed.

The method we used here consists of dividing the time interval into N subintervals and approximating the trajectory and control functions by hybrid of block-pulse functions and orthonormal Taylor series in each subinterval. Indeed, by applying this new method, while the accuracy of the approximate solutions increase, but the CPU time and computer needed memory reduce, since the operational matrices have large number of zero elements and they are mostly sparse.

Note that using Taylor series in hybrid approximation has some advantages, some of these important ones are

(a) They allow for incredibly accurate (depending on the number of terms) estimates of common functions.

(b) They provide for integration and differentiation of functions to arrive at representations of other functions.

The obtained results in Example 1,2 of this article and comparing these results by those obtained by some other techniques in Borzabadi & Asadi (2013), Gollman *et al.* (2009), Koshkouei *et al.* (2012), Mohan & Kar (2012) confirm the usefulness of using Taylor series.

The paper is organized as follows: in Section 2, we describe the basic properties of the hybrid functions of block-pulse and orthonormal Taylor series required for our subsequent development. Section 3 is devoted to the formulation of the optimal control problem. In Section 4, we apply the proposed numerical method to some control systems, and we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Hybrid functions and their properties

This section consists of basic definitions and properties of hybrid of block-pulse functions and orthonormal Taylor series, operational matrices of integration, product, dual and delay. Without loss of generality and for the sake of simplicity, we choose time interval equal to $[0, 1]$.

In general, if the time interval be $[0, t_f]$, one can normalize this to $[0, 1]$ by the following map:

$$\tau = \frac{t}{t_f} : [0, t_f] \to [0, 1].$$

2.1 *Hybrid of block-pulse functions and orthonormal Taylor series*

Hybrid functions $H_{n,m}(t)$, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M-1$, which have three arguments; n and m are the order of block-pulse functions and orthonormal Taylor series, respectively, and t is the normalized time, are defined on the interval $[0, 1]$ as

$$H_{n,m}(t) = \begin{cases} \sqrt{N}\mathrm{OT}_m(Nt - n + 1), & \frac{n-1}{N} \leqslant t < \frac{n}{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $\mathrm{OT}_m(t)$s are orthonormal Taylor series governed by the Gram–Schmidt orthonormalization process on $T_m = \{1, t, t^2, t^3, \ldots, t^m\}$, and the time interval is $[0, 1]$ with the weight function $w(t) = 1$ (Marcellan & Assche, 2006). For example, if $M = 5$, we have

$$\begin{aligned}
\mathrm{OT}_0(t) &= 1, \\
\mathrm{OT}_1(t) &= (2t-1)\sqrt{3}, \\
\mathrm{OT}_2(t) &= (6t^2 - 6t + 1)\sqrt{5}, \\
\mathrm{OT}_3(t) &= (2t-1)(10t^2 - 10t + 1)\sqrt{7}, \\
\mathrm{OT}_4(t) &= (70t^4 - 140t^3 + 90t^2 - 20t + 1)\sqrt{9}.
\end{aligned}$$

A function $f(t)$ belongs to the space $L^2[0, 1]$ may be expanded by hybrid functions as follows (Mohan & Kar, 2012)

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m} H_{n,m}(t). \tag{2}$$

By truncating the series (2), we can obtain an approximation for $f(t)$ as follows

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} H_{n,m}(t) = C^{\top} H(t), \tag{3}$$

where

$$C = [c_{1,0} c_{1,1} \dots c_{1,M-1} c_{2,0} c_{2,1} \dots c_{2,M-1} \dots c_{N,0} \dots c_{N,M-1}]^{\top},$$

and

$$\begin{aligned} H(t) = [&H_{1,0}(t) H_{1,1}(t) \dots H_{1,M-1}(t) H_{2,0}(t) H_{2,1}(t) \dots H_{2,M-1}(t) \\ &\dots H_{N,0}(t) \dots H_{N,M-1}(t)]^{\top}, \end{aligned} \tag{4}$$

while $c_{n,m}$, $n = 1, 2, \dots, N$, $m = 0, 1, \dots, M-1$ are the coefficients expansion of the function $f(t)$ in the nth subinterval $[(n-1)/N, n/N)$.

We have $c_{n,m} = \langle f(t), H_{n,m}(t) \rangle$ and $\langle . \rangle$ is the standard inner product on $L^2[0, 1]$.

2.2 *On the convergence of the method*

Let $K = L^2[0, 1]$, and the set

$$H(t) = \{H_{1,0}(t) H_{1,1}(t) \dots H_{1,M-1}(t) H_{2,0}(t) H_{2,1}(t) \dots H_{2,M-1}(t)\}, \quad 0 \leqslant t \leqslant 1$$

be a subset of K. Define

$$Y = \operatorname{Span}(H(t)).$$

Suppose $f(t) \in K$. Now we have Lemma 2.1.

LEMMA 2.1 For every arbitrary element $f(t) \in K$, there exists a unique best approximation out of Y such as $f_0 \in Y$, that is

$$\forall g \in Y \rightarrow \|f - f_0\|_2 \leqslant \|f - g\|_2.$$

Proof. See Kreyszig (1978) and Watson (1980). □

We know that $f_0 \in Y$, so there exist unique coefficients

$$\{c_{1,0} c_{1,1} \dots c_{1,M-1} c_{2,0} c_{2,1} \dots c_{2,M-1} \dots c_{N,0} \dots c_{N,M-1}\}$$

such that

$$f \simeq f_0 = C^{\top} H(t),$$

where

$$\begin{aligned} C &= [c_{1,0} c_{1,1} \dots c_{1,M-1} c_{2,0} c_{2,1} \dots c_{2,M-1} \dots c_{N,0} \dots c_{N,M-1}]^{\top}, \\ H(t) &= [H_{1,0}(t) H_{1,1}(t) \dots H_{1,M-1}(t) H_{2,0}(t) H_{2,1}(t) \dots H_{2,M-1}(t) \\ &\quad \dots H_{N,0}(t) \dots H_{N,M-1}(t)]^{\top}. \end{aligned}$$

If one assume

$$e(f) = \|f - C^{\top}H(t)\|_2^2 = \int_0^t (f(t) - C^{\top}H(t))^2\,\mathrm{d}t,$$

then $e(f)$ should reduce to zero when N and M tends to infinity. Thus by increasing N and M, one can find more accurate approximation.

2.3 *Operational matrix of integration*

We can approximate the integral of $H(t)$ defined in (4) as follows

$$\int_0^t H(s)\,\mathrm{d}s \simeq PH(t), \tag{5}$$

where P is $MN \times MN$ operational matrix for integration and is given as

$$P = \begin{pmatrix} A_1 & B_1 & B_1 & \dots & B_1 \\ 0 & A_1 & B_1 & \dots & B_1 \\ 0 & 0 & A_1 & \dots & B_1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & A_1 \end{pmatrix}, \tag{6}$$

where

$$B_1 = \frac{1}{N}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(M\times M)},$$

and

$$A_1 = \frac{1}{2N}\begin{pmatrix} 1 & \frac{\sqrt{1}\sqrt{3}}{1\times 3} & 0 & 0 & 0 & \dots & 0 \\ -\frac{\sqrt{1}\sqrt{3}}{1\times 3} & 0 & \frac{\sqrt{3}\sqrt{5}}{3\times 5} & 0 & 0 & \dots & 0 \\ 0 & -\frac{\sqrt{3}\sqrt{5}}{3\times 5} & 0 & \frac{\sqrt{5}\sqrt{7}}{5\times 7} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & 0 \\ 0 & 0 & 0 & \dots & \vdots & \dots & \frac{\sqrt{2M-1}\sqrt{2M-3}}{(2M-1)\times(2M-3)} \\ 0 & 0 & 0 & \dots & \vdots & -\frac{\sqrt{2M-1}\sqrt{2M-3}}{(2M-1)\times(2M-3)} & 0 \end{pmatrix}_{(M\times M)}.$$

2.4 *Dual and product operational matrices*

Since $H_{n,m}(t)$s are disjoint and orthonormal sets on [0, 1], so the dual operational matrix of $H(t)$ is

$$L = \int_0^1 H(t)H^\top(t)\,\mathrm{d}t = \mathbf{I},$$

where $\mathbf{I}$ is MN identity matrix. The following property of the product of two hybrid function vectors can also be used. Let

$$H(t)H^\top(t)C \simeq \tilde{C}H(t), \tag{7}$$

where $\tilde{C}$ is an $MN \times MN$ product operational matrix. To illustrate the calculation procedure, choose $N = 3$ and $M = 3$. Thus we have

$$C = [c_{1,0}c_{1,1}c_{1,2}\ldots c_{3,0}c_{3,1}c_{3,2}]^\top, \tag{8}$$

and

$$H(t) = [H_{1,0}(t)H_{1,1}(t)H_{1,2}(t)\ldots H_{3,0}(t)H_{3,1}(t)H_{3,2}(t)]^\top, \tag{9}$$

$$\left.\begin{aligned} H_{1,0} &= \sqrt{3} \\ H_{1,1} &= \sqrt{3}(6t-1)\sqrt{3} \\ H_{1,2} &= \sqrt{3}(54t^2-18t+1)\sqrt{5} \end{aligned}\right\} \quad 0 \leqslant t < \tfrac{1}{3},$$
$$\left.\begin{aligned} H_{2,0} &= \sqrt{3} \\ H_{2,1} &= \sqrt{3}(6t-3)\sqrt{3} \\ H_{2,2} &= \sqrt{3}(54t^2-54t+13)\sqrt{5} \end{aligned}\right\} \quad \tfrac{1}{3} \leqslant t < \tfrac{2}{3}, \tag{10}$$
$$\left.\begin{aligned} H_{3,0} &= \sqrt{3} \\ H_{3,1} &= \sqrt{3}(6t-5)\sqrt{3} \\ H_{3,2} &= \sqrt{3}(54t^2-90t+37)\sqrt{5} \end{aligned}\right\} \quad \tfrac{2}{3} \leqslant t \leqslant 1.$$

We also have

$$H(t)H^\top(t) = \begin{pmatrix} H_{1,0}H_{1,0} & H_{1,0}H_{1,1} & H_{1,0}H_{1,2} & & & & \\ H_{1,1}H_{1,0} & H_{1,1}H_{1,1} & H_{1,1}H_{1,2} & & & \bigcirc & \\ H_{1,2}H_{1,0} & H_{1,2}H_{1,1} & H_{1,2}H_{1,2} & & & & \\ & & & \ddots & & & \\ & \bigcirc & & & H_{3,0}H_{3,0} & H_{3,0}H_{3,1} & H_{3,0}H_{3,2} \\ & & & & H_{3,1}H_{3,0} & H_{3,1}H_{3,1} & H_{3,1}H_{3,2} \\ & & & & H_{3,2}H_{3,0} & H_{3,2}H_{3,1} & H_{3,2}H_{3,2} \end{pmatrix},$$

where $\bigcirc$ denotes zero matrix. By using the vector C in (8) the 9×9 matrix $\tilde{C}$ in (7) is

$$\tilde{C} = \begin{pmatrix} \tilde{C}_1 & 0 & 0 \\ 0 & \tilde{C}_2 & 0 \\ 0 & 0 & \tilde{C}_3 \end{pmatrix}, \tag{11}$$

where $\tilde{C}_i$s, $i = 1, 2, 3$ are 3×3 matrices given by

$$\tilde{C}_i = \begin{pmatrix} \sqrt{3}c_{i,0} & \sqrt{3}c_{i,1} & \sqrt{3}c_{i,2} \\ \sqrt{3}c_{i,1} & \sqrt{3}c_{i,0} + 2\dfrac{\sqrt{3}\sqrt{5}}{5}c_{i,2} & 2\dfrac{\sqrt{3}\sqrt{5}}{5}c_{i,1} \\ \sqrt{3}c_{i,2} & 2\dfrac{\sqrt{3}\sqrt{5}}{5}c_{i,1} & \sqrt{3}c_{i,0} + 2\dfrac{\sqrt{3}\sqrt{5}}{7}c_{i,2} \end{pmatrix}. \tag{12}$$

2.5 *Delay operational matrix*

The delay function $H(t-\tau)$ is the shifted of the function $H(t)$ defined in (4), along the time axis by τ. In other words, we have

$$H(t-\tau) = D_\tau H(t), \quad \tau \leqslant t \leqslant 1, \tag{13}$$

where D_τ is the delay operational matrix of hybrid functions. To find D_τ, we first choose N the order of block-pulse functions, in the following manner (Marzban & Razzaghi, 2004)

$$N = \begin{cases} \dfrac{1}{\tau}, & \dfrac{1}{\tau} \in \mathbb{Z}, \\ \left[\dfrac{1}{\tau}\right] + 1 & \text{otherwise}, \end{cases} \tag{14}$$

where [.] denotes greatest integer value.

Note that in the interval $\tau \leqslant t \leqslant 2\tau$, the terms $H_{1,m}(t)$ for $m = 0, 1, \ldots, M-1$ are non- zero and all other terms are zero. So if we expand $H_{1,m}(t)$ in terms of $H_{2,m}(t)$ then the coefficient is an $M \times M$ identity matrix since we have $H_{1,m}(t-\tau) = H_{2,m}(t)$. Similar manner can be used to all other intervals. Thus if we expand $H(t-\tau)$ in terms of $H(t)$, we find $NM \times NM$ matrix D_τ as

$$D_\tau = \begin{pmatrix} 0 & I & 0 & 0 & \ldots & 0 \\ 0 & 0 & I & 0 & \ldots & 0 \\ 0 & 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{15}$$

where I is M identity matrix.

3. Solving non-linear optimal control problems

Consider the following quadratic time-independent delay control system:

$$\min\ J=\frac{1}{2}x^{\top}(1)Sx(1)+\frac{1}{2}\int_0^1\{x^{\top}(t)Qx(t)+u^{\top}(t)Ru(t)\}\,\mathrm{d}t, \tag{16}$$

subject to

$$\begin{aligned}
&\dot{x}(t)=F(x(t),x(t-\tau_1),u(t),u(t-\tau_2)), \quad 0\leqslant t\leqslant 1,\\
&x(t)=h_1(t), \quad -\tau_1\leqslant t\leqslant 0,\\
&u(t)=h_2(t), \quad -\tau_2\leqslant t\leqslant 0,\\
&x(0)=x_0,
\end{aligned} \tag{17}$$

where Q and S are positive semi-definite matrices and R is symmetric positive definite; $x(t)\in R^p$, $u(t)\in R^q$ are state and control vectors, respectively, where the state $x(t)$ is a continuous function and the control $u(t)$ is a measurable piecewise continuous function; x_0 is a constant-specified vector; τ_1,τ_2 are constant positive scalars representing delays; $F(.)$ is assumed to be a continuously differentiable function of its arguments; $h_1(t),h_2(t)$ are arbitrary known functions. The problem is to find optimal $x(t)$ and $u(t)$, $0\leqslant t\leqslant 1$, satisfying (17) while minimizing (16).

Assume that

$$\begin{aligned}
&x(t)=[x_1(t)x_2(t)\dots x_p(t)]^{\top}, \quad u(t)=[u_1(t)u_2(t)\dots u_q(t)]^{\top},\\
&\hat{H}(t)=I_p\otimes H(t), \quad \hat{H_1}(t)=I_q\otimes H(t),
\end{aligned} \tag{18}$$

where I_p and I_q are respectively the p and q identity matrices, and $\otimes$ denotes Kronecker product (Kharatishvili, 1961). Here

$$\hat{H}(t)=I_p\otimes H(t)=\begin{pmatrix}H(t) & \dots & H(t)\\ \vdots & \vdots & \vdots\\ H(t) & \dots & H(t)\end{pmatrix}_{p\times p},$$

$$\hat{H}_1(t)=I_q\otimes H(t)=\begin{pmatrix}H(t) & \dots & H(t)\\ \vdots & \vdots & \vdots\\ H(t) & \dots & H(t)\end{pmatrix}_{q\times q}.$$

Matrices $\hat{H}(t)$ and $\hat{H}_1(t)$ are $pMN\times p$ and $qMN\times q$ matrices, respectively, while $H(t)$ is the vector function defined in (4).

Assume that the functions $x_i(t)$ and $u_j(t), i=1,2,\dots,p, j=1,2,\dots,q$, can be written in terms of hybrid functions as

$$\begin{aligned}
&x_i(t)=H^{\top}(t)X_i,\\
&u_j(t)=H^{\top}(t)U_j.
\end{aligned} \tag{19}$$

Using equations (18–19), we have

$$x(t) = \hat{H}^\top(t)X,$$
$$u(t) = \hat{H}_1^\top(t)U, \tag{20}$$

where

$$X = [X_1X_2 \dots X_p]^\top,$$
$$U = [U_1U_2 \dots U_q]^\top.$$

Similarly, we have (Marzban & Razzaghi, 2006)

$$x(0) = \hat{H}^\top(t)G,$$
$$h_1(t-\tau_1) = \hat{H}^\top(t)K, \tag{21}$$
$$h_2(t-\tau_2) = \hat{H}^\top(t)R,$$

where

$$G = [g_1g_2 \dots g_p]^\top,$$
$$K = [k_1k_2 \dots k_p]^\top,$$
$$R = [r_1r_2 \dots r_q]^\top.$$

are known vectors.

We can also write $x(t-\tau_1)$ and $u(t-\tau_2)$ in terms of hybrid functions as

$$x(t-\tau_1) = \begin{cases} \hat{H}^\top(t)K, & 0 \leqslant t \leqslant \tau_1, \\ \hat{H}^\top(t)\hat{D}_1^\top X, & \tau_1 \leqslant t \leqslant 1, \end{cases}$$
$$u(t-\tau_2) = \begin{cases} \hat{H}^\top(t)R, & 0 \leqslant t \leqslant \tau_2, \\ \hat{H}^\top(t)\hat{D}_2^\top U, & \tau_2 \leqslant t \leqslant 1, \end{cases}$$

where $\hat{D}_1 = I_p \otimes D_{\tau_1}$ and $\hat{D}_2 = I_q \otimes D_{\tau_2}$ and D_{τ_1}, D_{τ_2} are respectively delay operational matrices given in (15).

Moreover, we have

$$\int_0^t \hat{H}^\top(s)\,\mathrm{d}s = (I_p \otimes H^\top(t))(I_p \otimes P^\top) = \hat{H}(t)\hat{P}^\top,$$
$$\int_0^t x(s-\tau_1)\,\mathrm{d}s = \begin{cases} \hat{H}^\top(t)\hat{P}^\top K, & 0 \leqslant t \leqslant \tau_1, \\ \hat{H}^\top(t)Z_1K + \hat{H}^\top(t)\hat{P}^\top \hat{D_1}^\top X, & \tau_1 \leqslant t \leqslant 1, \end{cases} \tag{22}$$
$$\int_0^t u(s-\tau_2)\,\mathrm{d}s = \begin{cases} \hat{H}^\top(t)\hat{P}^\top R, & 0 \leqslant t \leqslant \tau_2, \\ \hat{H}^\top(t)Z_2R + \hat{H}^\top(t)\hat{P}^\top \hat{D_2}^\top U, & \tau_2 \leqslant t \leqslant 1, \end{cases}$$

where P is the operational matrix of integration given in (6), and the constant matrices Z_1, Z_2 are given as

$$\int_0^{\tau_i} \hat{H}^\top(t)\,\mathrm{d}t = \hat{H}^\top(t) Z_i, \quad i = 1, 2. \tag{23}$$

By integrating of $\dot{x}(t)$ in (17) from 0 to t and using (20–22), we can obtain a new system of equations

$$C^* = \hat{H}^\top(t)X - \hat{H}^\top(t)G - \int_0^t F(x(s), x(s-\tau_1), u(s), u(s-\tau_2))\,\mathrm{d}s = 0. \tag{24}$$

Similarly for J, we have

$$J = \tfrac{1}{2} X^\top (H(1)H^\top(1) \otimes S)X + \tfrac{1}{2} X^\top (L \otimes Q)X + \tfrac{1}{2} U^\top (L \otimes R)U, \tag{25}$$

where $L = \int_0^1 H(t)H^\top(t)\,\mathrm{d}t$ and indeed, we know that $L = I$, where I is MN identity matrix.

The delay optimal control problem has now been reduced to a parameter optimization problem that can be stated as follows Find X and U, so that $J(X,U)$ is minimized subject to the constraints in Equation (24).

Let

$$J^*(X,U,\lambda) = J(X,U) + \lambda^\top C^*, \tag{26}$$

where the vector λ represents the unknown Lagrange multipliers. Since the functional J in (16) is a convex function (as the special chosen of the matrices S, Q and R), so then the necessary and sufficient conditions for stationary are given by

$$\begin{aligned} \frac{\partial}{\partial X} J^*(X,U,\lambda) &= 0, \\ \frac{\partial}{\partial U} J^*(X,U,\lambda) &= 0, \\ \frac{\partial}{\partial \lambda} J^*(X,U,\lambda) &= 0. \end{aligned} \tag{27}$$

4. Illustrative examples

In this section, two examples are given to demonstrate the applicability, efficiency and accuracy of our proposed method.

4.1 *Example 1*

Consider the following non-linear optimal control system (Mohan & Kar, 2012)

$$\min\ J = \frac{1}{2}\int_0^1 \{x^2(t) + u^2(t)\}\,\mathrm{d}t, \tag{28}$$

subject to

$$\begin{aligned} \dot{x}(t) &= -x^2(t) + u(t), \quad 0 \leqslant t \leqslant 1, \\ x(0) &= 10. \end{aligned} \tag{29}$$

TABLE 1 *The approximated values of J in Example 1*

BP method (Mohan & Kar, 2012)	Our method
4.4951 ($N=3, M=10$)	4.48627 ($N=3.M=5$)
	4.48588 ($N=3.M=7$)
	4.48582 ($N=3.M=9$)

The problem is to find the optimal control pair $(x(.), u(.))$ which minimizes J in (28) subject to (29). We solve this problem by hybrid functions with $N=3$ and different values of M. Suppose that

$$x(t) = C^\top H(t), \quad u(t) = U^\top H(t), \quad x(0) = C_0^\top H(t). \tag{30}$$

By expanding $x(0)$ in terms of hybrid functions for $M=5, N=3$, we have

$$x(0) = \left[\frac{10\sqrt{3}}{3}, 0, 0, 0, 0, \frac{10\sqrt{3}}{3}, 0, 0, 0, 0, \frac{10\sqrt{3}}{3}, 0, 0, 0, 0\right]^\top = e_1^\top H(t). \tag{31}$$

If we integrate (29) from 0 to t and use (30–31), we have

$$\begin{aligned} \int_0^t \dot{x}(s)\,\mathrm{d}s &= -\int_0^t x^2(s)\,\mathrm{d}s + \int_0^t u(s)\,\mathrm{d}s, \\ x(t) - x(0) &= -\int_0^t C^\top H(s) H^\top(s) C\,\mathrm{d}s + \int_0^t U^\top H(s)\,\mathrm{d}s, \\ C^\top H(t) - e_1^\top H(t) &= -C^\top \underbrace{\left(\int_0^t H(s)H^\top(s)\,\mathrm{d}s\right)}_{K(t)} C + U^\top P H(t), \end{aligned}$$

so we obtain

$$C^* = C^\top H(t) - e_1^\top H(t) + C^\top K(t) C - U^\top P H(t) = 0, \tag{32}$$

And for J, we have

$$J = \tfrac{1}{2}(C^\top \tilde{E} C + U^\top \tilde{E} U), \tag{33}$$

where $\tilde{E}$ is the dual operational matrix of $H(t)$. Now, we have reduced system as follows

$$\begin{aligned} \min \quad & J = \tfrac{1}{2}(C^\top \tilde{E} C + U^\top \tilde{E} U), \\ \text{s.t} \quad & C^* = C^\top H(t) - e_1^\top H(t) + C^\top K(t) C - U^\top P H(t) = 0. \end{aligned}$$

Approximated values of the cost function J with $N=3$ and for $M=3,5,7$ are given in Table 1 and are compared with the solutions obtained in Mohan & Kar (2012). As we can see, our minimum value is better than Mohan & Kar (2012). The curves of state and control functions for $M=7$ are shown in Fig. 1.

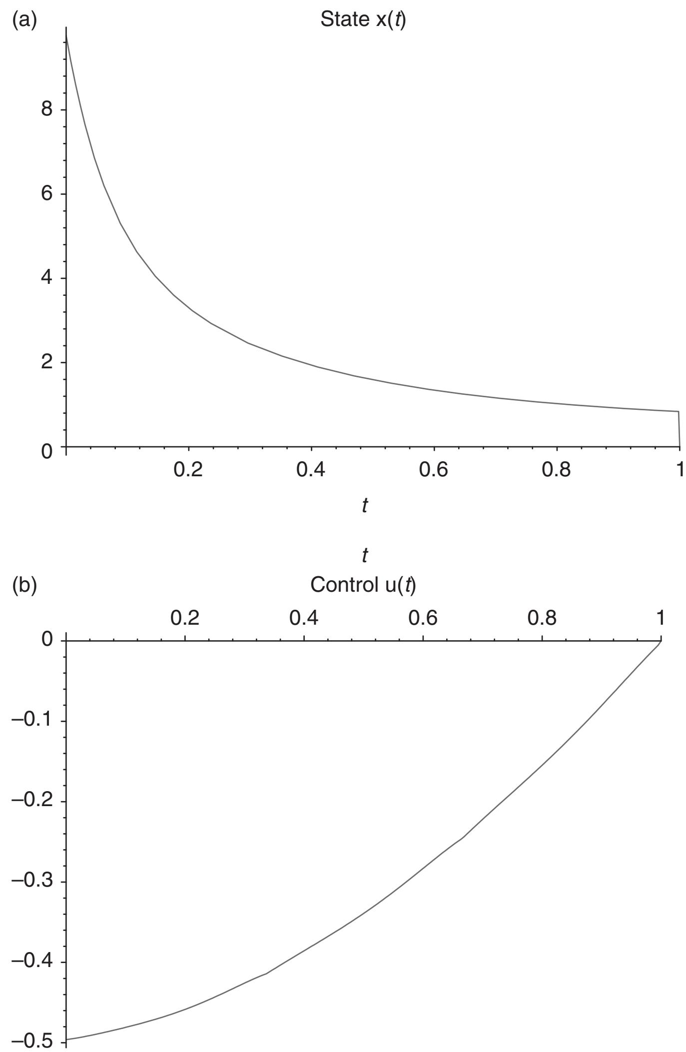

FIG. 1. State vector $x(t)$ and control $u(t)$ in Example 1. (a) State $x(t)$. (b) Control $u(t)$.

Approximated values for $x(t), u(t)$ with $N = 3,\ M = 5$ are as follows

$$x(t) = \begin{cases} 9.56496 - 72.38742t + 352.57378t^2 - 939.22282t^3 + 1019.68495t^4 & 0 \leqslant t < \frac{1}{3} \\ 6.02709 - 20.27660t + 36.82506t^2 - 34.81123t^3 + 13.43288t^4 & \frac{1}{3} \leqslant t < \frac{2}{3} \\ 4.33081 - 9.48652t + 10.62748t^2 - 6.07888t^3 + 1.44317t^4 & \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$

$$u(t) = \begin{cases} -0.49518 + 0.08547t + 0.55492t^2 - 0.38696t^3 + 0.48035t^4 & 0 \leqslant t < \frac{1}{3} \\ -0.49162 + 0.05796t + 0.61439t^2 - 0.28392t^3 + 0.10764t^4 & \frac{1}{3} \leqslant t < \frac{2}{3} \\ -0.44908 - 0.16412t + 1.03584t^2 - 0.61992t^3 + 0.19730t^4 & \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$

4.2 *Example 2*

Consider the following optimal control problem of the non-linear time-delay system (Gollman *et al.*, 2009; Koshkouei *et al.*, 2012; Borzabadi & Asadi, 2013)

$$\min \quad J = 3\int_0^1 \{x^2(t) + u^2(t)\}\,\mathrm{d}t \tag{34}$$

$$\begin{aligned} \text{s.t} \quad & \dot{x}(t) = 3x\left(t-\tfrac{1}{3}\right)u\left(t-\tfrac{2}{3}\right), \quad 0 \leqslant t \leqslant 1, \\ & x(t) = 1, \quad -\tfrac{1}{3} \leqslant t \leqslant 0, \\ & u(t) = 0, \quad -\tfrac{2}{3} \leqslant t \leqslant 0. \end{aligned} \tag{35}$$

Here we have different delays in state ($\tau_1 = \frac{1}{3}$) and control ($\tau_2 = \frac{2}{3}$). Suppose that

$$x(t) = C^\top H(t), \quad u(t) = U^\top H(t), \quad x(0) = C_0^\top H(t), \tag{36}$$

where $C^\top, U^\top, H(t), C_0$ are defined previously. Now we solve this problem. If we integrate $\dot{x}(t)$ from 0 to t and use (36), we have

$$\begin{aligned} \int_0^t \dot{x}(s)\,\mathrm{d}s &= 3\int_0^t x\left(s-\frac{1}{3}\right)u\left(s-\frac{2}{3}\right)\mathrm{d}s \\ x(t) - x(0) &= 3\int_0^t C^\top D_{\tau_1} H(s) H^\top(s) D_{\tau_2}^\top U\,\mathrm{d}s \\ &= 3C^\top D_{\tau_1} \underbrace{\left(\int_0^t H(s)H^\top(s)\,\mathrm{d}s\right)}_{M(t)} D_{\tau_2}^\top U, \\ &= 3C^\top D_{\tau_1} M(t) D_{\tau_2}^\top U, \end{aligned}$$

so we obtain

$$C^\top H(t) - C_0^\top H(t) = 3C^\top D_{\tau_1} M(t) D_{\tau_2}^\top U,$$

one may find

$$C^* = C^\top H(t) - C_0^\top H(t) - 3C^\top D_{\tau_1} M(t) D_{\tau_2}^\top U = 0. \tag{37}$$

For J, we have

$$\begin{aligned} J &= 3\int_0^1 \{C^\top H(t)H^\top(t)C + U^\top H(t)H^\top(t)U\}\,\mathrm{d}t \\ &= 3(C^\top \tilde{E} C + U^\top \tilde{E} U), \end{aligned}$$

Now we have reduced system as follows

$$\begin{aligned} \min \quad & J = 3(C^\top \tilde{E} C + U^\top \tilde{E} U), \\ \text{s.t} \quad & C^* = C^\top H(t) - C_0^\top H(t) - 3C^\top D_{\tau_1} M(t) D_{\tau_2}^\top U = 0. \end{aligned}$$

TABLE 2 *Approximated values of J in Example 2 with $N = 3$*

Method in Borzabadi & Asadi (2013)	Method in Koshkouei et al. (2012)	Method in Gollman et al. (2009)	Our method
2.761652	2.7640	3.1082	2.76147 ($M = 5$) 2.76141 ($M = 7$) 2.76140 ($M = 9$)

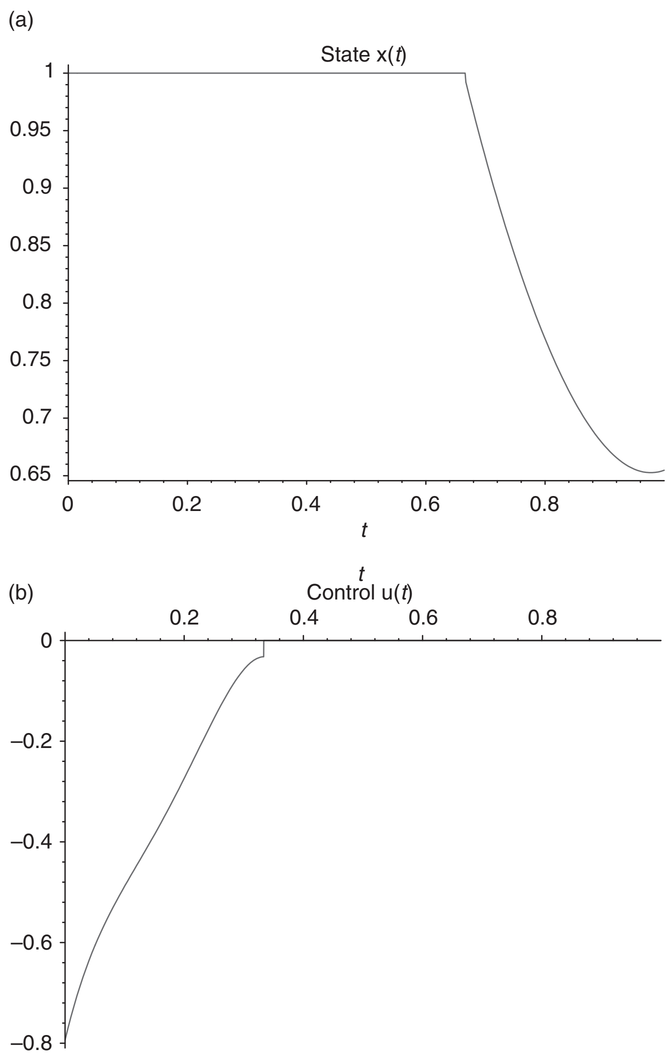

FIG. 2. State vector $x(t)$ and control $u(t)$ in Example 2. (a) State $x(t)$. (b) Control $u(t)$.

Approximate values of the cost function J with $N = 3$ and for $M = 3, 5, 7$ are given in Table 2 and are compared with the solutions obtained in Borzabadi & Asadi (2013), Gollman *et al.* (2009), Koshkouei *et al.* (2012). Clearly, the results are better than the approximate optimal value $J = 3.1082$ reported in Gollman *et al.* (2009) considering 60 000 grid points and CPU time of 65.8 s, and from Koshkouei *et al.* (2012) with $J = 2.7640$ with the CPU time on a Pentium 4 in 7 s and from Borzabadi & Asadi (2013) with 128 nodes in the Haar wavelets collocation method and by $J = 2.761652$ in 59.084 CPU second, because our result is closer to the optimal value of the objective function corresponding to analytical solution, i.e. 2.761594 (Gollman *et al.*, 2009) and the CPU time is about 4.98 s. The curves of state and control functions for $M = 5$ are shown in Fig. 2. We also have

$$x(t) = \begin{cases} 1.0 & 0 \leqslant t < \frac{1}{3} \\ 1.0 & \frac{1}{3} \leqslant t < \frac{2}{3} \\ 4.45454 - 10.02233t + 10.95736t^2 - 7.23520t^3 + 2.50035t^4 & \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$

$$u(t) = \begin{cases} -0.79369 + 4.91153t - 29.21415t^2 + 124.66044t^3 - 181.98995t^4 & 0 \leqslant t < \frac{1}{3} \\ 0 & \frac{1}{3} \leqslant t < \frac{2}{3} \\ 0 & \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

5. Conclusion

A new approach for computing optimal control of non-linear time delays systems with quadratic performance index has been proposed using hybrid of general block-pulse functions and orthonormal Taylor series. The operational matrices of integration, dual, product and delay are obtained and used to reduce the solution of optimal control problem to the solution of algebraic equations. The operational matrices of integration and product have many zeros and so they are sparse matrices which makes hybrid functions computationally very attractive. So the computational cost is decreased. Illustrative examples demonstrate that the method is valid.

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