Product of derivations on C*-algebras

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Abstract

Let $\mathfrak{A}$ be an algebra. A linear mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ is called a \textit{derivation} if $\delta(ab) = \delta(a)b + a\delta(b)$ for each $a, b \in \mathfrak{A}$. Given two derivations $\delta$ and $\delta'$ on a C*-algebra $\mathfrak{A}$, we prove that there exists a derivation $\Delta$ on $\mathfrak{A}$ such that $\delta \delta' = \Delta^2$ if and only if either $\delta' = 0$ or $\delta = s\delta'$ for some $s \in \mathbb{C}$.

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1. Introduction

Let $\mathfrak{A}$ be an algebra. A linear mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ is called a \textit{derivation} if it satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for each $a, b \in \mathfrak{A}$. When $\mathfrak{A}$ is a *-algebra, $\delta$ is called a *-\textit{derivation} if $\delta(a^*) = \delta(a)^*$ for each $a \in \mathfrak{A}$.

Let $\delta$ be a *-derivation on a C*-algebra $\mathfrak{A}$, then $\delta^2$ is a derivation if and only if $\delta = 0$. To see this, note that $\delta^2$ is a derivation if and only if

$$\delta^2(xy) = \delta^2(x)
\delta^2(x) + 2\delta(x)\delta(y) + x\delta^2(y) = \delta^2(xy) = \delta^2(x) + 2\delta(x)\delta(y) + x\delta^2(y).$$

The latter is equivalent to the fact that $\delta(x)\delta(y) = 0$ for each $x, y \in \mathfrak{A}$. Thus $\delta(x)\delta(x^*) = \delta(x)\delta(x^*) = 0$ for each $x \in \mathfrak{A}$. Hence $\|\delta(x)\|^2 = \|\delta(x)\delta(x^*)\| = 0$. This shows that $\delta(x) = 0$ for each $x \in \mathfrak{A}$.

As a typical example of a non-zero derivation in a non-commutative algebra, we can consider the \textit{inner derivation} $\delta_a$ implemented by an element $a \in \mathfrak{A}$ which is defined as $\delta_a(x) = xa - ax$ for each $x \in \mathfrak{A}$. Even for an inner derivation $\delta_a$ on an algebra $\mathfrak{A}$, it is very probable that $\delta_a^2$ is \textit{not} a derivation.

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These considerations show that the set of derivations on an algebra $\mathfrak{A}$ is not in general closed under product. There are various researches seeking for some conditions under which the product of two derivations will be again a derivation. Posner [9] was the first one who studied the product of two derivations on a prime ring. He showed that if the product of two derivations on a prime ring, with characteristic not equal to 2, is a derivation then one of them must be equal to zero. The same question has been investigated by several authors on various algebras, see for example [1, 2, 3, 5, 6, 7, 8] and references therein. In the realm of $C^*$-algebras, Mathieu [5] showed that, if the product of two derivations $\delta$ and $\delta'$ on a $C^*$-algebra is a derivation then $\delta\delta' = 0$. The same result was proved by Pedersen [8] for unbounded densely defined derivations on a $C^*$-algebra.

There are known algebras $\mathfrak{A}$ such that each derivation on $\mathfrak{A}$ is inner which is implemented by an element of the algebra $\mathfrak{A}$ or an algebra $\mathfrak{B}$ containing $\mathfrak{A}$. For example, each derivation on a von Neumann algebra $\mathfrak{M}$ is inner and is implemented by an element of $\mathfrak{M}$. Moreover, each derivation on a $C^*$-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathfrak{H}$ is inner and implemented by an element of the weak closure $\mathfrak{M}$ of $\mathfrak{A}$ in $\mathfrak{B}(\mathfrak{H})$ (See [4, 10]).

In the present paper, we are concerned with the following problem: “Given two derivations $\delta$ and $\delta'$ on a $C^*$-algebra $\mathfrak{A}$, find necessary and sufficient condition under which there exists a derivation $\Delta$ on $\mathfrak{A}$ satisfying $\delta\delta' = \Delta^2$.”

We affirm that the condition is: either $\delta' = 0$ or $\delta = s\delta'$ for some $s \in \mathbb{C}$. We do this in two steps; for the matrix algebra $M_n(\mathbb{C})$ and for an arbitrary $C^*$-algebra.

2. The equation for the case of matrix algebras

In this section we are mainly concerned with the structure of derivations on the matrix algebra $M_n(\mathbb{C})$. Let $A = [a_{ij}] \in M_n(\mathbb{C})$. We denote the diagonal matrix whose diagonal entries are $a_{ii}$ by $A^D$.

**Proposition 2.1.** Let $A = [a_{ij}], B = [b_{ij}] \in M_n(\mathbb{C})$. Then there exists a $C = [c_{ij}] \in M_n(\mathbb{C})$ such that $\delta_A\delta_B = \delta C^2$ if and only if either $\delta_B = 0$ or $\delta_A = s\delta_B$ for some $s \in \mathbb{C}$.

**Proof.** Let $\{E_{ij}\}_{1 \leq i, j \leq n}$ be the standard system of matrix units for $M_n(\mathbb{C})$. First we show that $a_{ik}b_{lj} = b_{ik}a_{lj}$ for all $1 \leq i, k, \ell, j \leq n$ if and only if $AXB = BXA$ for all $X \in M_n(\mathbb{C})$.

To see this, suppose that $a_{ik}b_{lj} = b_{ik}a_{lj}$ for all $1 \leq i, k, \ell, j \leq n$ then we can write

$$(E_{ii}AE_{kl})(E_{ell}BE_{jj}) = a_{ik}b_{lj}E_{ij} = b_{ik}a_{lj}E_{ij} = (E_{ii}BE_{kl})(E_{ell}AE_{jj}).$$

We thus have

$$(\sum_{i=1}^{n} E_{ii})AE_{kl}B(\sum_{j=1}^{n} E_{jj}) = (\sum_{i=1}^{n} E_{ii})BE_{kl}A(\sum_{j=1}^{n} E_{jj}).$$

This shows that $AE_{kl}B = BE_{kl}A$ for each $1 \leq k, \ell \leq n$. We can therefore deduce that $AXB = BXA$ for all $X \in M_n(\mathbb{C})$. On the other hand, if $AXB = BXA$ for all $X \in M_n(\mathbb{C})$, then

$$a_{ij}b_{kl}E_{id} = (E_{ii}AE_{jk})(E_{kk}BE_{ell}) = (E_{ii}BE_{jk})(E_{kk}AE_{ell}) = b_{ij}a_{kl}E_{id}.$$ 

We can assume that $a_{11} = b_{11} = c_{11} = 0$. This is due to the fact that $\delta_{A-a_{11}I} = \delta_A$, $\delta_{B-b_{11}I} = \delta_B$ and $\delta_{C-c_{11}I} = \delta_C$. Then $\delta_A\delta_B = \delta C^2$ if and only if

$$ABE_{kl} - AE_{kl}B - BE_{kl}A + E_{kl}BA = C^2E_{kl} - 2CE_{kl}C + E_{kl}C^2,$$
for each $1 \leq k, \ell \leq n$. This is equivalent to the fact that
\[
E_{ii}(ABE_{k\ell} - AE_{k\ell}B - BE_{k\ell}A + E_{k\ell}BA)E_{jj} = E_{ii}(C^2E_{k\ell} - 2CE_{k\ell}C + E_{k\ell}C^2)E_{jj},
\]
for each $1 \leq i, j, k, \ell \leq n$. Now for $i \neq k$ and $j \neq \ell$ we have
\[
(0 - a_{ik}b_{\ell j} - b_{ik}a_{\ell j} + 0)E_{ij} = (0 - 2c_{ik}c_{\ell j} + 0)E_{ij}.
\]
(2.1)

For $i \neq k$ and $j = \ell$ we have
\[
\sum_{m=1}^{n} a_{im}b_{mk} - a_{ik}b_{\ell m} - b_{ik}a_{\ell m} + 0)E_{i\ell} = \sum_{m=1}^{n} c_{im}c_{mk} - 2c_{ik}c_{\ell m} + 0)E_{i\ell}.
\]
(2.2)

For $i = k$ and $j \neq \ell$ we have
\[
(0 - a_{kk}b_{\ell j} - b_{kk}a_{\ell j} + \sum_{m=1}^{n} b_{\ell m}c_{mj})E_{kj} = (0 - 2c_{kk}c_{\ell j} + \sum_{m=1}^{n} c_{\ell m}c_{mj})E_{kj}.
\]
(2.3)

And finally for $i = k$ and $j = \ell$ we have
\[
\sum_{m=1}^{n} a_{km}b_{mk} - a_{kk}b_{\ell m} - b_{kk}a_{\ell m} + \sum_{m=1}^{n} b_{\ell m}c_{mj})E_{k\ell} = \sum_{m=1}^{n} c_{km}c_{mk} - 2c_{kk}c_{\ell m} + \sum_{m=1}^{n} c_{\ell m}c_{mj})E_{k\ell}.
\]
(2.4)

If $k \neq \ell$ then putting $i = \ell$ and $j = k$ in the equation (2.1) we have $c_{\ell k}^2 = a_{\ell k}b_{\ell k}$. Thus for $i \neq k$ and $j \neq \ell$ we have $(a_{ik}b_{\ell j} + b_{ik}a_{\ell j})^2 = 4c_{ik}^2c_{\ell j}^2 = 4a_{ik}b_{\ell k}a_{\ell j}b_{\ell j}$. This implies that
\[
a_{ik}b_{\ell j} = b_{ik}a_{\ell j}, \text{ for } i \neq k, j \neq \ell.
\]
(2.5)

Now, if $b_{\ell j} \neq 0$ for some $1 \leq \ell, j \leq n$ with $\ell \neq j$, then the equation
\[
a_{ik} = \frac{a_{\ell j}}{b_{\ell j}}b_{ik}, \text{ for } i \neq k,
\]
implies the existence of some $\alpha$ and $\beta$ with $|\alpha| + |\beta| \neq 0$ such that
\[
\alpha(A - AD) = \beta(B - BD).
\]
(2.6)

If $b_{\ell j} = 0$ for all $1 \leq \ell, j \leq n$ with $\ell \neq j$, then $B = BD$ and so the equation (2.6) holds for $\alpha = 0$ and any nonzero $\beta \in \mathbb{C}$.

Interchanging $\ell \leftrightarrow i, j \leftrightarrow k$ and $k \leftrightarrow \ell$ in (2.3) we have
\[
\sum_{m=1}^{n} b_{im}a_{mk} - a_{\ell m}b_{ik} - b_{\ell m}a_{ik} = \sum_{m=1}^{n} c_{im}c_{mk} - 2c_{\ell m}c_{ik}, \text{ for } i \neq k.
\]
(2.7)

It follows from (2.2) and (2.7) that
\[
\sum_{m=1}^{n} a_{im}b_{mk} = \sum_{m=1}^{n} b_{im}a_{mk}, \text{ for } i \neq k.
\]

Returning to the fact that $a_{im}b_{mk} = b_{im}a_{mk}$ for $m \neq i, k$, we have
\[
a_{ij}b_{ik} + a_{ik}b_{kk} = b_{ij}a_{ik} + b_{ik}a_{kk}, \text{ for } i \neq k.
\]
This implies that
\[ a_{ik}(b_{ii} - b_{kk}) = b_{ik}(a_{ii} - a_{kk}). \] (2.8)

Putting \( k = \ell \) in (2.4) we get
\[ \sum_{m=1}^{n} a_{km}b_{mk} - a_{kk}b_{kk} = \sum_{m=1}^{n} c_{km}c_{mk} - c_{kk}c_{kk}. \]

Thus it follows from (2.4) that
\[ a_{kk}b_{kk} - a_{kk}b_{\ell\ell} - b_{kk}a_{\ell\ell} + b_{\ell\ell}a_{\ell\ell} = c_{kk}c_{kk} - 2c_{kk}c_{\ell\ell} + c_{\ell\ell}c_{\ell\ell}. \]

For \( \ell = 1 \) we have
\[ c_{kk}^2 = a_{kk}b_{kk}, \]
and then \( a_{kk}b_{\ell\ell} + b_{kk}a_{\ell\ell} = 2c_{kk}c_{\ell\ell} \). Thus for all \( 1 \leq k, \ell \leq n \) we have \( (a_{kk}b_{\ell\ell} + b_{kk}a_{\ell\ell})^2 = 4c_{kk}^2c_{\ell\ell}^2 = 4a_{kk}b_{kk}a_{\ell\ell}b_{\ell\ell} \). This implies that
\[ a_{kk}b_{\ell\ell} = b_{kk}a_{\ell\ell}, \text{ for all } k, \ell. \]

A similar argument as about the equation (2.5) implies the existence of some \( \alpha' \) and \( \beta' \) with \( |\alpha'| + |\beta'| \neq 0 \) such that
\[ \alpha'A^D = \beta'B^D. \]

Using (2.8) we have
\[ b_{jj}a_{ik}(b_{ii} - b_{kk}) = b_{ik}b_{jj}(a_{ii} - a_{kk}) = b_{ik}a_{jj}(b_{ii} - b_{kk}). \]

Now let \( B^D \notin CI \). Then \( b_{ii} \neq b_{kk} \) for some \( i \) and \( k \). This shows that \( b_{jj}a_{ik} = a_{jj}b_{ik} \). So we have \( \alpha = \alpha' \) and \( \beta = \beta' \). By a similar argument we can say that if \( A^D \notin CI \) then \( \alpha = \alpha' \) and \( \beta = \beta' \). We therefore have

if \( A^D \notin CI \) or \( B^D \notin CI \) then \( \alpha A = \beta B \) for some \( \alpha \) and \( \beta \) with \( |\alpha| + |\beta| \neq 0 \).

On the other hand, if \( A^D = sI \) and \( B^D = tI \) for some \( s, t \in \mathbb{C} \) then
\[ \alpha'A^D + \alpha(A - A^D) = s(\alpha' - \alpha)I + \alpha A, \]
and
\[ \beta'B^D + \beta(B - B^D) = t(\beta' - \beta)I + \beta B. \]

Therefore \( s(\alpha' - \alpha)I + \alpha A = t(\beta' - \beta)I + \beta B. \) Summarizing these we can say that \( \delta_A\delta_B = \delta C^2 \) if and only if \( \alpha A = \beta B + rI \) for some \( \alpha, \beta, r \in \mathbb{C} \) with \( |\alpha| + |\beta| \neq 0 \). This is equivalent to the fact that either \( \delta_B = 0 \) or \( \delta_A = s\delta_B \) for some \( s \in \mathbb{C} \). \( \square \)

A natural question is the following: Is it true in general that \( \delta \delta' = \Delta^2 \) on an algebra \( \mathcal{A} \) is equivalent to either \( \delta' = 0 \) or \( \delta = s\delta' \) for some \( s \in \mathbb{C} \)? In this case we of course have \( \Delta = \sqrt{s} \delta' \). The following example shows that the answer is not affirmative in general.

**Example 2.2.** Let \( \mathcal{A} \) be the subalgebra of \( M_2(\mathbb{C}) \) generated by \( E_{11} \) and \( E_{12} \). If \( \delta = \delta_{E_{12}} \) and \( \delta' = \delta_{E_{11}} \), then for each \( X = xE_{11} + yE_{12} \in \mathcal{A} \) we have
\[ \delta \delta'(X) = \delta(xE_{11} + yE_{12} - xE_{11}) = \delta(yE_{12}) = 0. \]

Thus \( \delta \delta' = \delta_0^2 \). But \( \delta' 
eq 0 \) and \( \delta \) is not a multiple of \( \delta' \).
Lemma 2.3. Let $\mathcal{A}$ be the subalgebra of $M_2(\mathbb{C})$ generated by $E_{11}$ and $E_{12}$. Then each derivation on $\mathcal{A}$ is of the form $\delta = \delta_{cE_{12} - dE_{11}}$ for some $c, d \in \mathbb{C}$.

Proof. Let $\delta : \mathcal{A} \rightarrow \mathcal{A}$ be a derivation defined by $\delta(xE_{11} + yE_{12}) = f(x, y)E_{11} + g(x, y)E_{12}$. Since $\delta$ is linear, $$ f(x, y) = f(x, 0) + f(0, y) = xf(1, 0) + yf(0, 1). $$

We therefore have $f(x, y) = ax + by$ and $g(x, y) = cx + dy$ for some $a, b, c, d \in \mathbb{C}$. Moreover, $$ \delta \left( (xE_{11} + yE_{12}) (x'E_{11} + y'E_{12}) \right) = \delta(xE_{11} + yE_{12}) (x'E_{11} + y'E_{12}) + (xE_{11} + yE_{12}) \delta(x'E_{11} + y'E_{12}) $$
implies
$$ f(xx', xy')E_{11} + g(xx', xy')E_{12} = f(x, y)x'E_{11} + f(x, y)y'E_{12} + x f(x', y')E_{11} + xg(x', y')E_{12}. $$

We thus have
$$ f(xx', xy') = f(x, y)x' + xf(x', y'), $$
$$ g(xx', xy') = f(x, y)y' + xg(x', y'). $$

By using the fact that $f(x, y) = ax + by$ and $g(x, y) = cx + dy$, we have $f(x, y) = 0$. Whence $\delta = \delta_{cE_{12} - dE_{11}}$. □

Proposition 2.4. Let $\mathcal{A}$ be the subalgebra of $M_2(\mathbb{C})$ generated by $E_{11}$ and $E_{12}$ and $\delta, \delta'$ be two derivations on $\mathcal{A}$. Then $\delta \delta' = \Delta^2$ if and only if $\delta' = 0$ or $\delta' = \delta_{\alpha' E_{12}}$ for some $\alpha' \in \mathbb{C}$ implies $\delta = \delta_{\alpha E_{12}}$ for some $\alpha \in \mathbb{C}$, or equivalently $\delta' = 0$ or $\delta'^2 = 0$ implies $\delta^2 = 0$.

Proof. Let $\delta = \delta_{\alpha E_{12} - \beta E_{11}}, \delta' = \delta_{\alpha' E_{12} - \beta' E_{11}}$ and $\Delta = \delta_{\tau E_{12} - \kappa E_{11}}$. Then $\delta \delta' = \Delta^2$ if and only if $rs = \beta \alpha'$ and $s^2 = \beta \beta'$. The latter is equivalent to the fact that $\delta' = 0$ or $\delta' = \delta_{\alpha' E_{12}}$ for some $\alpha' \in \mathbb{C}$ implies $\delta = \delta_{\alpha E_{12}}$ for some $\alpha \in \mathbb{C}$. On the other hand, a derivation $\delta$ on $\mathcal{A}$ is of the form $\delta_{\lambda E_{12}}$ for some $\lambda \in \mathbb{C}$ if and only if $\delta^2 = 0$. □

3. Derivations on $C^*$-algebras

Theorem 3.1. Let $\mathfrak{A}$ be a $C^*$-algebra and $\delta, \delta'$ be two derivations on $\mathfrak{A}$. Then there exists a derivation $\Delta$ on $\mathfrak{A}$ such that $\delta \delta' = \Delta^2$ if and only if either $\delta' = 0$ or $\delta = \delta' s^2$ for some $s \in \mathbb{C}$.

Proof. Let $\mathfrak{A}$ act faithfully on the Hilbert space $\mathfrak{H}$ with the orthonormal basis $\{\xi_i\}_{i \in I}$. For a bounded operator $T \in B(\mathfrak{H})$, let $T_{ij} = (T\xi_j, \xi_i)$ for $i, j \in I$. We thus have $T\xi_j = \sum_{i \in I} t_{ij} \xi_i$ and we can write $T = [t_{ij}]_{i, j \in I}$. The latter is called the matrix representation of $T$. For $i, j \in I$, let $E_{ij} = B(\mathfrak{H})$ be the operator defined by $E_{ij}\xi_j = \xi_i$ and $E_{ij}\xi_k = 0$ for $k \neq j$. Then we have $T = \sum_{p \in I} \sum_{q \in I} t_{pq} E_{qp}$ for every $T \in B(\mathfrak{H})$.

By the Kadison-Sakai theorem $[4][10]$, $\delta = \delta_R, \delta' = \delta_S$ and $\Delta = \delta_T$ for some $R, S$ and $T$ in $B(\mathfrak{H})$. Thus $\delta \delta' = \Delta^2$ if and only if
$$ RSE_{k\ell} - RE_{k\ell}S - SE_{k\ell}R + E_{k\ell}SR = T^2E_{k\ell} - 2TE_{k\ell}T + E_{k\ell}T^2, $$
for each $k, \ell \in I$. This is equivalent to the fact that
$$ E_{ii}(RSE_{k\ell} - RE_{k\ell}S - SE_{k\ell}R + E_{k\ell}SR)E_{jj} = E_{ii}(T^2E_{k\ell} - 2TE_{k\ell}T + E_{k\ell}T^2)E_{jj}, $$
for each $i, j \in I$. This is equivalent to the fact that
$$ E_{ii}(RSE_{k\ell} - RE_{k\ell}S - SE_{k\ell}R + E_{k\ell}SR)E_{jj} = E_{ii}(T^2E_{k\ell} - 2TE_{k\ell}T + E_{k\ell}T^2)E_{jj}, $$
for each $i, j \in I$.
for each \(i, j, k, \ell \in I\). For \(i \neq k\) and \(j \neq \ell\) we have
\[
r_{ik} s_{lj} + s_{ik} r_{lj} = 2 t_{ik} t_{lj}.
\]
Similarly, for \(i \neq k\) and \(j = \ell\) we have
\[
\sum_{m \in I} r_{im} s_{mk} - r_{ik} s_{\ell\ell} - s_{ik} r_{\ell\ell} = \sum_{m \in I} t_{im} t_{mk} - 2 t_{ik} t_{\ell\ell}.
\]
Also, for \(i = k\) and \(j \neq \ell\) we have
\[
- r_{kk} s_{lj} - s_{kk} r_{lj} + \sum_{m \in I} s_{\ell m} r_{mj} = -2 t_{kk} t_{lj} + \sum_{m \in I} t_{\ell m} t_{mj}.
\]
And finally for \(i = k\) and \(j = \ell\) we have
\[
\sum_{m \in I} r_{km} s_{mk} - r_{kk} s_{\ell\ell} - s_{kk} r_{\ell\ell} + \sum_{m \in I} s_{\ell m} r_{m\ell} = \sum_{m \in I} t_{km} t_{mk} - 2 t_{kk} t_{\ell\ell} + \sum_{m \in I} t_{\ell m} t_{m\ell}.
\]
Now a similar verification as in Proposition 2.1 implies the result. □

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