$k$-Combinations of an unlabelled graph

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ABSTRACT. In this paper we extend the notion of the binomial coefficient ($^n_k$) into a new notion ($^{|G|}_k$), where $[G]$ is an unlabelled graph with $n$ vertices and $0 \leq k \leq n$. We call ($^{|G|}_k$) as the graph binomial coefficient and a version of the graph binomial expansion is also studied. As an application of this notion, we enumerate the number of ways to color vertices of a path and beads of a necklace.

1. INTRODUCTION AND PRELIMINARIES

Let $n$ be a positive integer and $0 \leq k \leq n$. The binomial coefficient ($^n_k$) is the number of $k$-combinations of a set with $n$ elements. This is equal to $\frac{n!}{k!(n-k)!}$ and satisfies the recursive relation ($^n_k$) = ($^{n-1}_{k-1}$) + ($^{n-1}_k$). The summation $\sum_{k=0}^{n} (\begin{pmatrix} n \cr k \end{pmatrix})$ is then equal to the number of ways to choose a subset of a fixed set with $n$ elements which is obviously equal to $2^n$.

The mentioned fixed set with $n$ elements can be vertices of a given labelled graph. But if we omit the labels then the number of $k$-combinations is not necessarily equal to ($^n_k$). For a simple example, if we have an unlabelled path with 3 vertices, then the number of 2-combinations is not 3. In fact the two ends of the path play the same role.

Let $G$ be a graph with $n$ vertices labelled by $1, 2, \ldots, n$. If we ignore the labels we have an unlabelled graph, denoted by $[G]$, with $n$ vertices. We can formally say that $[G]$ is the class of all graphs $G'$ which are isomorphic to $G$. Whence, as a good question we can enumerate the number of $k$-combinations of an unlabelled graph $[G]$. We denote this number by ($^{|G|}_k$) and we aim to find some formulas for this. We can also evaluate $\sum_{k=0}^{n} (\begin{pmatrix} |G| \cr k \end{pmatrix})$ for a given graph $G$. The number can be interpreted as the number of ways to color the vertices of $[G]$ with two different colors. We apply this for some special cases such as paths, directed cycles and indirected cycles.

In the following, we use Burnside’s Lemma, [4], [2] and [6], which asserts that if a group $G$ acts on a set $X$, then the number of orbits of $G$ is equal to $\frac{1}{|G|} \sum_{g \in G} |X_g|$, where $X_g$ is the set of all $x \in X$ with $(g, x) = x$. To see a simple proof of Burnside’s Lemma the reader is referred to [1]. A discussion about Polya Enumeration Theorem, [7], which uses Burnside’s Lemma, can be found in [8].

Recall that the complement of a graph $G$ is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$. An automorphism of a graph $G = (V, E)$ is a permutation $\sigma$ of the vertex set $V$, such that the pair of vertices $(u, v)$ form an edge if and only if the pair $(\sigma(u), \sigma(v))$ also form an edge. The set of all automorphisms of a graph $G$, with the operation of composition of permutations, is a permutation group which is denoted by $\text{Aut}(G)$. See [3] for the terminology and main
results of permutation group theory. A graph and its complement have the same automorphism group. Frucht [5] proved that every group is the automorphism group of a graph. Moreover, if the group is finite, the graph can be taken to be finite.

Furthermore, recall that a graph \(G\) is called \textit{vertex transitive} if for each two vertices \(u\) and \(v\) of \(G\) there is an automorphism \(\sigma \in \text{Aut}(G)\) such that \(\sigma(u) = v\).

2. AN EXPLICIT FORMULA

In the following, for a labelled graph \(G\) we denote the class of all graphs \(G'\) which are isomorphic to \(G\) by \([G]\). This is called the \textit{unlabelled graph induced by} \(G\).

**Definition 2.1.** Let \([G]\) be an unlabelled graph with \(n\) vertices, where \(n\) is a positive integer. For \(0 \leq k \leq n\), a \(k\)-\textit{combination} of \([G]\) is a way of selecting members from \([G]\), such that the order of members in the selection does not matter. The number of \(k\)-combinations of \([G]\) is denoted by \((|G|)_k\) (read as \([G]\) choose \(k\)) and is called the \textit{graph binomial coefficient}.

**Example 2.1.** Let \(n\) be a positive integer. For the complete graph \(K_n\) and the star graph \(K_{1,n-1}\) we have \((|K_n|)_k = 1\) and \((|K_{1,n-1}|)_k = 2\) for each \(1 \leq k \leq n\).

Though for a vertex transitive graph \([G]\) the graph binomial coefficient \((|G|)_1\) is 1, but \((|G|)_2\) can be a number other than 1.

**Example 2.2.** Let \(Q_3\) be the 3-dimensional cube with vertices labelled as

\[a = 000, b = 001, c = 010, d = 011, e = 100, f = 101, g = 110, h = 111,\]

where two vertices are adjacent if and only if they differ in just one position. If we ignore the labels then there are three 2-combinations of \(|Q_3|\) which are \(ab, ad\) and \(ah\). Note that any other 2-combination is isomorphic to these.

We have the following two obvious results.

**Proposition 2.1.** Let \(G\) be a labelled graph with \(n\) vertices and let \(0 \leq k \leq n\). Then

\[
\left(\frac{|G|}{k}\right) = \left(\frac{|G|}{n-k}\right) = \left(\frac{|\overline{G}|}{k}\right),
\]

where \(\overline{G}\) is the complement of \(G\).

**Proposition 2.2.** Let \(G\) and \(G'\) be two labelled graphs with \(n\) vertices and let \(0 \leq k \leq n\). If \(\text{Aut}(G) \simeq \text{Aut}(G')\) then

\[
\left(\frac{|G|}{k}\right) = \left(\frac{|G'|}{k}\right).
\]

**Theorem 2.1.** Let \(G = (V, E)\) be a labelled graph with \(n\) vertices and let \(1 \leq k \leq n\). Then

\[
\left(\frac{|G|}{k}\right) = \frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} |V^k_\sigma|,
\]

where \(V^k_\sigma = \{v_1, \ldots, v_k\} \subseteq V : \sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\}\).

**Proof.** Let \(X\) be the set of \(k\)-subsets of \(V\). Then \(\text{Aut}(G)\) acts on \(X\) by the rule \((\sigma, A) = \sigma(A)\) for each \(A \in X\). Now, by the Burnside’s Lemma, the number of orbits of \(X\) under \(\text{Aut}(G)\), which is equal to \((|G|)/k\), is

\[
\frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} |V^k_\sigma|,
\]

where \(V^k_\sigma\) is the set of all members of \(X\) which are fixed under \(\sigma\).  \(\Box\)
Corollary 2.1. Let $P_n$ be the labelled path with $n$ vertices and let $1 \leq k \leq n$. Then

$$\binom{|[P_n]|}{k} = \begin{cases} \frac{1}{2} \binom{n}{k} + \frac{1}{2} \binom{n}{k} & \text{if } n \text{ is even and } k \text{ is odd} \\ \frac{1}{2} \binom{n}{k} + \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} & \text{otherwise} \end{cases}$$

Proof. There are two automorphisms for $P_n$: the identity automorphism $\iota$ and the automorphism $\alpha$ defined by $\alpha(i) = n + 1 - i$ for $1 \leq i \leq n$. For $\iota$ we obviously have $|V_\iota^k| = \binom{n}{k}$ and for $\alpha$ we see that a subset $\{v_1, \ldots, v_k\}$ of $V$ is in $V_\alpha^k$ if and only if $i \in \{v_1, \ldots, v_k\}$ implies $n + 1 - i \in \{v_1, \ldots, v_k\}$. If $n$ is even and $k$ is odd, the latter is impossible and for the other cases we should choose $\lfloor \frac{k}{2} \rfloor$ of the members of $\{v_1, \ldots, v_k\}$ from $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and the remainder should be chosen by symmetry. Now we can apply Theorem 2.1 to see the result.

Corollary 2.2. Let $\overrightarrow{C}_n$ be the labelled cycle with $n$ vertices which is clockwise directed and let $1 \leq k \leq n$. Then

$$\binom{|\overrightarrow{C}_n|}{k} = \frac{1}{n} \sum_{d \mid \gcd(n,k)} \varphi(d) \binom{n}{k} \frac{d}{n}.$$ 

Proof. We know that $\text{Aut}(\overrightarrow{C}_n)$ is the cyclic group generated by the permutation $\alpha = (12 \ldots n)$. Thus $\text{Aut}(\overrightarrow{C}_n) = \{\alpha, \alpha^2, \ldots, \alpha^n\}$. This group has $\varphi(d)$ elements of order $d$ for each divisor $d$ of $n$. An element of order $d$ has $\frac{n}{d}$ cycles of length $d$. For a subset $\{v_1, \ldots, v_k\}$ of $V$ and $\sigma \in \text{Aut}(\overrightarrow{C}_n)$, we have $\sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\}$ if and only if these $k$ elements consist of full cycles of $\alpha$. Whence if $d$ does not divide $k$ then $V_\sigma^k$ is empty and if $d \mid n$ then choosing a subset $\{v_1, \ldots, v_k\}$ with the property $\sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\}$ is equivalent to choosing $\frac{k}{d}$ cycles of the $\frac{n}{d}$ cycles of $\sigma$.

Corollary 2.3. Let $C_n$ be the labelled cycle with $n$ vertices and let $1 \leq k \leq n$. Then

$$\binom{|C_n|}{k} = \begin{cases} \frac{1}{2^n} \sum_{d \mid \gcd(n,k)} \varphi(d) \left(\frac{n}{d}\right)^k + \frac{1}{2} \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} - 1 & \text{if } n \text{ is even and } k \text{ is odd} \\ \frac{1}{2^n} \sum_{d \mid \gcd(n,k)} \varphi(d) \left(\frac{n}{d}\right)^k + \frac{1}{2} \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} & \text{otherwise} \end{cases}$$

Proof. $\text{Aut}(G)$ is the dihedral group consisting of a cyclic subgroup of order $n$ and $n$ reflections. If $n$ is odd then a reflection consists of a cycle with order one and $\frac{n-1}{2}$ cycles of order two. And if $n$ is even then we have $\frac{n}{2}$ reflections with $\frac{n}{2}$ cycles of order two and $\frac{n}{2}$ reflections with two cycles of order one and $\frac{n-2}{2}$ cycles of order two. Now we can do as in the previous corollary.

3. Two Recursive Formulas

A famous recursive relation for the binomial coefficient is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. This simply says that a $k$-combination of the set $[n] = \{1, 2, \ldots, n\}$ can be chosen in two ways: $\{n\}$ union by a $(k-1)$-combination of the set $[n-1]$ or a $k$-combination of the set $[n-1]$. Using this idea, we aim to give a recursive formula for the graph binomial coefficient. Prior to this, we need some preliminaries.

Definition 3.2. Let $G = (V, E)$ be a labelled graph with $n$ vertices, where $n$ is a positive integer, and let $H$ be a vertex induced subgraph of $G$. For $0 \leq k \leq n$, a $k$-combination of $H$ in $[G]$ is a way of selecting members from $H$, such that the order of members in
the selection does not matter. The number of $k$-combinations of $H$ in $[G]$ is denoted by \( \binom{|H \subseteq G|}{k} \) and is called the graph binomial coefficient of $H$ with respect to $[G]$.

**Example 3.3.** Let $G = (V, E)$ be the graph with
$$V = \{1, 2, 3, 4, 5, 6\}, \quad E = \{12, 23, 31, 34, 45, 56, 64\}.$$ If $H$ is the triangle $\{1, 2, 3\}$ then $\binom{|H|}{1}$ is 1, but $\binom{|H \subseteq G|}{1}$ is 2, since we have two different 1-combinations 1 and 3 of $H$.

**Theorem 3.2.** Let $G = (V, E)$ be a labelled graph with $n$ vertices, $1 \leq k \leq n$ and let $H$ be a vertex induced subgraph of $G$. Then
$$\binom{|H \subseteq G|}{k} = \frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} |H_\sigma^k|,$$
where $H_\sigma^k = \{\{v_1, \ldots, v_k\} \subseteq H : \sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\}\}$.

**Proof.** Let $X$ be the set of $k$-subsets of $H$. Then $\text{Aut}(G)$ acts on $X$ by the rule $(\sigma, A) = \sigma(A)$ for each $A \subseteq H$. Now, by the Burnside’s Lemma, the number of orbits of $X$ under $\text{Aut}(G)$, which is equal to $\binom{|H \subseteq G|}{k}$, is $\frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} |H_\sigma^k|$, where $H_\sigma^k$ is the set of all members of $X$ which are fixed under $\sigma$. \(\square\)

**Definition 3.3.** Let $G = (V, E)$ be a graph, $v$ be a vertex of $G$ and let $\sigma \in \text{Aut}(G)$. We denote the set of all $u \in V$ such that $u$ and $v$ are in the same cycle of $\sigma$, in the cyclic representation of $\sigma$, by $\text{Cycle}(v, \sigma)$. The set $\cup_{\sigma \in \text{Aut}(G)} \text{Cycle}(v, \sigma)$, denoted by $\text{Tran}_G(v)$, is called the $v$-transitive subset of $G$. The $v$-transitive subset $\text{Tran}_G(v)$ of $G$ is called strongly transitive if for each $u_1, u_2 \in \text{Tran}_G(v)$ and $u \in G$, there is a $\sigma \in \text{Aut}(G)$ such that $\sigma(u_1) = u_2$ and $\sigma(u) = u$. For a vertex induced subgraph $H$ of $G$ we say that $H$ is $v$-transitive if there is a $\sigma \in \text{Aut}(G)$ with $H = V(v, \sigma)$. The set of $v$-transitive vertex induced subgraphs of $G$ is denoted by $\mathcal{T}_G(v)$.

**Example 3.4.** Let $G = (V, E)$ be the graph with
$$V = \{1, 2, 3, 4, 5\}, \quad E = \{12, 13, 23, 24, 35, 45\}.$$ Then $\text{Tran}_G(1) = \{1\}, \text{Tran}_G(2) = \{2, 3\}$ and $\text{Tran}_G(4) = \{4, 5\}$. Here, $\text{Tran}_G(1)$ is strongly transitive but $\text{Tran}_G(2)$ and $\text{Tran}_G(4)$ are not. To see this note that for $2, 3 \in \text{Tran}_G(2)$ and $4 \in G$ there is no $\sigma \in \text{Aut}(G)$ with $\sigma(2) = 3$ and $\sigma(4) = 4$.

The following result is something similar to the recursive relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

**Theorem 3.3.** Let $G = (V, E)$ be a graph with $n$ vertices, $v$ be a fixed vertex of $G$ and let $1 \leq k \leq n$. Then
$$\binom{|G|}{k} = \sum_{H \in \mathcal{T}_G(v)} \left( \binom{|H^c \subseteq G|}{k} \right),$$
where $H^c$ is the vertex induced subgraph of $G$ whose vertex set is the complement of the vertex set of $H$.

**Proof.** Let $\sigma \in \text{Aut}(G)$ and $H = V(v, \sigma)$. Then
$$V_\sigma^k = \{\{v_1, \ldots, v_k\} : H \subseteq \{v_1, \ldots, v_k\}, \sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\}\} \cup \{\{v_1, \ldots, v_k\} : H \cap \{v_1, \ldots, v_k\} = \emptyset, \sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\}\} = \{\{v_1, \ldots, v_k\} : \{v_1, \ldots, v_k\} \subseteq H^c, \sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\}\} \cup \{\{v_1, \ldots, v_k\} : \{v_1, \ldots, v_k\} \subseteq H^c, \sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\}\}.$$
The above equality is true since for a subset \( \{v_1, \ldots, v_k\} \) of \( V \) with \( \sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_k\} \), the set should contain a full cycle of \( \sigma \) or none of the members of a cycle.

Since the union is disjoint, we have

\[
|V^k_\sigma| = |(H^c)^{k-|H|}_\sigma| + |(H^c)^k_\sigma|.
\]

We now can apply Theorem 3.2. \( \square \)

For the binomial coefficient \( \binom{n}{k} \) we have also the recursive relation

\[
\binom{n}{k} = \sum_{t=0}^{k} \binom{k}{t} \binom{n-t}{k-t},
\]

where \( t \) can be any fixed integer with \( 0 \leq t \leq n \). This simply says that choosing a \( k \)-combination from a group of \( t \) boys and \( n-t \) girls is equivalent to choosing \( \ell \) boys and \( k-\ell \) girls, where \( \ell \) can be 0 or 1 or \ldots or \( t \).

**Theorem 3.4.** Let \( G = (V, E) \) be a graph with \( n \) vertices, \( v \) be a fixed vertex of \( G \) and let \( 1 \leq k \leq n \). If \( \text{Tran}_G(v) \) is strongly transitive then

\[
\left( \left\lceil \frac{|\text{Tran}_G(v)|}{k} \right\rceil \right) = \sum_{\ell=0}^{\left\lceil \frac{|\text{Tran}_G(v)|}{k} \right\rceil} \left( \left\lceil \frac{|\text{Tran}_G(v) \subseteq G|}{\ell} \right\rceil \right) \left( \left\lceil \frac{|(\text{Tran}_G(v))^c \subseteq G|}{k-\ell} \right\rceil \right).
\]

**Proof.** Let \( \{v_1, \ldots, v_k\} \) be a subset of \( V \). Moreover, suppose that \( \{v_1, \ldots, v_k\} \cap \text{Tran}_G(v) = \{v_1, \ldots, v_\ell\} \), where \( 0 \leq \ell \leq |\text{Tran}_G(v)| \). Then \( \sigma(\{v_1, \ldots, v_k\}) = \{v_1, \ldots, v_\ell\} \) if and only if \( \sigma(\{v_1, \ldots, v_\ell\}) = \{v_1, \ldots, v_\ell\} \), \( \sigma(\{v_{\ell+1}, \ldots, v_k\}) = \{v_{\ell+1}, \ldots, v_k\} \).

This shows that

\[
|V^k_\sigma| = \sum_{\ell=0}^{\left\lceil \frac{|\text{Tran}_G(v)|}{k} \right\rceil} |(\text{Tran}_G(v))^{\ell}_\sigma| \times |((\text{Tran}_G(v))^c)^{k-\ell}_\sigma|.
\]

Theorem 2.1 and Theorem 3.2 now give the result. \( \square \)

### 4. Graph Binomial Expansion

Recall that the binomial expansion says \( \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a + b)^n \). In this section we want to find a graph version of the binomial expansion.

**Definition 4.4.** Let \( G = (V, E) \) be a labelled graph with \( n \) vertices, where \( n \) is a positive integer, and let \( H \) be a vertex induced subgraph of \( G \). We denote the summation \( \sum_{k=0}^{n} \binom{H \subseteq G}{k} a^k b^{n-k} \) by \( P_{[H \subseteq G]}(a, b) \). The expansion is called the **graph binomial expansion of** \( H \) with respect to \( [G] \). For the case \( H = G \) we simply write \( P_{[G]}(a, b) \) instead of \( P_{[G \subseteq G]}(a, b) \).

**Proposition 4.3.** Let \( G \) be a graph with \( n \) vertices. The number of ways to color vertices of \( [G] \) with two colors is \( P_{[G]}(1, 1) \).

As a corollary, using Corollaries 2.1, 2.2 and 2.3, we can compute the number of ways to color \( P_n, C_n \) (a necklace with rotations but without reflections) or \( C_n \) (a necklace with rotations and reflections) with two colors. For example, we have the following.

**Corollary 4.4.** Let \( C_n \) be the labelled cycle with \( n \) vertices which is clockwise directed. Then the number of ways to color vertices of \( C_n \) with two colors is

\[
1 + \sum_{k=1}^{n} \frac{1}{n} \sum_{d \mid \gcd(n, k)} \varphi(d) \binom{n}{\frac{n}{d}} \binom{\frac{n}{d}}{k}.
\]

Furthermore, as a corollary of Theorem 3.4, we can easily prove the following result.
**Theorem 4.5.** Let $G$ be a graph with $n$ vertices and $v$ be a fixed vertex of $G$. If $\text{Tran}_G(v)$ is strongly transitive then

$$P_{[G]}(a, b) = P_{[\text{Tran}_G(v) \subseteq G]}(a, b)P_{[(\text{Tran}_G(v))^c \subseteq G]}(a, b).$$

**Example 4.5.** Let $G = (V, E)$ be the graph with

$$V = \{1, 2, 3, 4, 5, 6, 7\}, \quad E = \{12, 23, 31, 34, 45, 56, 67, 74\}.$$  

Then

$$P_{[G]}(a, b) = a^7 + 5a^6b + 12a^5b^2 + 18a^4b^3 + 18a^3b^4 + 12a^2b^5 + 5ab^6 + b^7.$$  

Considering $u = 1$ we have $\text{Tran}_G(u) = \{1, 2\}$ which is strongly transitive. We have

$$P_{[\text{Tran}_G(u) \subseteq G]}(a, b)(a, b) = a^2 + ab + b^2$$

and

$$P_{[(\text{Tran}_G(u))^c \subseteq G]}(a, b) = a^5 + 4a^4b + 7a^3b^2 + 7a^2b^3 + 4ab^4 + b^5.$$  

Note that

$$a^7 + 5a^6b + 12a^5b^2 + 18a^4b^3 + 18a^3b^4 + 12a^2b^5 + 5ab^6 + b^7 = (a^2 + ab + b^2)(a^5 + 4a^4b + 7a^3b^2 + 7a^2b^3 + 4ab^4 + b^5).$$  

On the other hand, considering $v = 6$ we have $\text{Tran}_G(v) = \{6\}$ which is strongly transitive. We have

$$P_{[\text{Tran}_G(v) \subseteq G]}(a, b)(a, b) = a + b$$

and

$$P_{[(\text{Tran}_G(v))^c \subseteq G]}(a, b) = a^6 + 4a^5b + 8a^4b^2 + 10a^3b^3 + 8a^2b^4 + 4ab^5 + b^6.$$  

Note that

$$a^7 + 5a^6b + 12a^5b^2 + 18a^4b^3 + 18a^3b^4 + 12a^2b^5 + 5ab^6 + b^7 = (a + b)(a^6 + 4a^5b + 8a^4b^2 + 10a^3b^3 + 8a^2b^4 + 4ab^5 + b^6).$$  

**Example 4.6.** Let $G$ be the graph introduced in Example 3.4. Then

$$P_{[G]}(a, b) = a^5 + 3a^4b + 6a^3b^2 + 6a^2b^3 + 3ab^4 + b^5.$$  

Considering $v = 4$ we have $\text{Tran}_G(v) = \{4, 5\}$ which is not strongly transitive. We have

$$P_{[\text{Tran}_G(v) \subseteq G]}(a, b)(a, b) = a^2 + ab + b^2$$

and

$$P_{[(\text{Tran}_G(v))^c \subseteq G]}(a, b) = a^3 + 2a^2b + 2ab^2 + b^3.$$  

Note that

$$a^5 + 3a^4b + 6a^3b^2 + 6a^2b^3 + 3ab^4 + b^5 \neq (a^2 + ab + b^2)(a^3 + 2a^2b + 2ab^2 + b^3).$$

**Remark 4.1.** Let $G$ be a graph with vertices $1, 2, \ldots, n$. We add $i$ loop to vertex $i$ of $G$ to make a new graph $G'$. Then $\text{Aut}(G')$ is the identity group, since no two vertices of $G'$ are transitive to each other. This guarantees that $\binom{[G']}{k} = \binom{n}{k}$ for each $0 \leq k \leq n$. Thus

$$P_{[G']}(a, b) = \sum_{k=0}^{n} \binom{n}{k}a^k b^{n-k}.$$  

On the other hand, for each $v \in G'$ we have $\text{Tran}_{G'}(v) = \{v\}$ which is strongly transitive and so $P_{[\text{Tran}_{G'}(v) \subseteq G']} (a, b) = a + b$. Thus $P_{[G']}(a, b) = (a + b)^n$. This agrees to the famous binomial expansion.

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REFERENCES


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