POSITIVE BLOCK MATRICES

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ABSTRACT. Let $C$ and $D$ be positive operators such that $C \leq D$ and $D$ be invertible. We show that there exists a trace preserving unital completely positive map $\Phi_{C,D} : \mathcal{B}(H) \to \mathcal{B}(H)$ such that the block operator matrices

$$
\begin{pmatrix}
\Phi_{C,D}(A) & C \\
C & \Phi_{C,D}(B)
\end{pmatrix}
$$

are positive, for all positive operators $A$ and $B$ such that $D = A^\sharp B$.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. An operator $A$ is called positive if $\langle Ax, x \rangle \geq 0$ holds for every $x \in \mathcal{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ we say $A \geq B$ if $A - B \geq 0$. Choi in [3], showed that for operators $A, X$, and invertible operator $B$ in $\mathcal{B}(\mathcal{H})$, the block matrix

$$
\begin{pmatrix}
A & X \\
X^* & B
\end{pmatrix}
$$

(1.1)

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in $M_2(B(H))$ is positive if and only if $A$ and $B$ are positive operator and
\[ A \geq XB^{-1}X^*. \] (1.2)

For positive operators $A$ and $B$, assume that
\[ A \sharp B = \max \left\{ C \geq 0 \mid \begin{pmatrix} A & C \\ C & B \end{pmatrix} \right\}. \] (1.3)

It is known that $A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}B A^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ for two positive invertible operators $A$ and $B$ and in general case if $A$ and $B$ are positive, then
\[ A \sharp B = \lim_{n \to \infty} (A + \frac{1}{n})^{\frac{1}{2}}(B + \frac{1}{n}). \]

We recall that, a linear map $\Phi$ between two $C^*$ algebras $A$ and $B$ is said to be completely positive if for each $n \in \mathbb{N}$, the linear map $\Phi^n : M_n(A) \to M_n(B)$ defined by
\[ \Phi^n([a_{i,j}]) = [\phi(a_{i,j})] \]
is positive.

2. MAIN RESULTS

**Theorem 2.1.** Let $X$ and $Y$ be invertible operators in $B(H)$ such that $||X||, ||Y|| \leq 1$. Then there exists unital completely positive linear maps $\Phi_X, \Phi_Y : B(H) \to B(H)$ such that for all positive operators $A$ and $B$, the following operator matrix is positive.
\[ \begin{pmatrix} \Phi_Y(A) & Y(A \sharp B)X^* \\ X(A \sharp B)Y^* & \Phi_X(B) \end{pmatrix}. \]

Also, if $H$ is finite dimensional then $\Phi_X$ and $\Phi_Y$ are trace preserving.

**Theorem 2.2.** Let $C$ and $D$ be positive operators such that $C \leq D$ and $D$ be invertible. Then there exists a unital completely positive map $\Phi_{C,D} : B(H) \to B(H)$ such that the following statements are hold:
(i) If $C$ is invertible, $\Phi$ is normal and faithful.
(ii) If $H$ is finite dimensional, then $\Phi_{C,D}$ is a trace preserving map.
(iii) If $T$ commute with $C$ and $D$, then $\Phi(T) = T$.
(iv) For any positive operators $A$ and $B$ such that $D = A \sharp B$, the block matrix
\[ \begin{pmatrix} \Phi_{C,D}(A) & C \\ C & \Phi_{C,D}(B) \end{pmatrix} \]
are positive.
Theorem 2.3. Let $A$, $B$, and $C$ be positive operators such that $C \leq A \sharp B$. Then there exists a unital completely positive map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that the block matrix
\[
\begin{pmatrix}
\Phi(A) & C \\
C & \Phi(B)
\end{pmatrix},
\]
is positive. Moreover, if $\mathcal{H}$ is finite dimensional then $\Phi$ is trace preserving.

In Theorem 2.2, if we assumed that $D = 1$, then we have the following corollary.

Corollary 2.4. Let $0 \leq C \leq 1$. Then there exists a unital completely positive map $\Phi_C : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that for each positive invertible operator $A$, the block matrix
\[
\begin{pmatrix}
\Phi_C(A) & C \\
C & \Phi_C(A^{-1})
\end{pmatrix},
\]
is positive. Moreover, if $\mathcal{H}$ is finite dimensional then $\Phi$ is trace preserving.

References