A Note On The Stahl’s Theorem

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Abstract

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$ and $A$ be a bounden operator in $\mathcal{H}$. Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and $\Phi$ be a positive linear functional on $\mathcal{B}(\mathcal{H})$. We show that, if $f$ is a non-negative operator decreasing function, then the function $t \to \Phi(f(A + tB))$ can be written as a Laplace transform of a positive measure.

Keywords and phrases: BMV conjecture, Laplace transform, Operator monotone functions.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$ and $A$ be a bounden operator in $\mathcal{H}$. An operator $A$ is called positive if $\langle Ax, x \rangle \geq 0$ holds for every $x \in \mathcal{H}$ and then we writ $A \geq 0$. For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ we say $A \geq B$ if $A - B \geq 0$. For a continuous function $f$ and a self adjoint operator $A$ with spectra in domain of $f$, the operator $f(A)$ is defined by standard functional calculus. In particular, if $\mathcal{H}$ is a finite dimensional Hilbert space and $A$ has the spectral decomposition $A = \sum_{i=1}^{n} \mu_{i}P_{i}$, where $P_{i}$ are the projections corresponding to the eigenspaces of eigenvalue $\mu_{i}$, then

$$f(A) = \sum_{i=1}^{n} f(\mu_{i})P_{i}.$$ 

A function $f$ on the interval $I$ is called operator increasing if $A \geq B$ implies that $f(A) \geq f(B)$, for each self adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ with spectra in $I$. Also, $f$ is called operator decreasing function if $-f$ is operator increasing function.

The Bessis-Moussa-Villani conjecture states that for a self adjoint Matrix $A$ and positive Matrix $B$, the function $f(t) = \text{Tr}\left(\exp^{A-tB}\right)$ can be represented as the Laplace transform

$$f(t) = \int_{0}^{\infty} \exp^{-tx} d\mu(x),$$

(1)
of a positive measure $\mu$ on $[0, \infty)$ [1]. This conjecture has attracted a lot of attention in mathematics and physics. Despite a lot of afford to prove the conjecture, it remained open until 2012. Eventually, Stahl [5] proved this conjecture. In [3], Hansen got a similar results.

Theorem 1.1. [3] If $f$ is a non-negative operator decreasing function on $[0, \infty)$, then for positive matrices $A, B$ the map $t \to \text{Tr} f(A + tB)$ can be written as the Laplace transform of a positive measure.

In proof of this theorem, Hansen used the theory of Frechet differentials and the Bernstein theorem highlighting the measure in (1) exists if and only if $f$ is completely monotone or $(-1)^n f^n(t) \geq 0$, for each $n = 0, 1, 2, \ldots$ and $t > 0$.

In this note, we extended the results of Hansen and prove that for an arbitrary Hilbert space $\mathcal{H}$ and any positive linear functional $\Phi$ on $\mathbb{B}(\mathcal{H})$. Also, in a special case, we obtained a partial extension of a famous equivalent statement for Bessis-Moussa-Villani conjecture which state for each $p \geq 0$ and positive semi-definite matrices $A$ and $B$, the function $h_p(t) = \text{Tr} (A + tB)^p$ is completely monotone [4]. Indeed, we show that for a positive linear functional $\Phi$ on $\mathbb{B}(\mathcal{H})$, the function $\phi_p(t) = \Phi ((A + tB)^p)$ is completely monotone for each $-1 \leq p \leq 0$ and positive operators $A, B \in \mathbb{B}(\mathcal{H})$.

2. Main Results

The following theorem states the main results.

Theorem 2.1. Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive operators and $f$ be an operator decreasing function on $(0, \infty)$. For any positive linear functional $\Phi$ on $\mathbb{B}(\mathcal{H})$ the function $\phi(t) = \Phi (f(A + tB))$ is operator decreasing. In particular, if $f$ is non-negative then $\phi$ is completely monotone.

By replacing $f$ by $-f$ we obtain the following corollary.

Corollary 2.2. Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive operators and $f$ be an operator increasing function on $(0, \infty)$. Then, for any positive linear functional $\Phi$ on $\mathbb{B}(\mathcal{H})$ the function $\phi(t) = \Phi (f(A + tB))$ is an operator increasing function.

Example 2.3. The functions $\ln(t + 1)$ is a non-negative operator increasing function on $[0, \infty)$. Hence, for positive operators $A, B$ and positive linear functional $\Phi$, the function $t \to \Phi (f(A + tB))$ is a non-negative operator increasing function on $[0, \infty)$.

The function $f(t) = t^p$ is operator increasing for $0 < p < 1$ and operator decreasing for $-1 \leq p < 0$. Therefore, As an example we the following corollary is given.

Corollary 2.4. Let $A, B$ be positive operators in $\mathbb{B}(\mathcal{H})$ and $-1 \leq p \leq 1$. For a positive linear functional $\Phi$ on $\mathbb{B}(\mathcal{H})$, the function $\phi(t) = \Phi ((A + tB)^p)$ is operator increasing if $0 < p < 1$ and operator decreasing if $-1 \leq p \leq 0$. 


References


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