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Numerical solutions for solving a class of fractional optimal control problems via fixed-point approach

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Abstract In this paper, an optimization problem is performed to obtain an approximate solution for a class of fractional optimal control problems (FOCPs) with the initial and final conditions. The main characteristic of our approximation is to reduce the FOCP into a system of Volterra integral equations. Then by solving this new problem, based on minimization and control the total error, we transform the original FOCP into a discrete optimization problem. By obtaining the optimal solutions of this problem, we obtain the numerical solution of the original problem. This procedure not only simplifies the problem but also speeds up the computations. The numerical solutions obtained from the proposed approximation indicate that this approach is easy to implement and accurate when applied to FOCPs.

Keywords Riemann–Liouville fractional derivative · Fractional optimal control problem · Fractional differential equation · Volterra-integral equation

Mathematics Subject Classification 49L99 · 65L03

1 Introduction

Fractional differential equations (FDEs) consist of a fractional differentiation with specified value of unknown functions at more than one given point in the domain of the solution. The equations represent an important tool in technology, science and economics and engineering applications including population models, control engineering electrical network analysis, gravity, medicine [18,38]. Recently, numerical methods have been used widely to find the approximate solutions of these equations such as homotopy perturbation method (HPM) [25,33], the Adimian decomposition method (ADM) [32], the variational iteration method (VIM) [34], the generalized differential transformation method (GDTM) [35] and the frac-

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tional difference method (FDM) [33]. For existing theoretical work we refer the interested readers in [16,17,27,37,42]. Some numerical methods [12,15,31] allow solutions of the equations of arbitrary real order but they work properly only for relatively simple form of fractional equations. For that reason we need a method for solving such equations which will be effective, easy to use and applied for the equations in general form.

When the FDEs are used in conjunction with the performance index and a set of initial conditions, they lead to fractional optimal control problems (FOCPs) [2,5]. FOCPs can be defined with different definitions of fractional derivatives. In the last few decades, FOCPs have gained much attention for their many applications in engineering and physics (see [13,48]). For that reason, there has been significant interest in developing numerical schemes for their solution. Work on FOCPs and approximate solutions has started by [2,5], where the authors have achieved the necessary conditions of optimization for FOCPs with dependence on derivatives or integrals of fractional order, e.g., Riemann–Liouville fractional derivative (RLFD) (that it represents in the preliminary part) and also have solved the problem numerically by solving the necessary conditions. There also exist other numerical simulations for this purpose, such as [45]. The Hamiltonian system of equations for FOCPs has been solved approximately by [3,4], where the cost function is fixed to be quadratic in the input and obtaining the solution is reduced to finding the solution of a system of FDEs. Recent study in FOCPs is referred to [8,9,11,21,23,24,26,39,44,46,47]. In [10], a modified Grunwald–Letnikov approach was proposed for a class of FOCPs. In [6] a method for solving multi-dimensional fractional optimal control problems was presented. Dehghan et al. [14] focused on the numerical point of view, the Jacobi polynomials to solve fractional variational problems and FOCPs.

In this paper, we would like to investigate the possibility of presence numerical approximated solutions for a class of FOCPs. To proceed, we achieved the necessary conditions of optimization for this class of FOCPs with a system of FDEs. To solve this system, first using this fact that the initial value problem of a FDE is equivalent to a Volterra integral equation. In addition, some well-known ideas are provided in the context of Banach's fixed-point theorem that examines the existence solutions of this Volterra integral equation by using this equation and a joint application of minimization the total error, we transform the original system of FDEs into a discrete optimization problem in way by obtaining the optimal solutions of this problem; we obtain the approximate solution of the FOCP. We report some numerical examples that show the effectiveness of the proposed approximation by means of some comparisons with other existing methods. It is expected that some of results derived in this survey may find applications in the solution of certain FDEs and FOCPs. Also, we point out that our approach relies on the RLFD. For the sake of simplicity, we consider $0 < \alpha < 1$ and the lower limit of integration is zero. These considerations don't affect the generalization of the derivation procedure.

2 Preliminary considerations

In the next section, we present some important definitions and necessary preliminaries which are useful in what follows.

2.1 Fractional derivative and integrals

Here, we introduce some necessary preliminaries from fractional calculus. Several definitions of a fractional derivative have been proposed (see [9]). The left-RLFD can be defined as follows:

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \tag{2.1}$$

in which, $n-1 < \alpha \leq n$ is the order of fractional derivative, $t \in J = [0, 1]$ and $f : J \to \mathbb{R}$ is a continuous function. t Also, the right-RLFD can be defined by:

$$ {}_tD_1^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^1 (s-t)^{n-\alpha-1} f(s) ds. \tag{2.2}$$

We also have other definitions of Fractional derivative operator like Caputo, Grunwald–Letnikov [38] and Weyl–Marchaud [41].

Now, let $f(t) \in L_1[0, 1]$ and $\alpha > 0$. Then, in a similar way, the left fractional integral (LFI) is defined as follows:

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2.3}$$

and the right fractional integral (RFI) has the following definition:

$$ {}_tI_1^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} f(s) ds. \tag{2.4}$$

One can find more information concerning these properties in [36,41]. Hence, we mention the following mean theorem of this section:

Theorem 2.1 [28] *Let* $\alpha > 0$. *Then:*

(1) ${}_0D_t^n {}_0D_t^{\alpha} f(t) = {}_0D_t^{n+\alpha} \quad f(t), \in \mathbb{N}$.
(2) *The equality* ${}_0D_t^{\alpha} {}_0I_t^{\alpha} f(t) = {}_tD_1^{\alpha} {}_tI_1^{\alpha} f(t) = f(t)$ *holds for every* $f(t) \in L_1[0, 1]$.
(3) *For* $f(t) \in L_1[0, 1]$, $n = [\alpha] + 1$, *if* ${}_0I_t^{n-\alpha} f, {}_tI_1^{n-\alpha} f \in AC^{n-1}[0, 1]$, *then:*

$$\begin{cases} {}_0I_t^{\alpha} {}_0D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(\left(\frac{d}{dt}\right)^{n-k-1} \quad {}_0I_t^{n-\alpha} f(0)\right) \\ {}_tI_1^{\alpha} {}_tD_1^{\alpha} f(t) = f(t) \; - \sum_{k=0}^{n-1} \frac{(1-t)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(\left(-\frac{d}{dt}\right)^{n-k-1} \quad {}_tI_1^{n-\alpha} f(1)\right) \end{cases} \tag{2.5}$$

in which, $AC[0, 1]$ *means the space of functions* $f : [0, 1] \to \mathbb{R}$ *which are absolutely continuous and*

$$ AC^{n}[0, 1] = \left\{ f : [0, 1] \to \mathbb{R};\ \frac{d^{n-1}}{dt^{n-1}} f(t) \in AC[0, 1] \right\}. \tag{2.6}$$

2.2 Fixed-point method

In order to illustrate the basic concepts of fixed point method, we consider the following theorem and lemmas.

Theorem 2.2 [43] *Let* E *be a Banach space. Assume that* Ω *is an open bounded subset of* E *and let* $T : \bar{\Omega} \to E$ *be a completely continuous operator such that:*

$$ \| Tx \| \leq \| x \|, \quad \forall x \in \partial\Omega. \tag{2.7}$$

Then T *has a fixed point in* $\bar{\Omega}$.

Lemma 2.1 [49] *Let $\alpha > 0$, then the fractional differential equation ${}_0D_t^\alpha x(t) = 0$ has the following solution:*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, i = 0, \cdots, n-1, \tag{2.8}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of the real number α.

In view of Lemma 2.1, it follows that:

$${}_0I_t^\alpha {}_0D_t^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.9}$$

for some $c_i \in \mathbb{R}$, $i = 0, \cdots, n-1$, $n = [\alpha] + 1$.

Lemma 2.2 *Let $f(t, x(t))$ and $x(t)$ belong to $L_1[0, 1]$. Then $x(t)$ is a solution of the initial value problem*

$$\begin{aligned} {}_0D_t^\alpha x(t) = f(t, x(t)), \quad 0 < \alpha \leq 1, \\ {}_0D_t^{\alpha-1} x(t)|_{t=0} = c_0 \end{aligned} \tag{2.10}$$

if and only if $x(t)$ is a solution of the following Volterra integral equation:

$$x(t) = \frac{c_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds. \tag{2.11}$$

The value of $\alpha = 1$ corresponds to the ordinary Volterra integral equation.

Proof Performing the integral operator ${}_0I_t^\alpha$ on both sides of the Eq. (2.10) and using property (3) given in Theorem 2.1, we obtained Eq. (2.11). On the other hand, performing the operator ${}_0D_t^\alpha$ on both sides of Eq. (2.11) results in:

$${}_0D_t^\alpha x(t) = \frac{c_0}{\Gamma(\alpha)} {}_0D_t^\alpha t^{\alpha-1} + {}_0D_t^\alpha {}_0I_t^\alpha f(t, x(t)) = f(t, x(t)), \tag{2.12}$$

where we use the property (2) of Theorem 2.1 and the following fact:

$$\frac{{}_0D_t^\alpha t^{\alpha-1}}{\Gamma(\alpha)} = \frac{t^{\alpha-1-\alpha}}{\Gamma(0)} = \frac{t^{-1}}{\infty} = 0. \tag{2.13}$$

Now, it is sufficient to show that the solution of Eq. (2.11) satisfies the initial condition of problem (2.10). So, by performing the operator $D_{0,t}^{\alpha-j-1}$on both sides of (2.11), and using the following property:

$${}_0D_t^{\alpha-j-1} t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(j+1)} t^j, \tag{2.14}$$

for $0 \leq j < 0 < \alpha <$, we have:

$$\begin{aligned} {}_0D_t^{\alpha-j-1} x(t) &= \frac{c_0}{\Gamma(j+1)} t^j + {}_0D_t^{\alpha-j-1} {}_0I_t^\alpha f(t, x(t)) \\ &= \frac{c_0}{\Gamma(j+1)} t^j + {}_0D_t^{-j-1} f\ (t, x(t)\ . \end{aligned} \tag{2.15}$$

Taking a limit $t \to 0$ in the above equation and using the fact that:

$$\lim_{t \to 0} {}_0D_t^{-j-1} f(t, x(t) = 0, \tag{2.16}$$

the proof is completed. □

In this way, the numerical schemes for solving this type of Volterra integral equations can also be applied for the solution of problem (2.10). Kumar and Agrawal [29], smartly take advantage of this fact and designed a novel numerical approach. They divided the interval [0,1] into a set of small intervals, and between two successive intervals the unknown functions are approximated using the quadratic order polynomials. Whereas the classical polynomials work well for the numerical solution of conventional differential equations, their application for the fractional differential equations (FDEs) implies some difficulties such as the fractional integrals and derivatives of a classical polynomial are not polynomials, so we may not be able to obtain a good approximation for the fractional integrals and derivatives via the classical polynomials. However, the fractional order polynomials give more accurate results than the classical polynomials [20].

In this paper, we proposed a different approach for the numerical treatment of FOCPs. Our first result is based on Banach fixed point theorem that provides some conditions under which the problem (2.10) will admit at least one solution on [0,1].

Theorem 2.3 *Assume that there exists a constant K>0 such that:*

$$|f(t,x)-f(t,\bar{x})| \leq K|x-\bar{x}|, \tag{2.17}$$

for each $t \in [0,1]$*, and all* $x, \bar{x} \in \mathbb{R}$*. If* $\frac{K}{\Gamma(\alpha+1)} < 1$*, then, problem* (2.10) *has a unique solution on* $[0,1]$.

Proof At first, we transform the problem (2.10) into a fixed point problem. Consider the operator:

$$T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$$

defined by:

$$T(x(t)) = c_0\frac{t}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds. \tag{2.18}$$

Clearly, showing the existence of fixed-points of T is equivalent to showing the existence of solutions to the problem (2.10). We shall use the Banach contraction principle to prove that T has a fixed point.

Obviously, T is a continuous operator. In this way, let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,1],\mathbb{R})$. Then for each $t \in [0,1]$,

$$\begin{aligned} |T(x_n(t)) - T(x(t))| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x_n(s)) - f(s,x(s))|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s) \sup_{s\in[0,1]} |f(s,x_n(s)) - f(s,x(s))|ds. \end{aligned} \tag{2.19}$$

Since f is a continuous function, then we have:

$$\| T(x_n) - T(x) \|_\infty \to 0 \ \ as \ \ n \to \infty. \tag{2.20}$$

Let $x, y \in C([0,1],\mathbb{R})$. Then for each $t \in [0,1]$ we have:

$$\begin{aligned} &|Tx(t)) - T(y(t))| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|ds \\ &\leq \frac{K \| x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds = \frac{K \| x-y\|_\infty}{\Gamma(\alpha+1)}t^\alpha \leq \frac{K}{\Gamma(\alpha+1)} \| x-y \|_\infty . \end{aligned} \tag{2.21}$$

Thus:

$$\| T(x) - T(y) \|_\infty \leq \frac{K}{\Gamma(\alpha+1)} \| x - y \|_\infty . \tag{2.22}$$

So as a consequence of Banach fixed point theorem, we deduce that T has a fixed point which is a solution of problem (2.10). □

3 Solving a class of fractional optimal control problems

The goal of this section is solving a class of FOCPs by using the proposed approach for solving FDEs. Our problem formulation is as follow:

$$\begin{aligned} min \ \ J(u) &= \int_0^1 f(x(t), u(t), t)dt \\ \text{s.t.} \ \ {}_0D_t^\alpha x(\mathrm{t}) &= g(x(t), u(t), t) \\ {}_0D_t^{\alpha-1} x(0) &= x_0, \end{aligned} \tag{3.1}$$

where $x \in C^{1-\alpha}[0, 1]$ in which:

$$C^{n-\alpha}[0,1] = \left\{ x : [0,1] \to R^n : \frac{d^{n-1}}{dt^{n-1}}({}_0D_t^\alpha x(t)) \in L_1[0,1], \quad n-1 < \alpha < n \right\},$$

$u \in R^m$, x_0 is a constant, f and g are two Riemann integrable functions and the fractional derivative operator ${}_0D_t^\alpha$, $0 < \alpha < 1$ is defined in the Riemann–Liouville sense.

To obtain the necessary conditions of problem (3.1), we use the Lagrange multiplier technique for this problem and take the variations of the resulting equation. Using the integration by parts to modify the equations that it does not contain variations of a derivative term, impose the necessary terminal conditions, and set the coefficients of $\delta\lambda$, δx and δu to zero. From [3,7] the necessary conditions for the FOCP are given as the following system of FDEs:

$$\begin{aligned} {}_0D_t^\alpha x(\mathrm{t}) &= g(x, u, t) \\ {}_tD_1^\alpha \lambda(\mathrm{t}) &= \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial u} + \lambda \frac{\partial g}{\partial u} &= 0 \\ {}_0D_t^{\alpha-1} x(0) &= x_0, \quad {}_tD_1^{\alpha-1} \lambda(1) = 0, \end{aligned} \tag{3.2}$$

in which λ is the Lagrange multipliers. Equations (3.2) coincide with the classical ones as α approaches to 1.

So, as a special case, we describe the FOCP with the quadratic performance index as follows:

$$min \ J(u) = \frac{1}{2} \int_0^1 (x^T(t)Q(t)x(t) + u^T(t)R(t)u(t))dt, \tag{3.3}$$

and the system whose dynamics is described by the following linear FDE,

$${}_0D_t^\alpha x(t) = A(t)x(t) + B(t)u(t), \tag{3.4}$$

with the initial condition:

$$x(t_0) = x_0. \tag{3.5}$$

Here, as before, it is assumed that $J \in C^1[0, 1]$, $B(t) \neq 0$, $Q(t) \geq 0$, $Q(t_0) = 0$ and $R(t) > 0$ are, respectively, time-varying coefficients of the state and control variables in the cost function with continuous functions as their entries and $x(t) \in [0, 1]$.

Following the above approach, it can be shown that the Euler–Lagrange equations (3.2) for FOCP (3.3)–(3.5) leads to the following equations:

$$\begin{aligned} {}_0D_t^{\alpha} x(\mathrm{t}) &= A(t)x(t) + B(t)u(t), \quad {}_0D_t^{\alpha-1}x(0) = x_0, \\ {}_{\mathrm{t}}D_1^{\alpha}\lambda(t) &= Q(t)x(\mathrm{t}) + \lambda(\mathrm{t})A(t), \quad {}_tD_1^{\alpha-1}\lambda(1) = 0 \\ u(\mathrm{t}) &= -R^{-1}(t)B(t)\lambda(t) \end{aligned} \tag{3.6}$$

Under some conditions as can be seen from theorem 2.1, it is worthy to note that FOCP (3.3)–(3.5) can be converted to the equivalent system of Volterra integral equation. By applying a proper fractional integral operator for the first equation of (3.6), we have:

$${}_0I_t^{\alpha}\,{}_0D_t^{\alpha}x(\mathrm{t}) = {}_0I_t^{\alpha}(A(t)x(t) + B(t)u(t)). \tag{3.7}$$

According to Theorem 2.1 and definition of fractional integral it follows that:

$$x(t) - x_0\frac{t^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(A(s)x(s) + B(s)u(s))ds, \tag{3.8}$$

or equivalently it can be rewritten as follow:

$$x(t) = x_0\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(A(s)x(s) + B(s)u(s))ds. \tag{3.9}$$

As a similar way, the second equation of (3.6) can be converted to the following Volterra integral equation:

$$\lambda(t) = \frac{1}{\Gamma(\alpha)}\int_t^1 (s-t)^{\alpha-1}(Q(s)x(s) + \lambda(s)A(s))ds. \tag{3.10}$$

4 Approximation method

Now, we introduce a practical approach for approximating solutions of the Volterra integral equation (2.11). The main idea of this approach is to convert this equation into an equivalent optimization problem. So, consider the following theorem which we shall apply to formulate our approach.

Lemma 4.1 *If h(x) be continuous and non-negative function on* $[0, 1]$*, the necessary and sufficient condition for* $\int_0^1 \mathrm{h(x)dx} = 0$ *is that* $\mathrm{h(x)} \equiv 0$ *on* $[0, 1]$.

Proof See [40].

According to Lemma 4.1, the necessary and sufficient condition for $x(.)$ to be a solution of Eq. (2.11) is that the following optimization problem, that is called the total error optimization problem, has zero objective function:

$$\min_x \int_0^1 |x(t) - C_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, x(s))ds|dt. \tag{4.1}$$

In this numerical approximation, we can control the accuracy of the results. Indeed, if $x(.)$: $[0, 1], \to R$ is an ε-approximate solution of (2.11), then, we can write:

$$|x(t) - C_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds| \le \varepsilon, \quad \text{for all} \quad t \in [0, 1], \tag{4.2}$$

where $C_0 = c_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. Now, if we want the total error to be less than a given number ε, we solve the following optimization problem:

$$\min_x \|E(x,t)\|_{L_p}^p$$
$$s.t. \|E(x,t)\|_{L_p}^p < \varepsilon,$$

where $0 \le x, t \le 1$ for any $p \ge 1$ and;

$$E(x,t) = x(t) - C_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds.$$

□

Theorem 4.1 *Let $x(t)$ be a continuous function on* $[0, 1]$ *and a solution for Eq.* (2.11). *Then $x(t)$ is the optimal solution of* (4.2) *with zero objective function.*

Proof See [22,40] for more details.

Now, for solving the optimization problem (4.2), we discrete the integral as following:

$$\begin{aligned} &\min_x \int_0^1 \left| x(t) - C_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \right| dt \\ &= \min_x \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left| x(t) - C_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \right| dt \end{aligned} \tag{4.3}$$

where $t_0 = 0$, $t_n = 1$ and $t_i \in (0, 1)$, $i = 1, 2, \ldots, n-1$. By using trapezoidal rule in any subinterval for approximating integrals, problem (4.3) can be converted to following problem:

$$\begin{aligned} &min \frac{1}{2n} \sum_{i=1}^n \left| \{x(t_{i-1}) - C_1 - \frac{1}{\Gamma(\alpha)} \int_0^{t_{i-1}} (t_{i-1} - s)^{\alpha-1} f(s, x(s)) ds\} \right. \\ &\left. + \{x(t_i) - C_2 - \frac{1}{\Gamma(\alpha)} \int_0^{t_i} (t_i - s)^{\alpha-1} f(s, x(s)) ds\} \right|, \end{aligned} \tag{4.4}$$

where $C_1 = c_0 \frac{t_{i-1}^{\alpha-1}}{\Gamma(\alpha)}$ and $C_2 = c_0 \frac{t_i^{\alpha-1}}{\Gamma(\alpha)}$. Now we will define piecewise linear function $F_{\mathrm{n}}(\mathrm{x})$ on [0,1] as following:

$$F_{\mathrm{n}}(s) = \sum_{j=1}^n f(t_j, x_j) \chi_{[\frac{j-1}{n}, \frac{j}{n}]}(s), \tag{4.5}$$

where the characteristic function $\chi_A(x)$, $A \subseteq \mathbb{R}$, of this formula is as follow:

$$\chi_A(\mathrm{x}) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \tag{4.6}$$

By using approximation (4.5), problem (4.4) is equivalent with the following optimization problem:

$$min\frac{1}{2n}\sum_{i=1}^{n}\Bigg|\left\{x(t_{i-1})-C_1-\frac{1}{\Gamma(\alpha)}\int_0^{t_{i-1}}(t_{i-1}-s)^{\alpha-1}\sum_{j=1}^{i-1}f(t_j,x_j)X_{[\frac{j-1}{n},\frac{j}{n}]}(s)ds\right\}$$
$$+\left\{x(t_i)-C_2-\frac{1}{\Gamma(\alpha)}\int_0^{t_i}(t_i-s)^{\alpha-1}\sum_{j=1}^{i}f(t_j,x_j)X_{[\frac{j-1}{n},\frac{j}{n}]}(s)ds\right\}\Bigg| \tag{4.7}$$

and equivalently we will have:

$$min\frac{1}{2n}\sum_{i=1}^{n}\Bigg|\left\{x(t_{i-1})-C_1-\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i-1}f(t_j,x_j)\int_0^{t_{i-1}}(t_{i-1}-s)^{\alpha-1}X_{[\frac{j-1}{n},\frac{j}{n}]}(s)ds\right\}$$
$$+\left\{x(t_i)-C_2-\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}f(t_j,x_j)\int_0^{t_i}(t_i-s)^{\alpha-1}X_{[\frac{j-1}{n},\frac{j}{n}]}(s)ds\right\}\Bigg| \tag{4.8}$$

Calculating the integrals in this problem so we have:

$$min\frac{1}{2n}\sum_{i=1}^{n}\Bigg|\left\{x(t_{i-1})-C_1-\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i-1}f(t_j,x_j)p_{i,j}\right\}$$
$$+\left\{x(t_i)-C_2-\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}f(t_j,x_j)q_{i,j}\right\}\Bigg|, \tag{4.9}$$

in which:

$$p_{i,j}=\frac{\mathrm{h}^\alpha}{\alpha}\begin{cases}1, & j=i-1\\ (\mathrm{i}-\mathrm{j})^\alpha-(\mathrm{i}-\mathrm{j}-1)^\alpha, & j<i-1\end{cases}$$
$$q_{i,j}=\frac{\mathrm{h}^\alpha}{\alpha}\begin{cases}1, & j=i\\ (\mathrm{i}-\mathrm{j}+1)^\alpha-(\mathrm{i}-\mathrm{j})^\alpha, & j<i.\end{cases} \tag{4.10}$$

In the easier way, optimization problem (4.9) is equivalent with the following problem:

$$min\frac{1}{2n}\sum_{i=1}^{n}\left|x(t_{i-1})+x(t_i)-C_1-C_2-\frac{\mathrm{h}^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^{i}f(t_j,x_j)k_{i,j}\right| \tag{4.11}$$

where,

$$k_{i,j}=\begin{cases}1, & j=i\\ 2^\alpha, & j=i-1\\ (\mathrm{i}-\mathrm{j}+1)^\alpha-(\mathrm{i}-\mathrm{j}-1)^\alpha, & \mathrm{j}<i-1.\end{cases} \tag{4.12}$$

We can change the optimization problem (4.11) to a linear programming problem with the following change of variable:

$$\min\frac{1}{2n}\sum_{i=1}^{n}(u_i+v_i) \tag{4.13}$$

s.t.

$$x(t_{i-1}) + x(t_i) - C_1 - C_2 - \frac{\mathrm{h}^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=1}^{i} f(t_j, x(t_j))k_{i,j} = u_i - v_i, i = 1, \ldots, n,$$
$$u_i, v_i \geq 0, i = 1, \ldots, n,$$

where in, $k_{i,j}$ will be obtained from (4.12) and C_1, C_2 are achieved as before. (4.13)

Finally, by obtaining the solution of problem (4.13), we recognize the approximate solution of Volterra integral equation (2.11).

Based on the above concepts, the key to the derivation of the approach for numerical solution of FOCP (3.3) – (3.5) is to replace system (3.6) by the following equivalent optimization problem:

$$min \int_0^1 \left(\left| {}_0D_t^{\alpha} x(t) - A(t)x(\mathrm{t}) - B(t)u(t) \right| + \left| {}_tD_1^{\alpha} \lambda(\mathrm{t}) - Q(t)x(t) - \lambda(\mathrm{t})A(t) \right| \right) dt$$
$${}_0D_t^{\alpha-1} x(0) = x_0, {}_tD_1^{\alpha-1} \lambda(1) = 0. \tag{4.14}$$

To solve this optimization problem, by approximating integrals as before and using Eqs. (3.9)–(3.10), we transformed (4.14) to the following optimization problem:

$$min \int_0^1 \left(\left| x(t) - x_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (A(s)x(s) + B(s)u(s))ds \right| \right.$$
$$\left. + \left| \lambda(t) - \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} (Q(s)x(s) + \lambda(\mathrm{s})A(s))ds \right| \right) dt. \tag{4.15}$$

Then, problem (4.15) can be converted to a linear programming problem with changes of variables as before. □

5 Numerical examples

This section is devoted to illustrate the validity and accuracy of the proposed numerical approach for FDEs and FOCPs by three examples. The calculations are performed using the Matlab software.

Example 5.1 Consider the following FDE:

$${}_0D_t^{\alpha} x(\mathrm{t}) = -x^2(t) + 1 \tag{5.1}$$

where $0 < \alpha \leq 1$ and the primary condition $x(0) = 0$. The exact solution for this equation when $\alpha = 1$ is given by:

$$x(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{5.2}$$

Now, assume that $\bar{x}(t_j)$ and $x(t_j)$ for $\mathrm{j} = 1, \ldots, \mathrm{N}$, are the approximated and exact solutions of Eq. (5.1), respectively. We defined the absolute error of approximation as follow:

Table 1 Absolute errors of $x(t)$ at $\alpha = 1$ and various choices of N for Example 5.1

Time	N = 10	N = 20	N = 40	N = 50
0.1	3.5704×10^{-2}	2.8621×10^{-4}	5.8437×10^{-4}	1.0514×10^{-4}
0.2	3.2556×10^{-2}	1.0206×10^{-3}	9.4608×10^{-4}	3.9260×10^{-4}
0.3	3.9002×10^{-2}	2.1021×10^{-3}	1.5114×10^{-3}	8.2355×10^{-4}
0.4	2.7051×10^{-2}	3.3940×10^{-3}	2.1601×10^{-3}	1.3437×10^{-3}
0.5	4.3865×10^{-2}	4.7504×10^{-3}	2.8451×10^{-3}	1.8939×10^{-3}
0.6	2.1284×10^{-2}	6.0381×10^{-3}	3.4986×10^{-3}	2.4190×10^{-3}
0.7	4.8219×10^{-2}	7.1535×10^{-3}	4.0663×10^{-3}	2.8756×10^{-3}
0.8	1.6987×10^{-2}	8.0292×10^{-3}	4.5129×10^{-2}	3.2350×10^{-3}
0.9	5.0815×10^{-2}	8.6339×10^{-3}	8.3631×10^{-2}	3.4835×10^{-3}
1.0	1.4855×10^{-2}	8.9669×10^{-3}	8.5387×10^{-3}	3.6203×10^{-3}

Table 2 Absolute errors of $x(t)$ at $\alpha = 1$ and various choices of N for Example 5.1

N	E_{min}	E_{max}
10	1.4855×10^{-3}	5.0815×10^{-2}
20	8.2753×10^{-5}	8.9669×10^{-3}
40	6.4989×10^{-5}	8.5387×10^{-3}
50	5.3273×10^{-6}	3.6203×10^{-3}

$$E = \max_j(\|x(t_j) - \bar{x}(t_j)\|), \quad \mathrm{j} = 1, \ldots, \mathrm{N}, \tag{5.3}$$

where $\|.\|$ is any norm in R space, such as usual norm.

Table 1, show the maximum absolute errors of state variable x(t) obtained using proposed approximation for $\alpha = 1$ and different values of N. As can be seen, from Table 1 it is clear that we obtain the approximate solutions with high accuracy in some points and in others it does not. As we mentioned before, since the proposed approach is based on minimization the total error, therefore solutions are close to exact solutions in some points and in others are not close. So we define the error norms as:

$$E_{max} = \max_j(|x(t_j) - \bar{x}(t_j)|), \quad \mathrm{j} = 1, \ldots, \mathrm{N}, \tag{5.4}$$

and

$$E_{min} = \min_j(|x(t_j) - \bar{x}(t_j)|), \quad \mathrm{j} = 1, \ldots, \mathrm{N}. \tag{5.5}$$

In order to demonstrate that our scheme is more accurate, we compute the error functions (5.4) and (5.5) and shows the results in Table 2.

As indicated in Table 2, by increasing the value of N we get better solutions. Behaviors of the numerical solutions of this problem with different values of α are given in Fig. 1 that shows as the value of α approaches into 1, the numerical solutions for this fractional differential equations approach into the analytical solution for $\alpha = 1$.

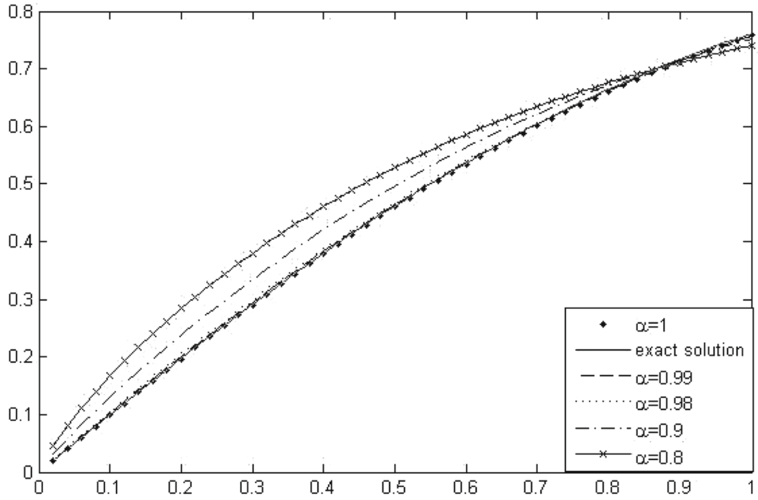

Fig. 1 Exact and approximation solutions at different values of α and N = 50 for Example 5.1

Example 5.2 Consider the following fractional initial value problem:

$$ {}_0D_t^{\alpha} x(t) = \frac{\Gamma(9)}{\Gamma(9-\alpha)} t^{8-\alpha} + t^8 + \frac{9}{4} t^8 + \frac{9}{4} \Gamma(\alpha + 1) - x(t). \tag{5.6} $$

with the initial condition $x(0) = 0$ and $0 < \alpha < 1$. By using the following formula

$$ {}_0D_t^{\alpha} t^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu - \alpha + 1)} t^{\mu-\alpha}, \tag{5.7} $$

it can be checked that the exact solution of the initial value problem (5.6) is

$$ x(t) = t^8 + \frac{9}{4} t^{\alpha}. \tag{5.8} $$

In our numerical approximation, the values of $x(t)$ at points $\{t_j\}_{j=0}^{N}$ are calculated by solving problem (4.13). The error functions for this example are obtained in Table 3. From these results, we can see that the results are in good agreement with [30].

Tables 4 and 5 show the maximum absolute errors of the variable $x(t)$ obtained using (5.4) and (5.5) and various choices of N and α. As can be seen, with increasing the number iterations N, good approximations of the exact solution were achieved.

Figure 2 shows the approximate solution for $x(t)$ by solving optimization problem (4.13) for $N = 50$ and different values of α. It can be seen with increasing α, the approximate solutions coincide with the exact solutions.

Example 5.3 Consider the following FOCP ($0 < \alpha \leq 1$):

$$ \min\ \mathrm{J(u)} = \frac{1}{2} \int_0^1 (x^2(\mathrm{t}) + \mathrm{u(t)})\mathrm{dt} \tag{5.9} $$

$$ \text{s.t.}\quad {}_0D_t^{\alpha} x(t) = -x(t) + u(t) $$

$$ x(0) = 1,\quad u(1) = 0. $$

Table 3 Absolute errors of $x(t)$ at different values of N and $\alpha = 1$ for Example 5.2

Time	N = 10	N = 20	N = 40	N = 50
0.05	2.586×10^{-5}	1.048×10^{-5}	4.492×10^{-6}	3.791×10^{-7}
0.10	2.886×10^{-4}	1.245×10^{-5}	5.607×10^{-5}	4.384×10^{-6}
0.15	4.438×10^{-4}	1.969×10^{-5}	9.129×10^{-5}	7.189×10^{-6}
0.20	3.104×10^{-4}	1.409×10^{-4}	6.652×10^{-5}	5.257×10^{-5}
0.25	1.411×10^{-4}	6.514×10^{-4}	3.111×10^{-4}	2.464×10^{-5}
0.30	4.881×10^{-4}	2.281×10^{-4}	1.097×10^{-4}	8.712×10^{-4}
0.35	1.396×10^{-3}	6.587×10^{-3}	3.189×10^{-4}	2.534×10^{-4}
0.40	3.474×10^{-3}	1.651×10^{-3}	8.032×10^{-3}	6.389×10^{-4}
0.45	7.771×10^{-3}	3.716×10^{-3}	1.814×10^{-3}	1.444×10^{-3}
0.5	1.597×10^{-3}	7.679×10^{-3}	3.758×10^{-3}	2.993×10^{-4}

Table 4 Absolute errors of $x(t)$ at different values of N and α =1 for Example 5.2

N	E_{min}	E_{max}
10	2.586×10^{-10}	1.597×10^{-3}
20	3.149×10^{-11}	7.679×10^{-4}
40	1.215×10^{-11}	3.758×10^{-4}
50	1.901×10^{-12}	2.993×10^{-4}

Table 5 Absolute errors of $x(t)$ with different values of α and N = 100 for Example 5.2

α	E_{min}	E_{max}
1	1.901×10^{-13}	2.993×10^{-4}
0.9	1.269×10^{-10}	2.894×10^{-4}
0.8	9.763×10^{-12}	2.764×10^{-4}
0.7	3.421×10^{-8}	2.587×10^{-4}
0.6	3.885×10^{-14}	2.381×10^{-4}
0.5	2.576×10^{-7}	2.053×10^{-4}

The exact solution for this problem when $\alpha = 1$ is given in [1]:

$$x(t) = \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t), \tag{5.10}$$

and;

$$u(t) = (1 + \beta\sqrt{2})\cosh(\sqrt{2}t) + (\beta\sqrt{2})\sinh(\sqrt{2t}), \tag{5.11}$$

where,

$$\beta = -\frac{\cosh(\sqrt{2}) + (\sqrt{2})\sinh(\sqrt{2})}{(\sqrt{2})\cosh(\sqrt{2}) + \sinh(t\sqrt{2})} \cong -0.9799.$$

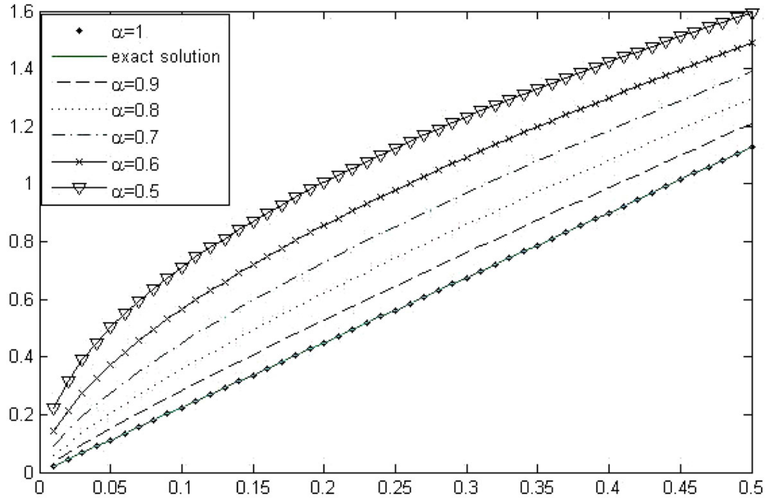

Fig. 2 Exact and approximation solutions at different values of α and N = 50 for Example 5.2

From Eq. (3.6), the necessary optimality conditions are yielded as follows:

$$\begin{aligned} {}_0D_t^{\alpha}x(\mathrm{t}) &= -x(t) + u(t) \\ {}_tD_1^{\alpha}u(t) &= -x(t) - u(t) \\ {}_0D_t^{\alpha-1}x(0) &= 1, {}_tD_1^{\alpha-1}u(1) = 0. \end{aligned} \tag{5.12}$$

By using the proposed method, the Lagrange equations (5.12) are converted into the following optimization problem:

$$\begin{aligned} min \int_0^1 \Bigg(\left| x(t) - 1 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(u(s) - x(s))ds \right| \\ + \left| u(t) + \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1}(x(s) + u(s))ds \right| \Bigg) dt, \end{aligned} \tag{5.13}$$

After some manipulations, optimization problem (5.13) reduces to:

$$\begin{aligned} min \frac{1}{2n} \sum_{i=1}^{n} \Bigg\{ \left| x(t_{i-1}) + x(t_i) - 2 - \frac{\mathrm{h}^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=1}^{i} (u_j - x_j) k_{i,j} \right| \\ + \left| u(t_{i-1}) + u(t_i) + \frac{\mathrm{h}^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=i}^{n} (u_j + x_j) l_{i,j} \right| \Bigg\} \end{aligned} \tag{5.14}$$

where $k_{i,j}$ is defined as (4.12) and $l_{i,j}$ is given as follow: (5.14)

$$l_{i,j} = \begin{cases} 1, & j = i \\ (\mathrm{j} - \mathrm{i} + 1)^{\alpha} - (\mathrm{j} - \mathrm{i} - 1)^{\alpha}, & \mathrm{j} > i. \end{cases}$$

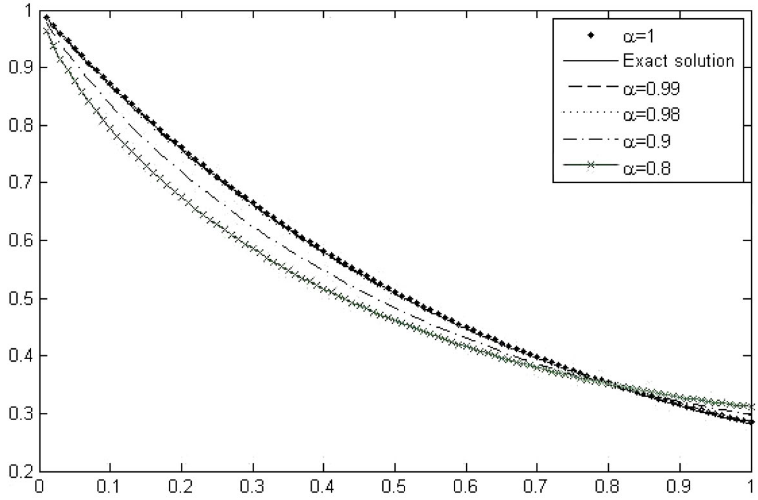

Fig. 3 Exact and approximate solutions of $x(t)$ at N = 100 and different values of α for Example 5.3

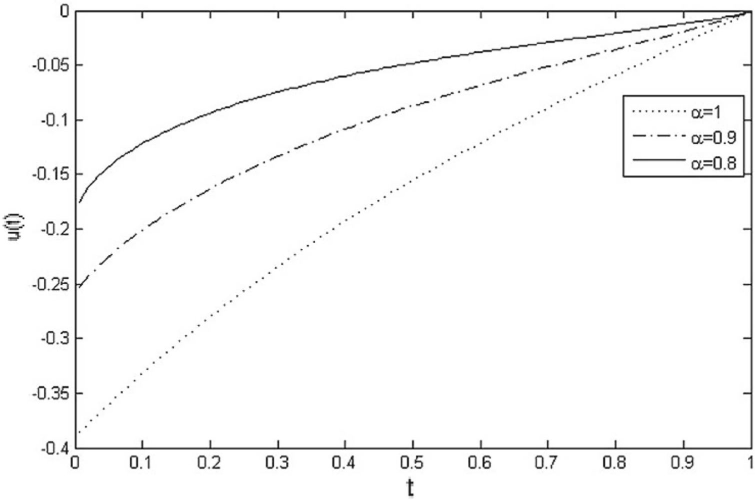

Fig. 4 Approximate solutions of $u(t)$ for N = 100 and different values of α for Example 5.3

We can convert problem (5.14) to a linear programming problem with the following change of variable:

$$\min \frac{1}{2n}\sum_{i=1}^{n}(u_i + v_i + w_i + p_i) \tag{5.15}$$

s.t.

$$x(t_{i-1}) + x(t_i) - 2 - \frac{\mathrm{h}^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=1}^{i}(u_j - x_j)k_{i,j} = u_i - v_i,\ i = 1, \ldots, n,$$

Table 6 Absolute errors of $x(t)$ and $u(t)$ at $\alpha = 1, t = 0.5$ and different values of N for Example 5.3

N	$x(t)$	$u(t)$
10	3.40323×10^{-3}	4.21340×10^{-2}
20	2.21262×10^{-4}	2.51714×10^{-4}
50	2.64992×10^{-6}	1.32072×10^{-7}
100	1.43634×10^{-9}	3.97889×10^{-9}

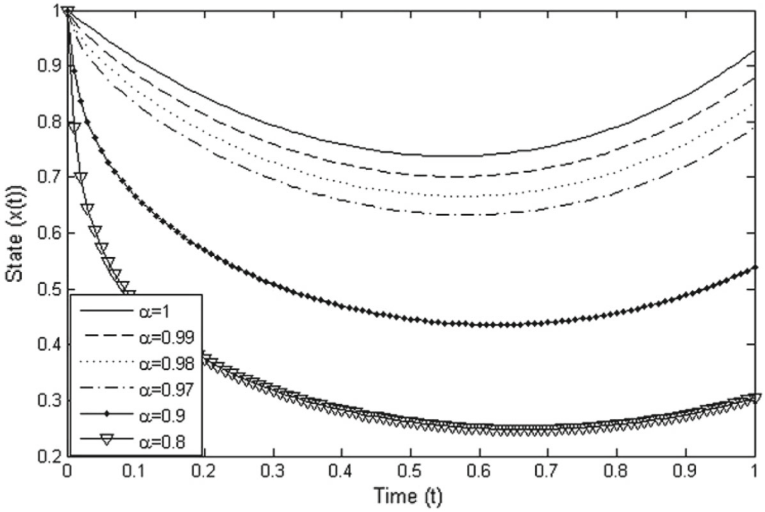

Fig. 5 Approximate solutions of $x(t)$ at N = 100 and different values of α for Example 5.4

$$u(t_{i-1}) + u(t_i) + \frac{\mathrm{h}^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=i}^{n} (u_j + x_j) l_{i,j} = w_i - p_i, i = 1, \ldots, n,$$
$$u_i, v_i, w_i, p_i \geq 0, i = 1, \ldots, n.$$

Figures 3 and 4 show the approximate solutions for the state $x(t)$ and control $u(t)$ for $N = 50$ and different values of α. It reveals that as α approaches to 1, the classical solution of problem (5.15) is recovered and leads to a good approximations with the exact state and control variables. Table 6 show the maximum absolute errors of $x(t)$ and $u(t)$ by using the approximated approach of this paper at $\alpha = 1$ and various choices of N.

Example 5.4 Consider the following FOCP in which $0 < \alpha \leq 1$ (see [19]):

$$\min \quad \mathrm{J(u)} = \frac{1}{2} \int_0^1 (x^2(\mathrm{t}) + \mathrm{u}^2(\mathrm{t}))\mathrm{dt} \tag{5.16}$$
$$\text{s.t.} \quad {}_0D_t^{\alpha} x(t) = t x(t) + u(t)$$
$$x(0) = 1, \quad u(1) = 0.$$

Figures 5 and 6 demonstrates the approximation of $x(t)$ and $u(t)$ for different values of α. Our results are in good agreement with the experimental results [5,10] that solved this problem by a different way.

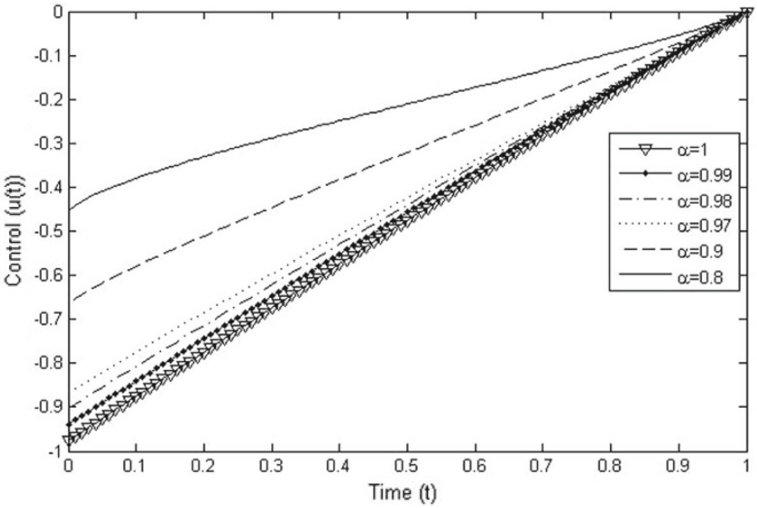

Fig. 6 Approximate solutions of $u(t)$ for N $= 100$ and different values of α for Example 5.4

6 Conclusion

The main focus of this paper is to present a numerical approximation for solving a class of FOCPs. In this approach, the properties of the RLFD are used to reduce the given FOCP into a system of Volterra integral equations that can be easily implemented. We state and prove the existence and uniqueness theorem of solutions for this system by using one of the most important theorems in functional analysis which is the fixed-point theorem. Then, we reduce this system of Volterra integral equations to an optimization problem. By discretizing the new problem and solving it, we obtained the best approximate solution of the original FOCP. Some examples are given and the numerical simulations are also provided to illustrate the effectiveness of the new approach. It is expected that as α is an integer number the presented approximation is equivalent to well-known results for optimal control problems of an integer order.

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