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NONLINEAR DYNAMIC ANALYSIS OF AN ELASTICALLY RESTRAINED CANTILEVER TAPERED BEAM

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Abstract: An analytical simulation of an elastically restrained tapered cantilever beam is performed. Five different analytical methods are applied to solve the dynamic model of the nonlinear oscillation equation. Analytical relationships between the natural frequency and the initial amplitude are obtained. The present solutions are compared with the exact solution, and excellent agreement is noted.

Keywords: nonlinear oscillation, analytical methods, periodic solution.

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INTRODUCTION

As cantilever beams are among the basic structural forms, dynamic behaviors of these units have been studied extensively. Tapered members not only reduce the dead weights and, consequently, increase the load capacities, but also are economical and aesthetical; thus, these members have been traditionally chosen in structural design.

Many engineering structures, such as offshore structure piles, oil platform supports, oil-loading terminals, tower structures, and moving arms, are modeled as tapered beams. As these structures are usually subjected to various excitation loads, such as wind loads, wave loads, and other excitations, the calculation of their nonlinear natural frequencies is required for their design recommendations.

Dozens of brilliant books were written on different aspects of vibrations both from theoretical and practical points of view. At least the classical works by Lord Rayleigh [1], Poincare [2], and Timoschenko et al. [3] should be mentioned here.

The dynamic response of the beams at large amplitudes of vibrations can be encountered in many engineering applications. In such cases, it is of interest to know how far the characteristics of the dynamic response deviate from those defined via the linear theory. The problem of vibrations of beams with different cross sections was recently investigated by many researchers with the use of nonlinear models [4]. The most important and fundamental step in analyzing an engineering problem is to derive equations governing the motion and dynamics of the system. These equations lead to the formation of ordinary or partial differential equations and different types of linear and nonlinear equations in general. The well-elaborated mathematical theory of linear differential equations enables us to solve, with sufficient accuracy, almost all problems evoked by the fast development of various types of machines, mechanisms, machine tools, robots, traffic means, etc.

The increased requirements on the speed, power, long life, and reliability of mechanical systems change the situation and prove that linear models of mechanical systems ensure only the first approximation of real process.

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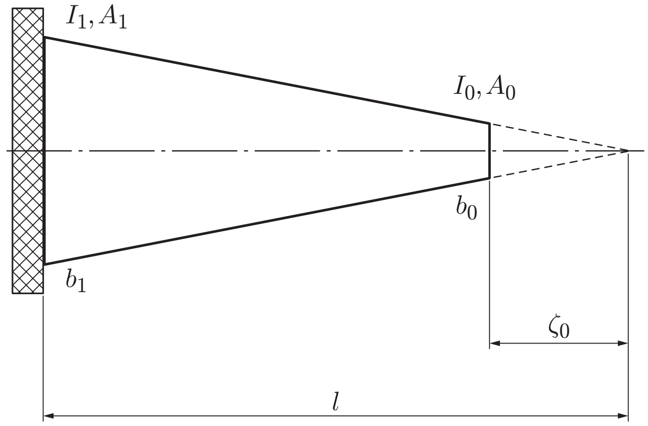

Fig. 1. Schematic of a tapered beam.

By means of the linear theory, we cannot explain all phenomena. The world around us and we ourselves are inherently nonlinear. For this reason, it is necessary to use nonlinear models in the form of ordinary or partial differential and integrodifferential equations, differential equations with time-varying parameters, with random properties, with delayed arguments, etc. Unfortunately, the development of the corresponding theoretical framework and mathematical language is still in its infancy.

Various methods for solving equations with weak nonlinearity have been developed. The most often applied ones are the Bogoliubov–Mitropolski method [5], Lindstead–Poincare method, perturbation methods [6], normal mode method [7], etc. If the nonlinearity is strong, the following methods are usually applied: harmonic balance method [8], energy balance method [9–12], homotopy perturbation method [13–21], Hamiltonian approach [22–25], variational approach [26, 27], amplitude frequency formulation [28–30], coupling of homotopy-variational method [31, 32], and other classical methods [33–40].

1. MATHEMATICAL MODEL

The model of the system mentioned in the previous Section of this paper is presented in Fig. 1. The physical properties, the modulus of elasticity E, and the density of the beam material are constant. The beam thickness and width vary linearly along the beam axis. The restrained end of the beam is modeled by a torsional spring with a rigidity K_r in combination with a translational spring with a rigidity K_t. The cross-sectional area and the moment of inertia at the large end are A_1 and I_1, respectively [41].

The thickness of the beam is assumed to be small as compared to its length, so that the effects of rotary inertia and shear deformation can be ignored. The beam transverse vibrations can be considered to be purely planar, and the amplitude of vibrations may reach large values.

The potential energy of the system consists of the strain energy due to the bending deformation and the elastic energy stored in the torsional and translational springs with rigidities K_r and K_t. It can be written as [41]

$$V = \frac{EI}{2}\int_0^1 I(\xi)R^2\,d\xi + \frac{1}{2}\left(K_r v'^2 + K_t v^2\right)\Big|_{\xi=1},$$

where $R = \lambda\varphi'$ is the curvature of the neutral axis of the beam, $\lambda = 1/l$, v is the transverse deflection, $\xi = s/l$, the prime means the derivative with respect to the dimensionless length ξ, and φ is the change in the slope along the beam [41].

The nonlinear curvature can be expressed as

$$R^2 = \lambda^4(v''^2 + \lambda^2 v''^2 v'^2).$$

The kinetic energy T of the beam can be written as

$$T = \frac{\rho l}{2} \int_0^1 A(\xi)[\dot{u}^2 + \dot{v}^2]\, d\xi,$$

where

$$u = \frac{1}{2} \int_0^{\zeta} \Big(\lambda v'^2 + \frac{\lambda^3}{4} v'^4\Big)\, d\zeta$$

is the axial shortening of the beam due to bending deformation. The Lagrangian of the beam can expressed as

$$L = T - V.$$

To perform separation of variables, the transverse deflection can be written as

$$v(\zeta, t) = \varphi_i(\zeta) q(t),$$

where $\varphi(\zeta)$ is the normalized self-similar assumed mode shape of the beam and $q(t)$ is an unknown time modulation of the assumed deflection mode $\varphi_i(\zeta)$. In the present work, the function $\varphi_i(\zeta)$ for a double-tapered beam is taken in the form

$$\varphi_i(\zeta) = \xi^{-1}[C_1 J_2(Z) + C_2 Y_2(Z) + C_3 I_2(Z) + C_4 K_2(Z)],$$

where J and Y are the Bessel functions of the first and second kind, respectively, I and K are the modified Bessel functions of the first and second kind, respectively, and $C_1, \ldots, C_4$ are arbitrary constants to be determined by imposing the boundary conditions at both ends of the beam. The following notations are introduced [41]:

$$\beta_1 = \int_0^1 A^* \varphi^2\, d\xi, \qquad \beta_2 = \int_0^1 A^* \Big(\int_0^{\xi} \varphi'^2\, d\chi\Big)^2 d\xi,$$

$$\beta_3 = \int_0^1 I^* \varphi''^2\, d\xi + \frac{K_t L^3}{EI}\, \varphi^2(1) + \frac{K_r L}{EI}\, \varphi'^2(1), \qquad \beta_4 = \int_0^1 I^* \varphi'^2 \varphi''^2\, d\xi.$$

We have $A_1^* = A_1 \xi^2$ and $I_1^* = I_1 \xi^4$ for the double-tapered beam and $A_1 \xi$ and $I_1 \xi^3$ for the single-tapered wedge beam. The Euler–Lagrangian relation

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) - \frac{\partial L}{\partial q} = 0$$

yields the following nonlinear dimensionless unimodal equation of motion:

$$\beta_1 \ddot{q} + \beta_2 (q^2 \ddot{q} + q \dot{q}^2) + \beta_2 (\beta_3 q + 2\beta_4 q^3) = 0. \tag{1}$$

It is to be noted that some of the coefficients β_i in Eq. (1) in general may have large values. Therefore, for convenience, Eq. (1) is scaled and converted to the dimensionless form [41]:

$$(1 + \varepsilon_1 q^2)\, \frac{d^2 q}{dt^{*2}} + \varepsilon_1 q \Big(\frac{dq}{dt^*}\Big)^2 + q + \varepsilon_2 q^3 = 0; \tag{2}$$

$$\varepsilon_1 = \frac{\beta_2}{\beta_1}, \qquad \varepsilon_2 = \frac{2\beta_4}{\beta_3}, \qquad t^* = \sqrt{\frac{\beta_2 \beta_3}{\beta_1}}\, t$$

(ε_1 and ε_2 are dimensionless coefficients).

Equation (2) describes the nonlinear planar flexural free vibrations of the inextensible tapered beam. Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude A of oscillations. Thus, the following initial conditions are considered:

$$q(0) = A, \qquad \frac{dq}{dt^*}(0) = 0. \tag{3}$$

2. VARIATIONAL APPROACH (VA)

The functional corresponding to Eq. (2) can be obtained by using the semi-inverse method [27]:

$$J = \int_0^{2\pi/\omega} \Big(-\frac{1}{2}\Big(\frac{d}{dt^*}\,q(t^*)\Big)^2 + \frac{1}{2}\,q(t^*)^2 - \frac{1}{2}\,\varepsilon_1 q(t^*)^2\Big(\frac{d}{dt}\,q(t^*)\Big)^2 + \frac{1}{4}\,\varepsilon_2 q(t^*)^4\Big)\,dt^*. \tag{4}$$

Its approximate solution can be expressed as

$$q(t^*) = A\cos{(\omega t^*)}. \tag{5}$$

Substituting Eq. (5) into Eq. (4), we obtain

$$J = -\frac{1}{8}\,A^4\omega\pi\varepsilon_1 + \frac{3}{16}\,\frac{A^4\pi\varepsilon_2}{\omega} - \frac{1}{2}\,A^2\omega\pi + \frac{1}{2}\,\frac{A^2\pi}{\omega}.$$

Using the Ritz method $\partial J/\partial A = 0$, we obtain the relationship between the frequency and amplitude:

$$\omega_{\mathrm{VA}} = \frac{\sqrt{2}}{2}\sqrt{\frac{3A^2\varepsilon_2 + 4}{A^2\varepsilon_1 + 2}}\,.$$

3. AMPLITUDE-FREQUENCY FORMULATION (AFF)

To solve nonlinear problems, an amplitude-frequency formulation for nonlinear oscillators was proposed by He [30]. According to this amplitude-frequency formulation, $q_1(t^*) = A\cos{(t^*)}$ and $q_2(t^*) = A\cos{(\omega t^*)}$ serve as the trial functions. Substituting q_1 and q_2 into Eq. (17), we obtain the residuals

$$R_1 = -(1 + \varepsilon_1 A^2\cos{(t^*)^2})A\cos{(t^*)} + \varepsilon_1 A^3\cos{(t^*)}\sin{(t^*)^2} + A\cos{(t^*)} + \varepsilon_2 A^3\cos{(t^*)^3}; \tag{6}$$

$$R_2 = -(1 + \varepsilon_1 A^2\cos{(\omega t^*)^2})A\cos{(\omega t^*)}\omega^2 + \varepsilon_1 A^3\cos{(\omega t^*)}\sin{(\omega t^*)^2}\omega^2 + A\cos{(\omega t^*)} + \varepsilon_2 A^3\cos{(\omega t^*)^3}. \tag{7}$$

According to [42], residuals (6) and (7) are replaced by the weighted residuals

$$R_{11} = \frac{4}{T_1}\int_0^{T_1/4} R_1\cos{(t^*)}\,dt^*, \qquad T_1 = 2\pi,$$

$$R_{22} = \frac{4}{T_2}\int_0^{T_2/4} R_2\cos{(\omega t^*)}\,dt^*, \qquad T_2 = \frac{2\pi}{\omega},$$

for which the following relation is valid [42]:

$$\frac{R_{22} - R_{11}\omega^2}{R_{22} - R_{11}} = \frac{-A(3A^2\omega^2\varepsilon_2 - 3A^2\varepsilon_2 + 4\omega^2 - 4)/8}{-A(A^2\omega^2\varepsilon_1 - A^2\varepsilon_1 + 2\omega^2 - 2)/4} = \omega^2.$$

Then the approximate frequency can be obtained as

$$\omega_{\mathrm{AFF}} = \frac{\sqrt{2}}{2}\sqrt{\frac{3A^2\varepsilon_2 + 4}{A^2\varepsilon_1 + 2}}\,.$$

4. HAMILTONIAN APPROACH (HA)

Previously, He [11] had introduced the energy balance method (EBM) based on the collocation and Hamiltonian approaches. This is a kind of the energy method with vast applications in conservative oscillatory systems. In order to clarify this approach, the Hamiltonian of Eq. (2) can be written in the following form [22]:

$$H = \frac{1}{2}\Big(\frac{d}{dt^*}\,q(t^*)\Big)^2 + \frac{1}{2}\,q(t^*)^2 - \frac{1}{2}\,\varepsilon_1 q(t^*)^2\Big(\frac{d}{dt^*}\,q(t^*)\Big)^2 + \frac{1}{4}\,\varepsilon_2 q(t^*)^4. \tag{8}$$

The Hamiltonian (8) implies that the total energy of the system is unchanged during the oscillation. Let us introduce a new function [16]

$$\bar{H} = \int\limits_0^{T/4} \Big(\frac{1}{2}\Big(\frac{d}{dt^*}\,q(t^*)\Big)^2 + \frac{1}{2}\,q(t^*)^2 - \frac{1}{2}\,\varepsilon_1 q(t^*)^2\Big(\frac{d}{dt^*}\,q(t^*)\Big)^2 + \frac{1}{4}\,\varepsilon_2 q(t^*)^4\Big)\,dt^*, \qquad T = \frac{2\pi}{\omega}.$$

It is obvious that

$$\frac{\partial\bar{H}}{\partial T} = \frac{1}{4}\,H. \tag{9}$$

Equation (9) is equivalent to the equation

$$\frac{\partial}{\partial A}\Big(\frac{\partial\bar{H}}{\partial T}\Big) = \frac{\partial}{\partial A}\Big(\frac{\partial\bar{H}}{\partial\,(1/\omega)}\Big) = 0. \tag{10}$$

We consider the solution of Eq. (2) in the form

$$q(t^*) = A\cos{(\omega t^*)}. \tag{11}$$

Substituting Eq. (11) into Eq. (10), we obtain

$$-\frac{1}{8}\,A^3\pi\varepsilon_1\omega^2 - \frac{1}{4}\,A\pi\omega^2 + \frac{3}{16}\,A^3\pi\varepsilon_2 + \frac{1}{4}\,A\pi = 0. \tag{12}$$

Consequently, the approximate frequency can be found from Eq. (12):

$$\omega_{\mathrm{HA}} = \frac{\sqrt{2}}{2}\sqrt{\frac{3A^2\varepsilon_2 + 4}{A^2\varepsilon_1 + 2}}.$$

5. INTEGRAL ITERATION METHOD (IIM)

According to the integral iteration method, Eq. (2) can be rewritten in the following iteration form [26]:

$$q_{n+1} = -q_n - f\Big(q_n, \frac{dq_n}{dt^*}, \frac{d^2q_n}{dt^{*2}}\Big). \tag{13}$$

Substituting a trial function into Eq. (13) and integrating twice, we obtain q_{n+1}. For $n = 0$, we have

$$\frac{d^2q_1}{dt^{*2}} = -\varepsilon_1 q_0\Big(\frac{dq_0}{dt^*}\Big)^2 - q_0 - \varepsilon_2 q_0^3 - \varepsilon_1 q_0^2\,\frac{d^2q_0}{dt^{*2}}. \tag{14}$$

The simplest trial function is

$$q_0(t^*) = A\cos{(\omega t^*)}. \tag{15}$$

Substituting Eq. (15) into Eq. (14) and integrating twice, we obtain

$$q_1 = \frac{A}{\omega^2}\Big(-\frac{1}{2}\,A^2\varepsilon_1\omega^2 + \frac{3}{4}\,A^2\varepsilon_2 + 1\Big)\cos{(\omega t^*)} + \frac{A^3}{\omega^2}\Big(-\frac{\varepsilon_1}{18}\,\omega^2 + \frac{\varepsilon_2}{36}\Big)\cos{(3\omega t^*)} + Ct^* + D,$$

where C and D are integration constants, which are assumed to have zero values owing to the periodic behavior of the solution [26]. By equating the coefficients at $\cos{(\omega t^*)}$ in the expressions for u_0 and u_1, the approximate frequency can be obtained as

$$\omega_{\mathrm{IIM}} = \frac{\sqrt{2}}{2}\sqrt{\frac{3A^2\varepsilon^2 + 4}{A^2\varepsilon_1 + 2}}\,.$$

6. COUPLED HOMOTOPY-VARIATIONAL FORMULATION (CHV)

A coupling method of He's homotopy technique [13] and the variational approach was used in [31] to solve nonlinear oscillation systems. In contrast to traditional perturbation methods, the proposed method does not require a small parameter in the equation. In the CHV method, a homotopy with an embedding parameter $p \in [0, 1]$ is constructed to obtain the problem solution at $p = 0$. The next term is solved by using the variational approach, where the frequency of the nonlinear oscillator can be obtained. The following homotopy can be constructed for Eq. (2):

$$\frac{d^2}{dt^{*2}}\,q(t^*) + \omega^2 q(t^*) + p\Big(\varepsilon_1 q(t^*)^2\Big(\frac{d^2}{dt^{*2}}\,q(t^*)\Big) + \varepsilon_1 q(t^*)\Big(\frac{d}{dt^*}\,q(t^*)\Big)^2 + (1-\omega^2)q(t^*) + \varepsilon_2 q(t^*)^3\Big) = 0. \tag{16}$$

At $p = 0$, Eq. (16) transforms to the linearized equation $d^2q(t^*)/dt^{*2} + \omega^2 q(t^*) = 0$; at $p = 1$, it turns out to be the original one. Let us assume that the periodic solution of Eq. (2) can be written as a power series in p:

$$q(t^*) = q_0(t^*) + pq_1(t^*) + p^2 q_2(t^*) + \ldots . \tag{17}$$

Substituting Eq. (17) into Eq. (16) and collecting terms of the same power of p, we obtain

$$p^0\colon \qquad \omega^2 q_0(t^*) + \frac{d^2}{dt^2}\,q_0(t^*) = 0; \tag{18}$$

$$p^1\colon \quad \frac{d^2}{dt^{*2}}\,q_1(t^*) + \varepsilon_2 q_0(t^*)^3 + \varepsilon_1 q_0(t^*)^2\Big(\frac{d^2}{dt^{*2}}\,q_0(t^*)\Big) + \varepsilon_1 q_0(t^*)\Big(\frac{d}{dt^*}\,q_0(t^*)\Big)^2 + +\omega^2 q_1(t) + q_0(t) - \omega^2 q_0(t) = 0.$$

The solution of Eq. (18) is $q_0(t^*) = A\cos(\omega t^*)$, where the frequency ω is identified from the variational formulation for q_1:

$$J(q_1) = \int_0^{2\pi/\omega} \Big(-\frac{1}{2}\Big(\frac{d}{dt^*}\,q_1(t^*)\Big)^2 + \varepsilon_2 A^3\cos(\omega t^*)^3 q_1(t^*) - \varepsilon_1 A^3 \cos(\omega t^*)^3 \omega^2 q_1(t^*)$$

$$+ \varepsilon_1 A^3 \cos(\omega t^*)\sin(\omega t^*)^2\omega^2 q_1(t^*) + \frac{1}{2}\,\omega^2 q_1(t^*)^2 + A\cos(\omega t^*)q_1(t^*) - \omega^2 A\cos(\omega t^*)q_1(t^*)\Big)\,dt^*. \tag{19}$$

6.1. First-Order Analytical Approximation

A simple trial function is chosen:

$$q_1(t^*) = B_1\Big(\cos(\omega t^*) - \frac{1}{3}\,\cos(5\omega t^*)\Big).$$

Substitution of q_1 into Eq. (19) yields

$$J(\omega, B_1) = -\frac{1}{2}\,B_1\omega\varepsilon_1 A^3\pi + \frac{3}{4}\,\frac{B_1\varepsilon_2 A^3\pi}{\omega} - B_1\omega A\pi + \frac{B_1 A\pi}{\omega} - \frac{4}{3}\,B_1^2\omega\pi.$$

Solving the equations

$$\frac{\partial J}{\partial B_1} = 0, \qquad \frac{\partial J}{\partial \omega} = 0,$$

we obtain the approximate expression for the frequency:

$$\omega_{\text{CHV}} = \frac{\sqrt{2}}{2}\sqrt{\frac{3A^2\varepsilon_2 + 4}{A^2\varepsilon_1 + 2}}.$$

6.2. Second-Order Analytical Approximation

The accuracy of the first-order approximate solution can be dramatically improved if the trial function is chosen in the form

$$q_1(t^*) = B_1\Big(\cos(\omega t^*) - \frac{1}{3}\,\cos(3\omega t^*)\Big) + B_2\Big(\frac{1}{3}\cos(3\omega t^*) - \frac{3}{5}\cos(5\omega t^*) + \frac{5}{7}\cos(7\omega t^*)\Big). \tag{20}$$

Substitution of Eq. (20) into Eq. (19) yields

$$J(\omega, B_1, B_2) = -\frac{3}{4}\,\omega A^3\varepsilon\pi\rho B_1 - \frac{9}{28}\,\omega A^3\varepsilon\pi\rho B_2 + \frac{3}{4}\,\frac{A^3\varepsilon\pi B_1\omega_0^2}{\omega} + \frac{1}{28}\,\frac{A^3\varepsilon\pi B_2\omega_0^2}{\omega}$$
$$- \omega A\pi B_1 + \frac{A\pi B_1\omega_0^2}{\omega} - 12\omega\pi B_1^2 - 168\omega\pi B_1B_2 - \frac{28\,816}{49}\,\omega\pi B_2^2. \quad (21)$$

The stationary condition of Eq. (21) requires that [31]

$$\frac{\partial J}{\partial B_1} = 0, \qquad \frac{\partial J}{\partial B_2} = 0, \qquad \frac{\partial J}{\partial \omega} = 0.$$

The second-order approximate frequency can be obtained as

$$\omega_{\text{CHV2}} = \frac{1}{2}\,\frac{\sqrt{-(43\,034A^4\varepsilon_1^2 + 253\,304A^2\varepsilon_1 + 375\,056)(\alpha)}}{21\,517A^4\varepsilon_1^2 + 126\,652A^2\varepsilon_1 + 187\,528},$$

$$\alpha = 84\,843A^4\varepsilon_1\varepsilon_2 + 126\,652A^2\varepsilon_1 + 250\,854A^2\varepsilon_2 - 2\sqrt{\beta} + 375\,056,$$

$$\beta = (0.721\,698A^8\varepsilon_1^2\varepsilon_2^2 + 2.156\,56A^6\varepsilon_1^2\varepsilon_2 + 4.252\,91A^6\varepsilon_1\varepsilon_2^2 + 1.611\,53A^4\varepsilon_1^2$$
$$+ 12.710\,95A^4\varepsilon_1\varepsilon_2 + 6.294\,64A^4\varepsilon_2^2 + 9.500\,32A^2\varepsilon_1 + 18.816\,86A^2\varepsilon_2 + 14.066\,70)\cdot 10^{10}.$$

7. RESULTS AND DISCUSSION

In this Section, the applicability, accuracy, and effectiveness of the proposed approaches are illustrated by comparing the analytical approximate frequency and periodic solution with the exact solutions [1]. The nonlinear oscillator described in Eq. (2) is a conservative system. By integrating Eq. (2) and using the initial conditions (3), we obtain

$$\frac{1}{2}\,(1+\varepsilon_1 q^2)\Big(\frac{dq}{dt^*}\Big)^2 + \frac{1}{2}\,q^2 + \frac{1}{4}\,\varepsilon_2 q^4 = \frac{1}{2}\,A^2 + \frac{1}{4}\,\varepsilon_2 A^4. \quad (22)$$

It follows from Eq. (22) that

$$\frac{dq}{dt^*} = \pm\Big(\frac{2(A^2-q^2) + \varepsilon_2(\varepsilon_1^4 - q^4)}{2(1+\varepsilon_1 q^2)}\Big)^{1/2}.$$

The time required for u to change from 0 to A is one fourth (1/4) of the exact period $T_{exact}(A)$. Hence, we have

$$T_{exact}(A) = 4\int\limits_0^A \Big(\frac{2(1+\varepsilon_1 q^2)}{2(A^2-q^2) + \varepsilon_2(\varepsilon_1^4 - q^4)}\Big)^{1/2}\,dq.$$

Therefore, the exact frequency is

$$\omega_{exact} = 2\pi/T_{exact}.$$

The results obtained by the variational approach, Hamiltonian approach, amplitude-frequency formulation, integral iteration method, and first-order coupled homotopy-variational formulation are similar, and we called them the first-order approximation. Also, Karimpour et al. [4] obtained a similar result for this problem by the energy balance method. The relative error for any ε_1 and ε_2 is found to be

$$\lim_{A\to\infty}\frac{\omega_1}{\omega_{exact}} = 0.866\,03,$$

where $\omega_1 = \omega_{\text{EBM}} = \omega_{\text{VA}} = \omega_{\text{AFF}} = \omega_{\text{IIM}} = \omega_{\text{CHV}} = \omega_{\text{HA}}$.
The second-order approximation for any ε_1 and ε_2 is

$$\lim_{A\to\infty}\frac{\omega_{\text{CHV2}}}{\omega_{exact}} = 0{,}9992.$$

The frequencies calculated by the proposed methods and the numerical method are compared in the table. The error percentage is defined as

$$E_1 = \frac{|\omega_1 - \omega_{num}|}{\omega_{num}}\cdot 100, \qquad E_2 = \frac{|\omega_{\text{CHV2}} - \omega_{num}|}{\omega_{num}}\cdot 100.$$

Figures 2 and 3 present the frequency as a function of the amplitude for different values of ε_1 and ε_2. The analytical dependences $q(t^*)$ obtained by different methods are compared in Fig. 4. It can be concluded that the difference between the second-order analytical solution and the numerical solution is negligible.

Frequencies for $\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_1 = \varepsilon_2 = 2$, and different values of the amplitude

A	$\varepsilon_1 = \varepsilon_2 = 1$					$\varepsilon_1 = \varepsilon_2 = 2$				
	ω_1	ω_{CHV2}	ω_{num}	E_1	E_2	ω_1	ω_{CHV2}	ω_{num}	E_1	E_2
0.01	1.0000	1.0000	1.0000	0	0	1.0000	1.0000	1.0000	0	0
0.05	1.0003	1.0004	1.0004	0.0100	0	1.0006	1.0008	1.0008	0.0077	0
0.10	1.0012	1.0016	1.0015	0.0257	0.0149	1.0025	1.0033	1.0031	0.0626	0.0183
0.50	1.0274	1.0375	1.0370	0.9255	0.0447	1.0488	1.0685	1.0682	1.8153	0.0245
1.00	1.0801	1.1169	1.1163	3.2408	0.0509	1.1180	1.1810	1.1802	5.2674	0.0711
5.00	1.2095	1.3688	1.3680	11.5841	0.0618	1.2169	1.3865	1.3855	12.1712	0.0708
10.00	1.2207	1.3960	1.3949	12.4858	0.0761	1.2227	1.4009	1.3998	12.6502	0.0773
20.00	1.2237	1.4034	1.4023	12.7342	0.0777	1.2242	1.4047	1.4036	12.7789	0.0752
50.00	1.2246	1.4055	1.4043	12.7977	0.0869	1.2247	1.4057	1.4047	12.8167	0.0730
100.00	1.2247	1.4058	1.4045	12.8014	0.0945	1.2247	1.4059	1.4049	12.8248	0.0696

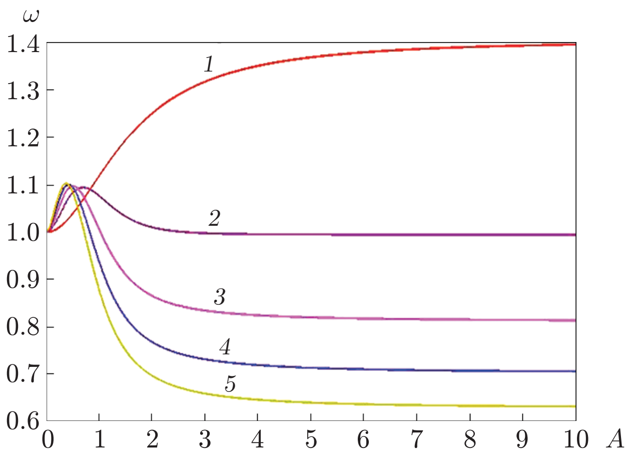

Fig. 2.

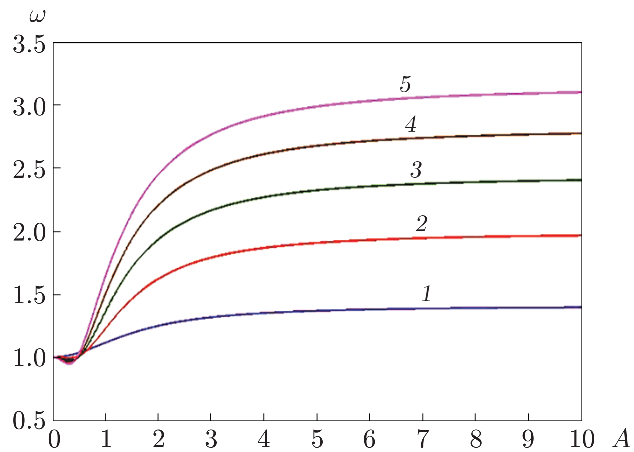

Fig. 3.

Fig. 2. Frequency versus amplitude for $\varepsilon_2 = 1$ and different values of ε_1: $\varepsilon_1 = 1$ (*1*), 2 (*2*), 3 (*3*), 4 (*4*), and 5 (*5*).

Fig. 3. Frequency versus amplitude for $\varepsilon_1 = 1$ and different values of ε_2: $\varepsilon_2 = 1$ (*1*), 2 (*2*), 3 (*3*), 4 (*4*), and 5 (*5*).

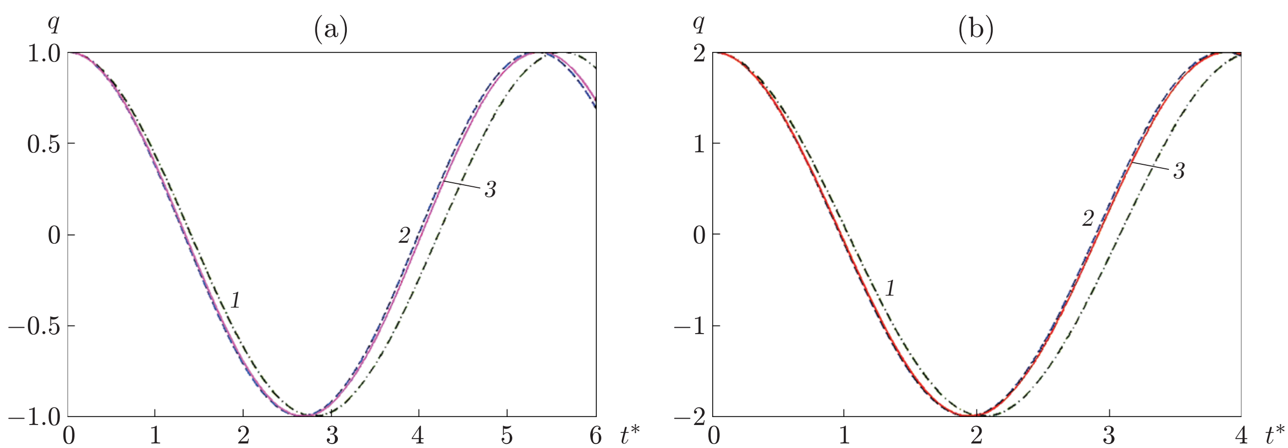

Fig. 4. Dependence $q(t^*)$ for $A = 1$ and $\varepsilon_1 = \varepsilon_2 = 2$ (a) and for $A = 2$, $\varepsilon_1 = 1$, and $\varepsilon_2 = 2$ (b): first approximation (*1*), second approximation (*2*), and numerical solution (*3*).

CONCLUSIONS

Five different methods are employed to propose the first-order and second-order approximate solutions for the analytical simulation of an elastically restrained tapered cantilever beam. The basic conclusions can be formulated as follows.

An elastically restrained tapered cantilever beam can be modeled with a system of nonlinear algebraic equations and a nonlinear differential equation. The present results reveal that these methods can be considered as a viable alternative to conventional methods for solving highly nonlinear oscillatory systems. In the coupled homotopy-variational formulation, the higher-order approximates can readily be obtained with higher accuracy. It is obvious that the variational approach provides us with a freedom of choosing the trial function and gives us more information regarding the relation between the frequency and amplitude. The second-order approximation can be obtained by using the coupled homotopy-variational formulation. The results obtained by different methods considered in the paper for the first-order approximation are similar to each other.

REFERENCES

1. J. W. S. Rayleigh, *The Theory of Sound* (Dover Publ., New York, 1877).
2. H. Poincaré, *Les méthodes Nouvelles de la Mèchanique Céleste. 3 v.* (Dover Publ., New York, 1957).
3. S. Timoschenko, D. H. Young, and W. Weaver Vibration Problems in Engineering (Wiley, New York, 1974).
4. S. Karimpour, S. S. Ganji, A. Barari, et al., "Nonlinear Vibration of an Elastically Restrained Tapered Beam," Sci. China **55** (10), 1925–1930 (2012).
5. N. Bogoliubov and J. A. Mitropolski, *Asymptotical Methods in the Theory of Nonlinear Oscillations* (Hindustan Publ. Co., Delhi, 1963).
6. A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations* (Wiley, New York, 1979).
7. R. Bhattacharya, P. Jainb, and A. Nair, "Normal Mode Localization for a Two Degrees-of-Freedom System with Quadratic and Cubic Non-Linearities," J. Sound Vibrat. **249**, 909–919 (2002).
8. S. K. Lai and C. W. Lim, "Nonlinear Vibration of a Two-Mass System with Nonlinear Stiffnesses," Nonlinear Dynamics **49**, 233–249 (2007).
9. D. D. Ganji, N. Ranjbar Malidarreh, and M. Akbarzade, "Comparison of Energy Balance Period for Arising Nonlinear Oscillator Equations (He's Energy Balance Period for Nonlinear Oscillators with and without Discontinuities)," Acta Appl. Math. **108** (2), 353–362 (2009).
10. I. Mehdipour, D. D. Ganji, and M. Mozaffari, "Application of the Energy Balance Method to Nonlinear Vibrating Equations," Current Appl. Phys. **10**, 104–112 (2010).
11. J. H. He, "Preliminary Report on the Energy Balance for Nonlinear Oscillations," Mech. Res. Comm. **29** (2/3), 107–111 (2002).
12. Y. Khan and A. Mirzabeigy, "Improved Accuracy of He's Energy Balance Method for Analysis of Conservative Nonlinear Oscillator," Neural Comput. Appl. **25**, 889–895 (2014).
13. J. H. He, "Homotopy Perturbation Technique," Comput. Math. Appl. Mech. Engng. **178** (3/4), 257–262 (1999).
14. J. H. He, "A Coupling Method of Homotopy Technique and Perturbation Technique for Nonlinear Problems," Intern. J. Nonlinear Mech. **35** (1), 115–123 (2000).
15. D. D. Ganji and A. Sadighi, "Application of He's Homotopy-Perturbation Method to Nonlinear Coupled Systems of Reaction-Diffusion Equations," Intern. J. Nonlinear Sci. Numer. Simulat. **7** (4), 411–418 (2006).
16. D. D. Ganji and M. Rafei, "Solitary Wave Solutions for a Generalized Hirota — Satsuma Coupled KdV Equation by Homotopy Perturbation Method," Phys. Lett. A **356** (2), 131–137 (2006).
17. L. Cveticanin, "The Homotopy-Perturbation Method Applied for Solving Complex Valued Differential Equations with Strong Cubic Nonlinearity," J. Sound Vibrat. **285**, 1171–1179 (2005).
18. L. Cveticanin, "Application of Homotopy-Perturbation to Non-Linear Partial Differential Equations," Chaos, Solitons Fractals. **40** (1), 221–228 (2009).
19. H. Tari, H.-J. Su, and T.-Y. Li, "A Constrained Homotopy Technique for Excluding Unwanted Solutions from Polynomial Equations Arising in Kinematics Problems," Mech. Machine Theory **45** (6), 898–910 (2010).
20. P. D. Ariel, "Extended Homotopy Perturbation Method and Computation of Flow Past a Stretching Sheet," Comput. Math. Appl. **58**, 2402–2409 (2009).

21. A. Fereidoon, Y. Rostamiyan, M. Akbarzade, and D. D. Ganji, "Application of He's Homotopy Perturbation Method to Nonlinear Shock Damper Dynamics," Arch. Appl. Mech. **80** (6), 641–649 (2010).
22. J. H. He, "Hamiltonian Approach to Nonlinear Oscillators," Phys. Lett. A **374**, 2312–2314 (2010).
23. S. Durmaz, S. Altay Demirbag, and M. O. Kaya, "High Order Hamiltonian Approach to Nonlinear Oscillators." Intern. J. Nonlinear Sci. Numer. Simulat. **11** (8), 565–570 (2010).
24. A. Yildirim, Z. Saadatnia, H. Askari, et al., "Higher Order Approximate Periodic Solutions for Nonlinear Oscillators with the Hamiltonian Approach," Appl. Math. Lett. **24**, 2042–2051 (2011(.
25. M. Bayat, I. Pakar, and L. Cveticanin, "Nonlinear Free Vibration of Systems with Inertia and Static Type Cubic Nonlinearities: An Analytical Approach," Mech. Machine Theory **77**, 50–58 (2014).
26. M. Akbarzade and Y. Khan, "Dynamic Model of Large Amplitude Non-Linear Oscillations Arising in the Structural Engineering: Analytical Solutions," Math. Comput. Modelling **55**, 480–489 (2012).
27. A. Yildirim, H. Askari, M. KalamiYazdi, and Y. Khan, "A Relationship between Three Analytical Approaches to Nonlinear Problems," Appl. Math. Lett. **25** (11), 1729–1733 (2012).
28. M. Akbarzade and A. Farshidianfar, "Application of the Amplitude-Frequency Formulation to a Nonlinear Vibration System Typified by a Mass Attached to a Stretched Wire," Intern. Appl. Mech. **50** (4), 476 (2014). DOI: 10.1007/s10778-014-0650-x.
29. L. Geng and X. C. Cai, "He's Frequency Formulation for Nonlinear Oscillators," Europ. J. Phys. **28**, 923–931 (2007).
30. J. H. He, "An Elementary Introduction to Recently Developed Asymptotic Methods and Nanomechanics in Textile Engineering," Intern. J. Modern Phys. B **22** (21), 3487–3578 (2008).
31. M. Akbarzade and J. Langari, "Determination of Natural Frequencies by Coupled Method of Homotopy Perturbation and Variational Method for Strongly Nonlinear Oscillators," J. Math. Phys. **52**, 023518 (2011).
32. Y. Khan, M. Akbarzade, and A. Kargar, "Coupling of Homotopy and the Variational Approach for a Conservative Oscillator with Strong Odd-Nonlinearity," Sci. Iran. A **19** (3), 417–422 (2012).
33. J. Sardanyés, C. Rodrigues, C. Januário, et al., "Activation of Effector Immune Cells Promotes Tumor Stochastic Extinction: A Homotopy Analysis Approach," Appl. Math. Comput. **252**, 484–495 (2015).
34. Y. Khan, K. Sayevand, M. Fardi, and M. Ghasemi, "A Novel Computing Multi-Parametric Homotopy Approach for System of Linear and Nonlinear Fredholm Integral Equations," Appl. Math. Comput. **249**, 229–236 (2014).
35. H. N. Hassan and M. A. El-Tawil, "A New Technique of using Homotopy Analysis Method for Second Order Nonlinear Differential Equations," Appl. Math. Comput. **219** (2), 708–728 (2012).
36. V. Marinca and N. Herisanu, "The Optimal Homotopy Asymptotic Method for Solving Blasius Equation," Appl. Math. Comput. **231**, 134–139 (2014).
37. T. E. Simos, "Bessel and Neumann Fitted Methods for the Numerical Solution of the Schrödinger Equation," Comput. Math. Appl. **42** (6/7), 833–847 (2001).
38. A. Jafarian, S. Measoomy, and S. Abbasbandy, "Artificial Neural Networks Based Modeling for Solving Volterra Integral Equations System," Appl. Soft Comput. **27**, 391–398 (2015).
39. M. H. Ghayesh, F. Alijani, and M. A. Darabi, "An Analytical Solution for Nonlinear Dynamics of a Viscoelastic Beam-Heavy Mass System," J. Mech. Sci. Technol. **25** (8), 1915–1923 (2011).
40. M. A. Z. Raja, J. A. Khan, A. M. Siddiqui, et al., "Exactly Satisfying Initial Conditions Neural Network Models for Numerical Treatment of First Painleve Equation," Appl. Soft Comput. **26**, 244–256 (2015).
41. M. S. Abdel-Jaber, A. A. Al-Qaisia, M. Abdel-Jaber, and R. G. Beale, "Nonlinear Natural Frequencies of an Elastically Restrained Tapered Beam," J. Sound Vibrat. **313**, 772–783 (2008).
42. J. H. He, "Some Asymptotic Methods for Strongly Nonlinear Equations," Intern. J. Modern Phys. B **20**, 1141–1199 (2006).