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Testing for independence against orthant dependence in multivariate models

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Abstract

A class of distribution-free tests for testing independence against positive quadrant dependence has been proposed by Kochar and Gupta (1987). In this paper, we consider a test of independence against orthant dependence. The proposed test is a generalization of Kochar and Gupta's tests to one multivariate dependence case. Also, we investigate two multivariate extensions of Kendall's tau as two competitors of this generalized class. The first option, developed by Joe (1990), is a direct generalization of Kendall's tau in the bivariate case and is based on Kochar and Gupta's approach. The second one, introduced by Kendall and Babington Smith (1940), is a coefficient of agreement defined as the average value of Kendall's tau taken over all possible pairs of variables. Finally, we compare the empirical power of multivariate extensions of Kendall's tau with that of the generalization of Kochar and Gupta's tests.

Keywords: Coefficient of agreement, Kendall's tau, Multivariate copulas, Orthant dependence, Test of independence, U-statistic.

Mathematics Subject Classification (2010): 62G10, 62H15.

1 Introduction

A class of distribution-free tests has been proposed by Kochar and Gupta (1987) for testing independence between two variables X and Y against positive quadrant dependence (PQD). Amini et al. (2010) and Amini et al. (2011) considered a dependence measure for generalized Farlie-Gumbel-Morgenstern (FGM) family in view of Kochar and Gupta (1987). Also, they compared Spearman's rho and Kendall's tau with this measure in FGM family.

An old and well known problem in non-parametric statistics is that of testing the null hypothesis of independence between $d \geq 2$ components of a multivariate vector with continuous distribution.

One of dependence concepts for multivariate case is orthant dependence defined as the following.

Definition 1.1. *(Ebrahimi and Ghosh, 1981) Let* $\mathbf{X} = (X_1, \ldots, X_d)$ *be a vector of random variables.* $\mathbf{X}$ *is positively lower orthant dependent (PLOD) if for all* $\mathbf{x} = (x_1, \ldots, x_d) \in R^d$,

$$P(\mathbf{X} \leq \mathbf{x}) \geq \prod_{i=1}^{d} P(X_i \leq x_i). \tag{1}$$

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X *is positively upper orthant dependent (PUOD) if for all* $\mathbf{x} = (x_1, \ldots, x_d) \in R^d$,

$$P(\mathbf{X} > \mathbf{x}) \geq \prod_{i=1}^{d} P(X_i > x_i). \tag{2}$$

X *is positively orthant dependent (POD) if for all* $\mathbf{x} \in R^d$, *both* (1) *and* (2) *hold.*

Negative lower orthant dependence ($NLOD$), negative upper orthant dependence ($NUOD$) and negative orthant dependence (NOD) are defined analogously, by reversing the inequalities in (1) and (2). The two cases (1) and (2) are equivalent only for $d = 2$.

Genest et al. (2011) considered the estimation of a real-valued dependence parameter in a multivariate copula model and compared the performance of the estimators resulting from the inversion of the these two versions of Kendall's tau based on copula models via simulation. In this paper, we extend the previous tests from bivariate to multivariate case for testing independence against orthant dependence and also used the estimators of Genest et al. (2011) to test our hypothesis.

The paper is organized as follows. In Section 2, we consider the problem of testing the independence null hypothesis (H_0), against the alternative hypothesis of strict POD (NOD) (H_1 (H_2)). In Section 3, we state two multivariate extensions of Kendall's tau. In Section 4, we consider the d-variate FGM copula and compare the empirical powers of the generalization of Kochar and Gupta's test with those of two multivariate extensions of Kendall's tau. Also, we investigate an example for illustrating our results.

2 A generalization of Kochar and Gupta's tests

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a vector of continuous random variables with distribution function H and marginals $F_1, \ldots, F_d$ and let $\bar{H}(x_1, \ldots, x_d) = P(X_1 > x_1, \ldots, X_d > x_d)$ denotes its survival function. We consider

$$d_k(\mathbf{x}) = H^k(\mathbf{x}) - \prod_{i=1}^{d} F_i^k(x_i), \qquad \forall \mathbf{x} \in R^d, \tag{3}$$

where $k \geq 1$ is a fixed integer. It is obvious that for all $k \geq 1$, if $X_1, \ldots, X_d$ are independent (under H_0), then $d_k(\mathbf{x}) = 0$ and if $\mathbf{X}$ is POD (NOD), then $d_k(\mathbf{x}) \geq (\leq) 0$ for all $\mathbf{x}$ in R^d and with a strict inequality over a set of nonzero probability.

The measure of deviation between independence and orthant dependence (OD) is considered as

$$D_k = \int_{R^d} d_k(\mathbf{x}) dH(\mathbf{x}) = D_{1k} - D_{2k}, \qquad \forall k \geq 1, \tag{4}$$

where

$$D_{1k} = \int_{R^d} H^k(\mathbf{x}) dH(\mathbf{x}) \quad = \quad P\left\{ \max_{1 \leq i \leq k} X_{i1} \leq X_{k+1,1}, \ldots, \max_{1 \leq i \leq k} X_{id} \leq X_{k+1,d} \right\}, \tag{5}$$

$$D_{2k} = \int_{R^d} \prod_{i=1}^{d} F_i^k(x_i) dH(\mathbf{x}) = \int_{R^d} \bar{H}(\mathbf{x}) \prod_{i=1}^{d} dF_i^k(x_i), \tag{6}$$

and $\mathbf{X}_i = (X_{i1}, \ldots, X_{id})$, for $i = 1, 2, \ldots, k+1$, are independent copies of $\mathbf{X} = (X_1, \ldots, X_d)$.

Under independence hypothesis (H_0), $D_{1k} = D_{2k} = (k+1)^{-d}$ and under hypothesis of strict POD (H_1), $D_{1k} > D_{2k} > (k+1)^{-d}$ and strict NOD (H_2), $D_{1k} < D_{2k} < (k+1)^{-d}$. Now, let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample from H, where $\mathbf{X}_i = (X_{i1}, \ldots, X_{id})$ for $i = 1, 2, \ldots, n$. Suppose that

$$\varphi_{k+1}\{\mathbf{X}_1, \ldots, \mathbf{X}_{k+1}\} = \begin{cases} 1, & (\max\{X_{11}, X_{21}, \ldots, X_{k+1,1}\}, \ldots, \max\{X_{1d}, X_{2d}, \ldots, X_{k+1,d}\}) \\ & \text{ belongs to the same vector } (X_1, \ldots, X_d) \\ 0, & \text{otherwise.} \end{cases}$$

The U-statistic estimator of D_{1k} is

$$U_n(k+1) = \frac{1}{\binom{n}{k+1}} \sum \varphi_{k+1}\{\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_{k+1}}\},$$

where the sum is over all combinations of $(k+1)$ integers $\{i_1, \ldots, i_{k+1}\}$ chosen out of $\{1, \ldots, n\}$.

Remark 2.1. *The large (small) values of $U_n(k+1)$ are significant for testing H_0 against H_1 (H_2).*

For testing H_0 against OD, we should obtain the asymptotic distribution of $U_n(k+1)$. It is gained in following theorem.

Theorem 2.2. *The asymptotic distribution of $\sqrt{n}(U_n(k+1) - EU_n(k+1))$ is normal with zero mean and variance $\sigma^2_{k+1} = (k+1)^2\gamma_1$ as $n \longrightarrow \infty$, where*

$$\gamma_1 = E\psi^2(\mathbf{X}_1) - E^2\psi(\mathbf{X}_1)$$

and

$$(\mathbf{x}_1) = E\varphi_{k+1}[\mathbf{x}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k+1}].$$

Proof. For proof, see Serfling (1980). □

Remark 2.3. *Under H_0, $E\psi(\mathbf{X}_1) = EU_n(k+1) = (k+1)^{-(d-1)}$ and*

$$\gamma_1 = \frac{(k+1)^d + 2^d k^2 + 2k}{(2k+1)^d (k+1)^d} - \left(\frac{1}{(k+1)^{d-1}}\right)^2 \tag{7}$$

and so $\sigma^2_{k+1} = (k+1)^2\gamma_1$.

3 Kendall's tau and coefficient of agreement

Copulas are a powerful tool for construction of multivariate distributions. Let H be a joint distribution function with marginals $F_1, \ldots, F_d$. Then there exists a d-copula C such that

$$H(x_1, \ldots, x_d) = C\left(F_1(x_1), \ldots, F_d(x_d)\right),$$

for all reals $(x_1, \ldots, x_d) \in R^d$. If $F_1, \ldots, F_d$ are all continuous, then C is unique and by Sklar's theorem (1959) the copula C is given by

$$C(u_1, \ldots, u_d) = H\left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right),$$

for any $\mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d$, where $F_1^{-1}, \ldots, F_d^{-1}$ denote the quasi-inverses of $F_1, \ldots, F_d$, respectively.

3.1 Kendall's tau

In bivariate case, Kendall's tau is

$$\begin{aligned}\tau_2 &= 4P(X_1 \le X_1^*, X_2 \le X_2^*) - 1 \\ &= 4\int_{[0,1]^2} C(u_1, u_2)dC(u_1, u_2) - 1,\end{aligned} \tag{8}$$

where (X_1^*, X_2^*) is an independent copy of (X_1, X_2).

Let (X_{i1}, X_{i2}), $i = 1, \ldots, n$, be a random sample from bivariate distribution H. The sample version of τ_2 is

$$\tau_{2,n} = \frac{4}{n(n-1)} \sum_{i \ne j} I(X_{i1} \le X_{j1}, X_{i2} \le X_{j2}) - 1, \tag{9}$$

where $I(A)$ denotes the indicator of the set A. Joe (1990) developed a multivariate generalization of Kendall's tau by a direct extension of (8) as

$$\tau_d = \frac{2^d P(\mathbf{X} \leq \mathbf{X}^*) - 1}{2^{d-1} - 1} = \frac{1}{2^{d-1} - 1}\left\{2^d \int_{[0,1]^d} C(\mathbf{u}) dC(\mathbf{u}) - 1\right\}, \tag{10}$$

where $\mathbf{X}^*$ is an independent copy of $\mathbf{X}$ and inequalities between vectors are considered to hold componentwise. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample from H, where $\mathbf{X}_i = (X_{i1}, \ldots, X_{id})$, for $i = 1, 2, \ldots, n$. Similarly, the d-dimensional version of $\tau_{2,n}$ is

$$\tau_{d,n} = \frac{1}{2^{d-1} - 1}\left\{\frac{2^d}{n(n-1)} \sum_{i \neq j} I(\mathbf{X}_i \leq \mathbf{X}_j) - 1\right\}. \tag{11}$$

The following remark states the relation of $\tau_{d,n}$ and $U_n(k+1)$.

Remark 3.1. *In d-dimensional case for $k = 1$,*

$$U_n(2) = \frac{1 + (2^{d-1} - 1)\tau_{d,n}}{2^{d-1}}. \tag{12}$$

3.2 Coefficient of agreement

Kendall and Babington Smith (1940) introduced a generalization of Kendall's tau as a coefficient of agreement. It is the average value of Kendall's tau taken over all possible pairs (X_r, X_s), where $r, s \in \{1, \ldots, d\}$, that is defined by

$$T_d(\mathbf{X}) = \frac{1}{d(d-1)} \sum_{r \neq s} \tau_2(X_r, X_s). \tag{13}$$

The sample version of $\tau_d(\mathbf{X})$ is

$$T_{d,n} = \frac{1}{d(d-1)} \sum_{r \neq s} \tau_{2,n}(r, s). \tag{14}$$

Remark 3.2. *When $d = 3$, Joe's extension of Kendall's tau (10) coincides with the coefficient of agreement (13); that is,*

$$\tau_3(\mathbf{X}) = \frac{1}{3}\left\{\tau_2(X_1, X_2) + \tau_2(X_1, X_3) + \tau_2(X_2, X_3)\right\}. \tag{15}$$

Genest et al. (2011) stated the asymptotic distribution of $T_{d,n}$ (Proposition 4, p. 167).

Remark 3.3. *According to Genest et al. (2011) and the independence test mentioned in Section 2, we have the independence copula $\prod_d = u_1 u_2 \ldots u_d$ under H_0 and as $n \to \infty$, $\sqrt{n} T_{d,n}$ converges weakly to a normal random variable with zero mean and variance*

$$\sigma^2(T_{d,n}) = \frac{8}{9d(d-1)}. \tag{16}$$

In the next section, we used $\tau_{d,n}$, $T_{d,n}$ and $U_n(k+1)$, $k = 1, 2, 3$, for testing H_0 against H_1 (H_2) and using the asymptotic distribution of these estimators under H_0 to get the critical values of the independence test. Then, we compare the power of these tests, empirically.

4 Simulation study

In this section, for power comparisons, we consider the d-variate FGM copulas. There exist several multivariate types of this family. We investigate two versions of d-variate FGM copula with a single real-valued parameter θ and with copula functions

$$C_1(u_1, u_2, \ldots, u_d) = \left(\prod_{i=1}^d u_i\right)\left\{1 + \theta \sum_{t=2}^d \sum_{1 \leq i_1 < \ldots < i_t \leq d} \prod_{j=1}^t (1 - u_{i_j})\right\} \tag{17}$$

Table 1: **The acceptable range of θ in d-variate FGM copula C_1 and C_2 for $d = 3, 6$.**

	Copula	
d	C_1	C_2
3	[-0.25,0.5]	[-0.33,1]
6	[-0.0175,0.2]	[-0.066,0.33]

and

$$C_2(u_1, u_2, \ldots, u_d) = \left(\prod_{i=1}^{d} u_i\right) \left\{1 + \theta \sum_{i<j} (1 - u_i)(1 - u_j)\right\}, \tag{18}$$

where $u_1, \ldots, u_d \in [0, 1]$ and the acceptable range of θ is obtained by

$$1 + \theta \sum_{t=2}^{d} \sum_{1 \leq i_1 < \ldots < i_t \leq d} \xi_{i_1} \xi_{i_2} \ldots \xi_{i_t} \geq 0, \qquad \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_d} \in \{-1, +1\}, \tag{19}$$

for C_1 and

$$1 + \theta \sum_{i<j} \xi_i \xi_j \geq 0, \qquad \xi_i, \xi_j \in \{-1, +1\}, \tag{20}$$

for C_2 (for more details, see Nelsen (2006), and Mari and Kotz (2001)).

Remark 4.1. *Using inequalities (19) and (20) for $d = 3, 6$, the acceptable range of θ is available in Table 1.*

For d-variate FGM copula, the empirical power of test statistics $\tau_{d,n}$, $T_{d,n}$ and $U_n(k+1)$ for $k = 1, 2, 3$, are calculated based on a simulation survey for 4000 independent replications using R software version 3.2.0 at 5% level of significance. The results are summarized in Tables 2-5 for $d = 3, 6$, sample sizes $n = 30, 50, 70, 100$ and d-variate FGM copulas C_1 and C_2. Table 2-5 show that for $d = 3, 6$ and copula function C_1 and C_2:

- the empirical sizes ($\theta = 0$) are close to 0.05.
- the power of all test statistics increases when n and $|\theta|$ increase.
- for $d = 3$ and all n and θ, $U_n(2)$ (or $\tau_{3,n}$) is the best test statistic.
- for $d = 6$ and all n, when $\theta > 0$, $U_n(2)$ (or $\tau_{6,n}$) is the best test statistic and when $\theta < 0$, $T_{6,n}$ is the best.
- the power of all test statistics for POD case is better than NOD case.

Tables 2 and 4 show that for $d = 3$, the powers of $U_n(2)$ (or $\tau_{3,n}$) and $T_{3,n}$ are the same (see Remark 3.2 and Remark 3.3). Also, Tables 3 and 5 indicate for $d = 6$ and POD case, $U_n(2)$ (or $\tau_{6,n}$) is better than test statistic $T_{6,n}$ and for NOD case, the test statistic $T_{6,n}$ is better than $U_n(2)$ (or $\tau_{6,n}$).

4.1 Example

We use the uranium exploration data set that is considered by Cook and Johnson (1986) and consist of 655 chemical analyses from water samples collected from the Montrose quadrangle of western Colorado. Concentrations were measured for the following elements: uranium (U), lithium (Li), cobalt (Co), potassium (K), caesium (Cs), scandium (Sc) and titanium (Ti).

We use the first 100 cases from two selected 3-tuple (U, Li, K) and (U, Li, Co) in the uranium exploration data. First, we test whether one can fit the 3-dimensional FGM distribution C_1 in (17) to (U, Li, K) and (U, Li, Co). Fasano and Franceschini (1987) proposed a generalization of the classical Kolmogorov-Smirnov test which is suitable to analyse random samples defined in two and more dimensional cases. In the three-dimensional case, we define D_n statistic as the absolute maximum difference

Table 2: **Empirical powers in the 3-variate FGM copula C_1 for OD case.**

		θ				
n	statistics	-0.25	-0.1	0	0.25	0.5
30	$U_n(2)$	**0.0762**	**0.0750**	0.0682	**0.1060**	**0.1547**
	$U_n(3)$	0.0382	0.0422	0.0780	0.0807	0.0870
	$U_n(4)$	0.0012	0.0005	0.0827	0.0735	0.0610
	$T_{d,n}$	0.0762	0.0750	0.0682	0.1060	0.1547
50	$U_n(2)$	**0.0845**	**0.0682**	0.0572	**0.1092**	**0.1837**
	$U_n(3)$	0.0430	0.0435	0.0610	0.0780	0.0790
	$U_n(4)$	0.0140	0.0182	0.0630	0.0642	0.0470
	$T_{d,n}$	0.0845	0.0682	0.0572	0.1092	0.1837
70	$U_n(2)$	**0.0972**	**0.0755**	0.0585	**0.1162**	**0.2162**
	$U_n(3)$	0.0490	0.0470	0.0630	0.0740	0.0870
	$U_n(4)$	0.0257	0.0280	0.0702	0.0557	0.0442
	$T_{d,n}$	0.0972	0.0755	0.0585	0.1162	0.2162
100	$U_n(2)$	**0.1112**	**0.0660**	0.0587	**0.1312**	**0.2595**
	$U_n(3)$	0.0550	0.0460	0.0690	0.0757	0.0855
	$U_n(4)$	0.0275	0.0330	0.0695	0.0565	0.0425
	$T_{d,n}$	0.1112	0.0660	0.0587	0.1312	0.2595

Table 3: **Empirical powers in the 6-variate FGM copula C_1 for OD case.**

		θ				
n	statistics	-0.015	-0.01	0	0.1	0.15
30	$U_n(2)$	0.0277	0.0278	0.0807	**0.1087**	**0.1312**
	$U_n(3)$	0.0000	0.0000	0.0720	0.0925	0.1030
	$U_n(4)$	0.0000	0.0000	0.0492	0.0627	0.0710
	$T_{d,n}$	**0.0471**	**0.0496**	0.0602	0.0827	0.0910
50	$U_n(2)$	0.0352	0.0360	0.0775	**0.1064**	**0.1345**
	$U_n(3)$	0.0000	0.0000	0.0740	0.0998	0.1202
	$U_n(4)$	0.0000	0.0000	0.0550	0.0764	0.092
	$T_{d,n}$	**0.0460**	**0.0445**	0.0640	0.0765	0.098
70	$U_n(2)$	0.0357	0.0392	0.0720	**0.1177**	**0.1410**
	$U_n(3)$	0.0000	0.0000	0.0807	0.1102	0.1365
	$U_n(4)$	0.0000	0.0000	0.0622	0.0880	0.1107
	$T_{d,n}$	**0.0435**	**0.0470**	0.0605	0.0890	0.0947
100	$U_n(2)$	0.0395	0.0400	0.0683	**0.1233**	**0.1463**
	$U_n(3)$	0.0000	0.0000	0.0726	0.1186	0.1430
	$U_n(4)$	0.0000	0.0000	0.0580	0.1090	0.1213
	$T_{d,n}$	**0.0485**	**0.0482**	0.0606	0.0820	0.1023

Table 4: **Empirical powers in the 3-variate FGM copula C_2 for OD case.**

		θ					
n	statistics	-0.3	-0.1	0	0.25	0.5	0.75
30	$U_n(2)$	**0.0975**	**0.0660**	0.0652	**0.1020**	**0.1525**	**0.2267**
	$U_n(3)$	0.0527	0.0375	0.0762	0.0910	0.1102	0.1465
	$U_n(4)$	0.0012	0.0005	0.0855	0.0882	0.0907	0.0995
	$T_{d,n}$	0.0975	0.0660	0.0652	0.1020	0.1525	0.2267
50	$U_n(2)$	**0.1085**	**0.0650**	0.0595	**0.1055**	**0.1840**	**0.2825**
	$U_n(3)$	0.0597	0.0432	0.0657	0.0875	0.1142	0.1485
	$U_n(4)$	0.0225	0.0190	0.0685	0.0822	0.0825	0.0917
	$T_{d,n}$	0.1085	0.0650	0.0595	0.1055	0.1840	0.2825
70	$U_n(2)$	**0.1232**	**0.0722**	0.0665	**0.1200**	**0.2185**	**0.3465**
	$U_n(3)$	0.0722	0.0490	0.0745	0.0927	0.1205	0.1775
	$U_n(4)$	0.0350	0.0285	0.0747	0.0865	0.0867	0.1012
	$T_{d,n}$	0.1232	0.0722	0.0665	0.1200	0.2185	0.3465
100	$U_n(2)$	**0.1370**	**0.0707**	0.0537	**0.1400**	**0.2640**	**0.4167**
	$U_n(3)$	0.0737	0.0470	0.0620	0.1005	0.1332	0.1875
	$U_n(4)$	0.0405	0.0342	0.0737	0.0802	0.0825	0.0935
	$T_{d,n}$	0.1370	0.0707	0.0537	0.1400	0.2640	0.4167

Table 5: **Empirical powers in the 6-variate FGM copula C_2 for OD case.**

		θ				
n	statistics	-0.06	-0.03	0	0.1	0.3
30	$U_n(2)$	0.0262	0.0257	0.0832	**0.0907**	**0.13675**
	$U_n(3)$	0.0000	0.0000	0.0782	0.0857	0.1077
	$U_n(4)$	0.0000	0.0000	0.0517	0.0520	0.0695
	$T_{d,n}$	**0.047**	**0.045**	0.0637	0.0697	0.1057
50	$U_n(2)$	0.0382	0.0372	0.0722	**0.0905**	**0.1315**
	$U_n(3)$	0.0000	0.0000	0.0710	0.0835	0.1125
	$U_n(4)$	0.0000	0.0000	0.0532	0.0577	0.0832
	$T_{d,n}$	**0.0465**	**0.0407**	0.0602	0.0717	0.1000
70	$U_n(2)$	0.0377	0.0355	0.0750	**0.0952**	**0.1375**
	$U_n(3)$	0.0000	0.0000	0.0590	0.0825	0.1297
	$U_n(4)$	0.0000	0.0000	0.0637	0.0590	0.0985
	$T_{d,n}$	**0.0517**	**0.0467**	0.0637	0.0700	0.1062
100	$U_n(2)$	0.0427	0.0422	0.0670	**0.0945**	**0.1560**
	$U_n(3)$	0.0000	0.0000	0.0727	0.0897	0.1552
	$U_n(4)$	0.0000	0.0000	0.0597	0.0690	0.1250
	$T_{d,n}$	**0.0525**	**0.0482**	0.0630	0.0795	0.1147

Table 6: **Goodness of fit test for two selected 3-tuple of variables in uranium data.**

3-tuple	$MLE(\theta)$	$-\ell$	D_n	p-value
(U, Li, K)	0.350	-2.808	1.002	0.856
(U, Li, Co)	-0.158	-0.531	0.917	0.905

(multiplied by $\sqrt{n}$) between the observed and predicted normalized integral distributions cumulated within the eight volumes of the three-dimensional space defined for each data point (X_i, Y_i, Z_i) by

$$(x < X_i, y < Y_i, z < Z_i),\ (x < X_i, y < Y_i, z > Z_i), \ldots,\ (x > X_i, y > Y_i, z > Z_i) \qquad i = 1, \ldots, n.$$

For the 3-dimensional FGM distribution C_1 in (17), Tabel 6 includes the maximum likelihood estimator (MLE) of θ, negative log-likelihood $(-\ell)$ of MLE of θ, D_n and associated p-value for POD and NOD cases. At 0.05 level, Table 6 shows that the 3-dimensional FGM distribution C_1 and $MLE(\theta) = 0.350$ is acceptable for (U, Li, k) and $MLE(\theta) = -0.158$ is acceptable for (U, Li, Co). Now, we test the independence hypothesis against the alternative hypothesis of strict POD (NOD). At level 0.05, the results are:

1. For (U, Li, K), we estimate the marginal distribution functions F_U, F_{Li} and F_K by their empirical distribution function and then compute the test statistics $U_n(2)$, $U_n(3)$, $U_n(4)$ and $T_{d,n}$. Table 7 includes the values of test statistics, their critical values and associated p-values. Table 7 shows the rejection of H_0 in favor of H_1 based on all test statistics. Moreover, because the associated p-values for $U_n(2)$ and $T_{d,n}$ is less than these of $U_n(3)$ and $U_n(4)$, the result is coincident with the empirical powers in Table 2.

2. For (U, Li, Co), we estimate the marginal distribution functions F_U, F_{Li} and F_{Co} by their empirical distribution function and then compute the test statistics $U_n(2)$, $U_n(3)$, $U_n(4)$ and $T_{d,n}$. Table 8 includes the values of test statistics, their critical values and associated p-values. According to

Table 7: **The results of independence tests for (U, Li, K) in POD case.**

	$U_n(2)$	$U_n(3)$	$U_n(4)$	$T_{d,n}$
Test statistic	0.332	0.170	0.110	0.116
Critical point	0.297	0.150	0.096	0.063
p-value	0.002	0.006	0.009	0.001

Table 8: **The results of independence tests for (U, Li, Co) in NOD case.**

	$U_n(2)$	$U_n(3)$	$U_n(4)$	$T_{d,n}$
Test statistic	0.221	0.106	0.073	-0.036
Critical point	0.202	0.072	0.029	-0.063
p-value	0.161	0.410	0.704	0.170

Table 8, we can't reject H_0 in favor of H_2 based on all test statistics. MLE of θ is close to zero in NOD case. That is, we have negative weak dependence. So, we expect that the tests don't lead to rejection of H_0 in favor of H_2. Also, because $U_n(2)$ and $T_{d,n}$ are good tests based on simulation results in Table 2, they have p-values less than other competitors.

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