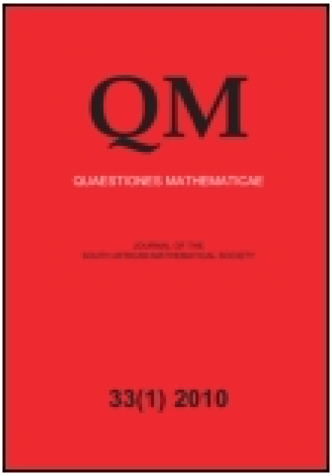

Quaestiones Mathematicae
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tqma20

Characterization of relative n-isoclinism of a pair of Lie algebras

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Published online: 10 Apr 2015.

Quaestiones Mathematicae **38**(1) 2015: 27–39.

http://dx.doi.org/10.2989/16073606.2014.981700

CHARACTERIZATION OF RELATIVE n-ISOCLINISM OF A PAIR OF LIE ALGEBRAS

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Abstract. In 1994, Moneyhun introduced and studied the concept of isoclinism in Lie algebras. Moghaddam and parvaneh in 2009 introduced and gave some properties of isoclinism of a pair of Lie algebras. In this paper we introduce the concept of relative n-isoclinism between two pairs of Lie algebras. They form equivalence classes and we show that each equivalent class contains an n-stem pair of Lie algebras. We also show that for a relative n-isoclinism family of Lie algebras $\mathcal{C}$ say, consists of (M, L) such that L is finitely generated and $[M,_n L]$ is finite dimension, then there exists an n-stem pair of Lie algebras $(R, S) \in \mathcal{C}$ such that

$$dim([R,_{n-1} S] = min\{dim([M,_{n-1} L]); for\ all\ (M, L) \in \mathcal{C}\}.$$

Mathematics Subject Classification (2010): Primary 17B30, 17B60, 17B99; Secondary 20E99, 20D15.

Key words: Lie algebra, n-isoclinism, relative n-isoclinism, n-stem Lie algebra.

1. Introduction. We consider all Lie algebras over a fixed field F and assume M to be an ideal of a Lie algebra L, with the Lie product $[\ ,\ ]$. Then (M, L) is said to be a pair of Lie algebras. The lower and the upper central series of L are defined inductively by $L^1 = L$, $L^{n+1} = [L^n, L]$, for all $n \geq 1$, and $Z_0(L) = \{0\}$, $Z_1(L) = Z(L)$, and $Z_{n+1}(L)/Z_n(L) = Z(L/Z_n(L))$, for all $n \geq 0$. Also one may consider

$$[M, L] = \langle [m, l]; \;\; m \in M, \; l \in L \rangle,$$
$$Z(M, L) = \{m \in M; \;\; [m, l] = 0, \; \forall l \in L\} = Z(L) \cap M,$$

to be the commutator and the centre of the pair of Lie algebras (M, L), respectively. We also denote $[M, \underbrace{L, \ldots, L}_{n-times}]$ by $[M,_n L]$ and

$$Z_n(L) \cap M = \{m \in M; \;\; [m, l_1, l_2, \ldots, l_n] = 0, \; \forall l_i \in L, \; 1 \leq i \leq n\}$$

by $Z_n(M, L)$, which is considered as the n^{th}-term of the upper central series of the pair of Lie algebras (M, L).
Clearly, $[M,_n L]$ and $Z_n(M, L)$ are both ideals of L and when $M = L$, they are L^{n+1} and $Z_n(L)$, the $(n+1)^{st}$ and n^{th}-term of the lower and the upper central series of L, respectively.

The notion of isoclinism for groups was first introduced by P. Hall [3] in 1940 and was generalized to n- isoclinism and isologism by other authors (see also [4], [5], [6]). In 1994, Moneyhun [9] introduced the Lie algebra's analogue of the concept and contrary to the group theory case, she showed that when two Lie algebras have the same finite dimensions then the two concepts of isoclinism and isomorphism are coincide.

In 2009, the first author and parvaneh [8] studied the concept of isoclinism of a pair of Lie algebras and gave some structural properties of the notion.

In the present article we introduce the concept of relative n-isoclinism of a pair of Lie algebras and give some properties of this new notion. Among other results, it is shown that if $\mathcal{C}$ is a relative n-isoclinism family of Lie algebras and (M, L) is a pair in $\mathcal{C}$ such that L is finitely generated and $[M,_n L]$ is of finite dimension, then there exists an n-stem pair $(R, S) \in \mathcal{C}$ such that

$$dim([R,_{n-1} S] = min\{dim([M,_{n-1} L]); \;\; for \; all \; (M, L) \in \mathcal{C}\}.$$

For a pair of Lie algebras (M, L), put $\overline{L} = \frac{L}{Z_n(M,L)}$ and $\overline{M} = \frac{M}{Z_n(M,L)}$. Let M_1 and M_2 be two ideals of the Lie algebras L_1 and L_2, respectively. Then the pair (α, β) is called a relative n-isoclinism between the pairs of Lie algebras (M_1, L_1) and (M_2, L_2), if the maps $\alpha : \overline{L_1} \longrightarrow \overline{L_2}$ with $\alpha(\overline{M_1}) = \overline{M_2}$ and $\beta : [M_1,_n L_1] \longrightarrow [M_2,_n L_2]$ are both isomorphisms, for which the following diagram commutes

$$\begin{array}{ccc} \overline{M_1} \oplus \underbrace{\overline{L_1} \oplus \ldots \oplus \overline{L_1}}_{n-times} & \stackrel{\alpha^{n+1}}{\rightarrow} & \overline{M_2} \oplus \underbrace{\overline{L_2} \oplus \ldots \oplus \overline{L_2}}_{n-times} \\ \downarrow & & \downarrow \\ [M_1,_n L_1] & \stackrel{\beta}{\rightarrow} & [M_2,_n L_2]. \end{array}$$

In this case, we write $(M_1, L_1) \sim_n (M_2, L_2)$. We show the map from $\overline{M_i} \oplus \underbrace{\overline{L_i} \oplus \ldots \oplus \overline{L_i}}_{n-times}$ $(i = 1, 2)$ to $[M_i,_n L_i]$ by $\gamma(n, M_i, L_i)$ $(i = 1, 2)$, which sends $(m_i, l_{i_1}, \ldots l_{i_n})$ to $[m_i, l_{i_1}, \ldots l_{i_n}]$ and if M_i is equal to L_i, the map is shown by $\gamma(n, L_i)$. Clearly, in this case we have the notion of n-isoclinism (see [10]) and when $n = 1$ one obtains the concept of isoclinism of the pairs of Lie algebras.

2. Some properties of relative n-isoclinism. In this section we give some properties of relative n-isoclinism of a pair of Lie algebras, which will be used in the next section. We also give some relation between the notions of n-isoclinism and relative n-isoclinism. Finally, it is shown that if L is a Lie algebra with an ideal M and a subalgebra H containing M such that $M \cap [H,_n L] = 0$, then $(\frac{H}{M}, \frac{L}{M}) \sim_n (H, L)$ and its converse is also true, when the subalgebra $[H,_n L]$ is of finite dimension.

The proof of the following useful lemma can be easily seen.

LEMMA 2.1. *Let (α, β) be a relative n-isoclinism between the pairs of Lie algebras (M_1, L_1) and (M_2, L_2). Then for all $x \in [M_1,_n L_1]$,*

1. $\alpha(x + Z_n(M_1, L_1)) = \beta(x) + Z_n(M_2, L_2)$;
2. $\beta([m_1, l_1 \ldots . l_{i-1}, x, l_{i+1}, \ldots l_n]) = [m_2, k_1, \ldots k_{i-1}, \beta(x), k_{i+1}, \ldots k_n]$, *where* $m_2 \in \alpha(m_1 + Z_n(M_1, L_1))$ *and* $k_j \in \alpha(l_j + Z_n(M_1, L_1))$; $1 \leq j \leq n$; $i \neq j$;
3. $\beta([M_1,_n L_1] \cap Z_i(M_1, L_1)) = [M_2,_n L_2] \cap Z_i(M_2, L_2)$, $n \geq i \geq 0$.

THEOREM 2.2. *The pairs of Lie algebras (M_1, L_1) and (M_2, L_2) are relative n-isoclinism if and only if there exist ideals N_1 and N_2 of L_1 and L_2 contained in $Z_n(M_1, L_1)$ and $Z_n(M_2, L_2)$, respectively, together with the isomorphisms $\alpha : \frac{L_1}{N_1} \longrightarrow \frac{L_2}{N_2}$, for which $\alpha(\frac{M_1}{N_1}) = \frac{M_2}{N_2}$ and $\beta : [M_1,_n L_1] \longrightarrow [M_2,_n L_2]$ given by $\beta([m_1, l_1, \ldots, l_n]) = [m_2, k_1, \ldots k_n]$, where $m_1 \in M_1$, $l_i \in L_1$ $1 \leq i \leq n$, and $m_2 \in M_2$, $k_i \in L_2$, $m_2 \in \alpha(m_1 + Z_n(M_1, L_1))$, and $k_i \in \alpha(l_i + Z_n(M_1, L_1))$.*

Proof. Let N_1 and N_2 be the ideals of L_1 and L_2, respectively, with the given isomorphisms α and β. By the definition, we must show that α induces an isomorphism from $\overline{L_1}$ onto $\overline{L_2}$. It is enough to show that $\alpha(\frac{Z_n(M_1,L_1)}{N_1}) = \frac{Z_n(M_2,L_2)}{N_2}$. So, let $k \in Z_n(M_2, L_2)$ and $l \in \alpha^{-1}(k + N_2)$. For $l_1, \ldots l_{n+1} \in L_1$, suppose that $k_i \in \alpha(l_i + N_1)$, $1 \leq i \leq n + 1$. It is clear that $l_i + l \in \alpha^{-1}((k_i + k) + N_2)$. Now

$$\begin{aligned}[l_1, \ldots l_i + l, \ldots l_{n+1}] &= \beta^{-1}([k_1, \ldots, k_i + k, \ldots, k_{n+1}]) \\ &= [l_1, \ldots l_i, \ldots l_{n+1}],\end{aligned}$$

which implies that $l \in Z_n(M_1, L_1)$, and hence $\frac{Z_n(M_2,L_2)}{N_2} \subseteq \alpha(\frac{Z_n(M_1,L_1)}{N_1})$. By a similar argument, we have $\alpha(\frac{Z_n(M_1,L_1)}{N_1}) \subseteq \frac{Z_n(M_2,L_2)}{N_2}$, which completes the proof. □

COROLLARY 2.3. *If $(M_1, L_1) \sim_n (M_2, L_2)$, then $(M_1, L_1) \sim_{n+1} (M_2, L_2)$.*

Proof. By the assumption, the isomorphisms $\alpha : \overline{L_1} \longrightarrow \overline{L_2}$, with $\alpha(\overline{M_1}) = \overline{M_2}$ and $\beta : [M_1,_n L_1] \longrightarrow [M_2,_n L_2]$ exist. Now, put $N_1 = Z_n(M_1, L_1) \subseteq Z_{n+1}(M_1, L_1)$ and $N_2 = Z_n(M_2, L_2) \subseteq Z_{n+1}(M_2, L_2)$. Then consider the isomorphism $\beta' : [M_1,_{n+1} L_1] \longrightarrow [M_2,_{n+1} L_2]$, such that $\beta'(x) = \beta(x)$, for all $x \in [M_1,_{n+1} L_1]$.

It is easily seen that the conditions of the above theorem are satisfied. Hence $(M_1, L_1) \sim_{n+1} (M_2, L_2)$. □

Corollary 2.4. *If $(M_1, L_1) \sim_n (M_2, L_2)$, then for all $0 \le i \le n$*

$$(\frac{M_1}{Z_i(M_1, L_1)}, \frac{L_1}{Z_i(M_1, L_1)}) \sim_{n-i} (\frac{M_2}{Z_i(M_2, L_2)}, \frac{L_2}{Z_i(M_2, L_2)}).$$

Proof. It is a routin exercise to show that for any pair of Lie algebras (M, L),

$$Z_j(\frac{M}{Z_i(M, L)}, \frac{L}{Z_i(M, L)}) \cong \frac{Z_{i+j}(M, L)}{Z_i(M, L)}. \quad (1)$$

Assume $\widetilde{L_j} = \frac{L_j}{Z_i(M_j, L_j)}$ and $\widetilde{M_j} = \frac{M_j}{Z_i(M_j, L_j)}$ $(j = 1, 2)$. By the assumption, the isomorphisms $\alpha : \overline{L_1} \longrightarrow \overline{L_2}$ with $\alpha(\overline{M_1}) = \overline{M_2}$ and $\beta : [M_1,_n L_1] \longrightarrow [M_2,_n L_2]$ exist. Now using (1), we obtain $\widetilde{\alpha} : \widetilde{L_1}/Z_{n-i}(\widetilde{M_1}, \widetilde{L_1}) \longrightarrow \widetilde{L_2}/Z_{n-i}(\widetilde{M_2}, \widetilde{L_2})$ such that $\widetilde{\alpha}(\widetilde{M_1}/Z_{n-i}(\widetilde{M_1}, \widetilde{L_1})) = \widetilde{M_2}/Z_{n-i}(\widetilde{M_2}, \widetilde{L_2})$ and $\widetilde{\beta} : [\widetilde{M_1},_{n-i} \widetilde{L_1}] \longrightarrow [\widetilde{M_2},_{n-i} \widetilde{L_2}]$, given by $\widetilde{\beta}(\widetilde{x}) = \beta(x) + Z_{n-i}(M_2, L_2)$. Now, Theorem 2.2 gives the result. □

In the next result of this section, we present some properties of relative n-isoclinism between the subalgebras of Lie algebras, which is similar to the work of Moghaddam and Parvaneh [8] for the isoclinism of a pair of Lie algebras.

Theorem 2.5. *Let (α, β) be a relative n-isoclinism between the pairs of Lie algebras (M_1, L_1) and (M_2, L_2).*

1. *If K_1 is a subalgebra of L_1 containing $Z_n(M_1, L_1)$ and K_2 is some subalgebra of L_2, such that $\alpha(\overline{K_1}) = \overline{K_2}$, where $\overline{K_i} = \frac{K_i}{Z_n(M_i, L_i)}$ $(i = 1, 2)$, Then $(K_1 \cap M_1, K_1) \sim_n (K_2 \cap M_2, K_2)$.*

2. *If K_1 is an ideal of L_1 contained in $[M_1,_n L_1]$, then*

$$(\frac{M_1}{K_1}, \frac{L_1}{K_1}) \sim_n (\frac{M_2}{\beta(K_1)}, \frac{L_2}{\beta(K_1)}).$$

Proof. (1) By Theorem 2.2, we consider $N_i = Z_n(K_i \cap M_i, L_i) \subseteq Z_n(K_i \cap M_i, K_i)$, $i = 1, 2$. Now, we define $\overline{\alpha} : \frac{K_1}{N_1} \longrightarrow \frac{K_2}{N_2}$ given by $\overline{\alpha}(k_1 + N_1) = k_2 + N_2$, where $k_2 \in \alpha(\overline{K_1})$ and $\overline{\beta} : [K_1 \cap M_1,_n K_1] \longrightarrow [K_2 \cap M_2,_n K_2]$ such that $\overline{\beta}(x) = \beta(x)$, for all $x \in [K_1 \cap M_1,_n K_1]$.

Clearly $\overline{\alpha}$ and $\overline{\beta}$ are monomorphisms, as α and β are. On the other hand $\overline{\alpha}$ is onto, since $\alpha(\overline{K_1}) = \overline{K_2}$ and $N_2 \subseteq Z_n(M_2, L_2)$. We only need to show that $\overline{\beta}$ is surjective. For doing this, assume $[m_2, k_{2_1}, k_{2_2}, \ldots, k_{2_n}]$ is an arbitrary generator of $[K_2 \cap M_2,_n K_2]$. By surjectivity of α, there exist $k_{1_1}, k_{1_2}, \ldots, k_{1_n} \in K_1$ such that $\alpha(k_{1_i} + Z_n(M_1, L_1)) = k_{2_i} + Z_n(M_2, L_2)$, for $1 \le i \le n$. Also since $\alpha(\overline{M_1}) = \overline{M_2}$

and $\alpha(\overline{K_1}) = \overline{K_2}$, there exists $m_2 \in M_2 \cap K_2$ such that $\alpha(m_1 + Z_n(M_1, L_1)) = m_2 + Z_n(M_2, L_2)$. By the assumption, the pair (α, β) is relative n-isoclinism and using the commutativity of the diagram in the definition,

$$\beta([m_1, k_{1_1}, k_{1_2}, \ldots, k_{1_n}]) = [m_2, k_{2_1}, k_{2_2}, \ldots, k_{2_n}].$$

Finally, one must show that $\overline{\alpha}(\frac{K_1 \cap M_1}{N_1}) = \frac{K_2 \cap M_2}{N_2}$, which is trivially true, by the above discussion.

(2) Clearly $\beta(K_1)$ is an ideal of L_2, and so put $\widetilde{L_1} = \frac{L_1}{K_1}$, $\widetilde{M_1} = \frac{M_1}{K_1}$, $\widetilde{L_2} = \frac{L_2}{\beta(K_1)}$ and $\widetilde{M_2} = \frac{M_2}{\beta(K_1)}$. Also assume that $\widetilde{N_1} = \frac{Z_n(M_1,L_1)+K_1}{K_1} \subseteq Z_n(\widetilde{M_1}, \widetilde{L_1})$, and $\widetilde{N_2} = \frac{Z_n(M_2,L_2)+\beta(K_1)}{\beta(K_1)} \subseteq Z_n(\widetilde{M_2}, \widetilde{L_2})$.

We may define

$$\widetilde{\alpha} : \frac{\widetilde{L_1}}{\widetilde{N_1}} \longrightarrow \frac{\widetilde{L_2}}{\widetilde{N_2}}$$

such that $\widetilde{\alpha}(\widetilde{l_1} + \widetilde{N_1}) = \widetilde{l_2} + \widetilde{N_2}$, where $\widetilde{l_1} \in \widetilde{L_1}$ and $l_2 \in \alpha(l_1 + Z_n(M_1, L_1))$. Also $\widetilde{\beta} : [\widetilde{M_1},_n \widetilde{L_1}] \longrightarrow [\widetilde{M_2},_n \widetilde{L_2}]$, given by $\widetilde{\beta}(\widetilde{x}) = \widetilde{\beta}(x + K_1) = \frac{\beta(x)+\beta(K_1)}{\beta(K_1)} = \widetilde{\beta(x)}$, for all generators x of $[M_1,_n L_1]$. The maps $\widetilde{\alpha}$ and $\widetilde{\beta}$ are both isomorphisms, as α and β are. We also have $\widetilde{\alpha}(\widetilde{M_1}/\widetilde{N_1}) = \widetilde{M_2}/\widetilde{N_2}$, since $\alpha(\overline{M_1}) = \overline{M_2}$. Clearly

$$\widetilde{\beta}([m_1, l_{1_1}, \ldots l_{1_n}] + K_1) = \widetilde{\beta}([\widetilde{m_1}, \widetilde{l_{1_1}}, \ldots \widetilde{l_{1_n}}]) = [\widetilde{m_2}, \widetilde{l_{2_1}}, \ldots \widetilde{l_{2_n}}],$$

where $\widetilde{m_2} \in \widetilde{\alpha}(\widetilde{m_1} + \widetilde{N_1})$, and $\widetilde{l_{2_i}} \in \widetilde{\alpha}(\widetilde{l_{1_i}} + \widetilde{N_1})$, for $1 \leq i \leq n$. Now, using Theorem 2.2, we deduce $(\frac{M_1}{K_1}, \frac{L_1}{K_1}) \sim_n (\frac{M_2}{\beta(K_1)}, \frac{L_2}{\beta(K_1)})$. □

The following theorem shows that, if (M_1, L_1) and (M_2, L_2) are two pairs of Lie algebras, such that $(M_1, L_1) \sim_n (M_2, L_2)$, then M_1 and M_2 are n-isoclinism Lie algebras.

THEOREM 2.6. *If (M_1, L_1) and (M_2, L_2) are relative n-isoclinism, then M_1 and M_2 are n-isoclinism Lie algebras.*

Proof. Let (α, β) be a relative n-isoclinism between the pairs of Lie algebras (M_1, L_1) and (M_2, L_2). Assume $K = Z_n(M_1, L_1) \subseteq Z_n(M_1)$ and $H = Z_n(M_2, L_2) \subseteq Z_n(M_2)$. Consider $\alpha' : \frac{M_1}{K} \longrightarrow \frac{M_2}{H}$ and $\beta' : M_1^{n+1} \longrightarrow M_2^{n+1}$, which are the restrictions of α and β to $\frac{M_1}{K}$ and M_1^{n+1}, respectively. Clearly both α' and β' are isomorphisms, and using the definition

$$\begin{aligned} \beta'([m_1, m_2 \ldots, m_{n+1}]) &= \beta([m_1, m_2, \ldots, m_{n+1}]) \\ &= \beta\gamma(n, M_1, L_1)(\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_{n+1}) \\ &= \gamma(n, M_2, L_2)\alpha^{n+1}(\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_{n+1}) \\ &= [k_1, k_2, \ldots, k_n], \end{aligned}$$

for all $m_1, m_2, \ldots m_{n+1} \in M_1^{n+1}$ and $k_1, k_2, \ldots k_{n+1} \in \alpha'(m_i + K)$. Using Theorem 2.2 and considering the case $L_1 = M_1$ and $M_2 = L_2$ we obtain $M_1 \sim_n M_2$. □

In the following result we give a sufficient condition, so that the n-isoclinism between Lie algebras gives rise to a relative n-isoclinism of the pairs of Lie algebras.

THEOREM 2.7. *Let (φ, ψ) be a pair of n-isoclinism between the Lie algebras L_1 and L_2 such that for any $l_1, l_1' \in L_1$, the equality $\varphi(l_1 + Z_n(L_1)) = \varphi(l_1' + Z_n(L_1))$ implies that $l_1 = l_1'$. Then for any ideal M_1 of L_1, there exists an ideal M_2 of L_2 such that $(M_1, L_1) \sim_n (M_2, L_2)$.*

Proof. Let (φ, ψ) be an n-isoclinism between the Lie algebras L_1 and L_2, and M_1 be an arbitrary ideal of L_1. Assume

$$M_2 = \{x \in L_2;\ \ \exists m_1 \in M_1,\ \ \varphi(m_1 + Z_n(L_1)) = x + Z_n(L_2)\}.$$

One can easily check that M_2 is an ideal of L_2, and hence let $\alpha : \overline{L_1} \longrightarrow \overline{L_2}$ be a map given by $\alpha(l_1 + Z_n(M_1, L_1)) = \varphi(l_1 + Z_n(L_1))$. As φ is an isomorphism, then so is α and since M_2 is an ideal of L_2, it follows that $\alpha(\overline{M_1}) = \overline{M_2}$.

Now, consider the following diagram

$$\begin{array}{ccc} \overline{M_1} \oplus \overline{L_1} \oplus \ldots \oplus \overline{L_1} & \overset{\alpha^{n+1}}{\rightarrow} & \overline{M_2} \oplus \overline{L_2} \oplus \ldots \oplus [\overline{L_2} \\ \downarrow & & \downarrow \\ [M_1,_n L_1] & \overset{\beta}{\rightarrow} & [M_2,_n L_2], \end{array}$$

in which β is the restriction of ψ to $[M_1,_n L_1]$. The map β is monomorphism, as ψ is. Assume $[m_2, l_{2_1}, l_{2_2}, \ldots, l_{2_n}]$ is any generator of $[M_2,_n L_2]$, then using the property of M_2 and φ being onto, there exist $l_{1_i} \in L_1$ $(1 \leq i \leq n)$ and $m_1 \in M_1$, such that $\varphi(m_1 + Z_n(L_1)) = m_2 + Z_n(L_2)) = \alpha(\overline{m}_1)$ and $\varphi(l_{1_i} + Z_n(L_1)) = l_{2_i} + Z_n(L_2)) = \alpha(\bar{l}_{1_i})$, for all $(1 \leq i \leq n)$.

By the commutativity of the diagram in the definition of n-isoclinism, we have

$$\begin{aligned} \beta([m_1, l_{1_1} \ldots, l_{1_n}]) &= \psi([m_1, l_{1_1} \ldots, l_{1_n}]) \\ &= \psi\gamma(n, L_1)(m_1 + Z_n(L_1), l_{1_1} + Z_n(L_1), \ldots, l_{1_n} + Z_n(L_1)) \\ &= \gamma(n, L_2)\varphi(m_1 + Z_n(L_1), l_{1_1} + Z_n(L_1), \ldots, l_{1_n} + Z_n(L_1)) \\ &= [m_2, l_{2_1}, l_{2_2}, \ldots, l_{2_n}]. \end{aligned}$$

This shows that β is onto and hence it is an isomorphism. Hence it is enough to show the commutativity of the diagram. So, for any generator $[\overline{m}_1, \bar{l}_{1_1}, \ldots, \bar{l}_{1_n}]$ of $\overline{M_1} \oplus \overline{L_1} \oplus \ldots \oplus \overline{L_1}$, we have

$$\begin{aligned} \beta\gamma(n, M_1, L_1)(\overline{m}_1, \bar{l}_{1_1} \ldots, \bar{l}_{1_n}) &= \beta([m_1, l_{1_1} \ldots, l_{1_n}]) \\ &= [m_2, l_{2_1}, l_{2_2}, \ldots, l_{2_n}] \\ &= \gamma(n, M_2, L_2)(\overline{m}_2, \bar{l}_{2_1} \ldots, \bar{l}_{2_n}) \\ &= \gamma(n, M_2, L_2)\alpha(\overline{m}_1, \bar{l}_{1_1} \ldots, \bar{l}_{1_n}), \end{aligned}$$

and hence the proof is complete. □

In the following it is shown that, if L_1 and L_2 are n-isoclinic, then under some condition this property inherits to the ideals of L_1 and L_2. The proof follows from Theorems 2.7 and 2.6

COROLLARY 2.8. *Let (φ, ψ) be an n-isoclinism between the Lie algebras L_1 and L_2 such that for all $l_1, l_1' \in L_1$, the equality $\varphi(l_1 + Z_n(L_1)) = \varphi(l_1' + Z_n(L_1))$ implies that $l_1 = l_1'$. Then for any ideal M_1 of L_1, there exists an ideal M_2 of L_2 such that $M_1 \sim_n M_2$.*

Finally in this section, we give the following result and has a useful corollary, which will be used in the next section.

THEOREM 2.9. *Let M and H be ideals of a Lie algebra L such that H contains M. Then, for all $n \geq o$*

$$(\frac{H}{M}, \frac{L}{M}) \sim_n (\frac{H}{M \cap [H,_n L]}, \frac{L}{M \cap [H,_n L]}).$$

Proof. Assume, $\widetilde{L} = \frac{L}{M}$, $\widetilde{H} = \frac{H}{M}$, $\widehat{L} = \frac{L}{M \cap [H,_n L]}$ and $\widehat{H} = \frac{H}{M \cap [H,_n L]}$. So, we must show that $(\widetilde{H}, \widetilde{L}) \sim_n (\widehat{H}, \widehat{L})$. We define the map $\alpha : \frac{\widetilde{L}}{Z_n(\widetilde{H}, \widetilde{L})} \longrightarrow \frac{\widehat{L}}{Z_n(\widehat{H}, \widehat{L})}$ given by $\alpha(\widetilde{l} + Z_n(\widetilde{H}, \widetilde{L})) = \widehat{l} + Z_n(\widehat{H}, \widehat{L})$, for all $\widetilde{l} \in \widetilde{L}$. It can be easily seen that α is a well-defined and one-to-one map. It is a Lie algebra homomorphism, since for all $\widetilde{x}, \widetilde{y} \in \widetilde{L}$,

$$\begin{aligned} \alpha((\widetilde{x} + \widetilde{y}) + Z_n(\widetilde{H}, \widetilde{L})) &= \alpha(\widetilde{x + y} + Z_n(\widetilde{H}, \widetilde{L})) \\ &= \widehat{x + y} + Z_n(\widehat{H}, \widehat{L}) \\ &= \alpha(\widetilde{x} + Z_n(\widetilde{H}, \widetilde{L})) + \alpha(\widetilde{y} + Z_n(\widetilde{H}, \widetilde{L})), \end{aligned}$$

and

$$\begin{aligned} \alpha([\widetilde{x} + Z_n(\widetilde{H}, \widetilde{L}), \widetilde{y} + Z_n(\widetilde{H}, \widetilde{L})]) &= \alpha([\widetilde{x}, \widetilde{y}] + Z_n(\widetilde{H}, \widetilde{L})) \\ &= \widehat{[x, y]} + Z_n(\widehat{H}, \widehat{L}) \\ &= [\widehat{x}, \widehat{y}] + Z_n(\widehat{H}, \widehat{L}) \\ &= [\widehat{x} + Z_n(\widehat{H}, \widehat{L}), \widehat{y} + Z_n(\widehat{H}, \widehat{L})] \\ &= [\alpha(\widetilde{x} + Z_n(\widetilde{H}, \widetilde{L})), \alpha(\widetilde{y} + Z_n(\widetilde{H}, \widetilde{L}))]. \end{aligned}$$

The surjectivity of α is obvious and hence it is an isomorphism. Thus the restriction of α to $\frac{\widetilde{H}}{Z_n(\widetilde{H}, \widetilde{L})}$ onto $\frac{\widehat{H}}{Z_n(\widehat{H}, \widehat{L})}$ is also an isomorphism. Now, consider $\beta : [\widetilde{H},_n \widetilde{L}] \longrightarrow [\widehat{H},_n \widehat{L}]$ given by $\beta([\widetilde{h}, \widetilde{l}]) = [\widehat{h}, \widehat{l}]$, for all $\widetilde{h} \in \widetilde{H}$, $\widetilde{l} \in \widetilde{L}^n$. Clearly, β is a Lie algebra epimorphism and it is one-to-one, since for every $h \in H$ and $l_i \in L$, $i = 1 \ldots n$, $\beta([\widetilde{h}, \widetilde{l_1} \ldots \widetilde{l_n}]) = 0$ implies that $[\widehat{h}, \widehat{l_1} \ldots \widehat{l_n}] = \widehat{0}$, which means that $[h, l_1, \ldots l_n] \in M \cap [H,_n L]$. Therefore $[\widetilde{h}, \widetilde{l_1} \ldots \widetilde{l_n}] = \widetilde{0}$ and hence β is an isomorphism. To complete the proof, it is enough to show that the following diagram is commutative.

$$\begin{array}{ccc} \frac{\widetilde{H}}{Z_n(\widetilde{H},\widetilde{L})} \oplus \frac{\widetilde{L}}{Z_n(\widetilde{H},\widetilde{L})} \oplus \dots \oplus \frac{\widetilde{L}}{Z_n(\widetilde{H},\widetilde{L})} & \overset{\alpha^{n+1}}{\rightarrow} & \frac{\widehat{H}}{Z_n(\widehat{H},\widehat{L})} \oplus \frac{\widehat{L}}{Z_n(\widehat{H},\widehat{L})} \oplus \dots \oplus \frac{\widehat{L}}{Z_n(\widehat{H},\widehat{L})} \\ \downarrow & & \downarrow \\ [\widetilde{H},_n \widetilde{L}] & \overset{\beta}{\rightarrow} & [\widehat{H},_n \widehat{L}]. \end{array}$$

This property follows, by using the definition of isomorphisms α and β. □

The following corollary is an immediate consequence of the above theorem.

Corollary 2.10. *Let L be a Lie algebra with an ideal M and a subalgebra H containing M. If $M \cap [H,_n L] = 0$, for all $n \geq 1$, then*

$$(\frac{H}{M}, \frac{L}{M}) \sim_n (H, L).$$

Conversely, if the subalgebra $[H,_n L]$ is of finite dimension and $(\frac{H}{M}, \frac{L}{M}) \sim_n (H, L)$, then $M \cap [H,_n L] = 0$.

3. The properties of relative n-stem pair of Lie algebras. The Lie algebra S is said to be a stem Lie algebra, if $Z(S) \subseteq [S, S]$. In [9], it is shown that every isoclinism family of Lie algebras contains at least one stem Lie algebra. In finite dimensional case, every stem Lie algebra is of the least dimension in the isoclinism family. It is defined in [8] that the pair of Lie algebras (R, S) is a stem pair, when $Z(R, S) \subseteq [R, S]$. It is also shown that each isoclinism pair contains at least one stem pair. In fact, if (R, S) is a pair of finite dimensional Lie algebras in the isoclinism family $\mathcal{C}$ say, then it is a stem pair if and only if $dim(S) = min\{dim(L) \ |(M, L) \in \mathcal{C}\}$. Also, a Lie algebra S is called an n-stem if $Z(S) \subseteq S^{n+1}$ (see [10]) and it is shown that each n-isoclinism family of Lie algebras contains an n-stem Lie algebra. In particular, if $\mathcal{C}$ is a family of n-isoclinism of finitely generated Lie algebras such that $dim(L^{n+1})$ is finite, for all $L \in \mathcal{C}$, then $\mathcal{C}$ contains an n-stem Lie algebra S say, so that $dim(S^n) = min\{dim(L^n) \ |L \in \mathcal{C}\}$.

The following definition is vital in our investigation.

Definition 3.1. The pair of Lie algebras (R, S) is called an n-stem pair, for $n \geq 1$, when $Z(R, S) \subseteq [R,_n S]$.

Clearly, if $n = 1$ then we obtain the notion as defined in [8] and when $R = S$ one has the concept in [10].

The following lemma gives a criterion for a pair of Lie algebras to be n-stem, for $n \geq 1$.

Lemma 3.2. *The pair of Lie algebras (R, S) is an n-stem pair if and only if S contains no ideal M say, for which $M \subseteq R$ with $M \cap [R,_n S] = 0$.*

Proof. We proceed by the way of contradiction and assume that the n-stem pair of Lie algebras (R, S) does not satisfy the conclusion. Hence, S contains an ideal M such that $M \subseteq R$ and $M \cap [R,_n S] = 0$. Then $M \subseteq Z_n(R,S)$, since for all $m \in M$ and $s_1 \dots s_n \in S$, we have $[m, s_1, \dots s_n] \in M \cap [R,_n S] = 0$. This implies that $m \in Z_n(S)$ and hence $m \in Z_n(S) \cap R = Z_n(R,S)$. Now, we claim that $M \cap Z_n(R,S) = 0$. To prove this, we use induction on n. For $n = 1$, we have $M \cap Z(R,S) \subseteq M \cap [R,_n S] = 0$. Assume $M \cap Z_{n-1}(R,S) = 0$. If $M \cap Z_n(R,S) \neq 0$, then there exists $m \in M$ such that for all $s_1, \dots s_n \in S$, $[m, s_1, s_2, \dots, s_n] = [[m, s_1], s_2, \dots, s_n] = 0$, so that $[m, s_1] \in M \cap Z_{n-1}(R,S)$. Now, by induction hypothesis $[m, s_1] = 0$ and hence $m \in M \cap Z(R,S) = 0$. Therefore $M = 0$, and so S has no non-trivial ideasl satisfying our condition.

Conversely, let R and S satisfy the assumption, and assume $Z(R,S) \nsubseteq [R,_n S]$. Consider $M = \langle x \rangle$, for $x \in Z(R,S) \backslash [R,_n S]$, then by the hypothesis $M \cap Z_n(R,S) \neq 0$, which gives the proof. □

THEOREM 3.3. *Let (M,L) be a pair of Lie algebras. Then the class of relative n-isoclinism pairs $\{(M,L)\}$ contains at least one n-stem pair.*

Proof. Consider the family of ideals $\mathcal{A} = \{N \mid N \trianglelefteq L;\ N \subseteq M,\ N \cap [M,_n L] = 0\}$. Clearly, $\mathcal{A}$ is non-empty, since $0 \in \mathcal{A}$. One notes that $\mathcal{A}$ is a partial ordered set by inclusion and hence using Zorn's lemma, it has a maximal member N, say. So by the definition, $N \cap [M,_n L] = 0$ and hence using Corollary 2.10, we have $(\frac{M}{N}, \frac{L}{N}) \sim_n (M,L)$. Now, let $\frac{K}{N} \trianglelefteq \frac{L}{N}$ such that $\frac{K}{N} \subseteq \frac{M}{N}$ and $\frac{K}{N} \cap [\frac{M}{N},_n \frac{L}{N}] = 0$. Thus $K \cap [M,_n L] \subseteq N$, so that $K \cap [M,_n L] \subseteq [M,_n L]$. This implies that $K \cap [M,_n L] \subseteq N \cap [M,_n L] = 0$. Therefore, $K \in \mathcal{A}$ so that $N \subseteq K$. By the maximality of N in $\mathcal{A}$, we have $K = N$ and hence the pair $(\frac{M}{N}, \frac{L}{N})$ satisfies the conditions of Lemma 3.2, and hence $(\frac{M}{N}, \frac{L}{N})$ is an n-stem pair in the relative n-isoclinism class of $\{(M,L)\}$. □

If $\mathcal{C}$ is a relative n-isoclinism family of Lie algebras pairs (M,L), such that $dim(L)$ and $dim([M,_n L])$ are finite, then we show that $\mathcal{C}$ contains at least one n-stem pair (R,S) say, such that

$$dim([R,_{n-1} L]) = min\{dim([M,_{n-1} L]);\ \ (M,L) \in \mathcal{C}\}.$$

To prove our main theorem in this section, we need the following lemmas.

LEMMA 3.4. *Let $l_1, l_2 \dots l_n$ be some elements of a Lie algebra L. Then for each $x \in L$, the commutator $[l_1, l_2, \dots, l_n, x]$ can be written as a finite sum of the form $\pm[x, l_{i_1}, l_{i_2} \dots l_{i_n}]$, where $l_{i_j} \in \{l_1, l_2, \dots, l_n\}$.*

Proof. The proof follows from Lemma 3.6 of [10]. □

LEMMA 3.5. *Let (M,L) be a pair of Lie algebras with $\frac{[M,_{n-1}L]}{Z(M,L) \cap [M,_{n-1}L]}$ to be of finite dimension, then the dimension of $[M,_n L]$ is also finite. The converse is true, when L is finitely generated.*

Proof. Put $A = Z(M,L) \cap [M,_{n-1} L]$ and take $\{x_1 + A, \ldots, x_k + A\}$ to be a basis for $\frac{[M,_{n-1}L]}{A}$, where $x_i = [m_i, l_{i_1}, \ldots, l_{i_{n-1}}] \in [M,_{n-1} L]$. Assume $y = [m, l_1, \ldots, l_n]$ is a generator of $[M,_n L]$, and hence $\eta = [m, l_1 \ldots l_{n-1}] \in [M,_{n-1} L]$ thus $\eta = \sum_{i=1}^{k} \lambda_i [m_i, l_{i_1}, \ldots, l_{i_{n-1}}] + a$, $a \in A$. Now by Lemma 3.4, we have

$$y = [\eta, l_n] = \sum_{i=1}^{k} \lambda_i [m_i, l_{i_1}, \ldots, l_{i_{n-1}}, l_n] = \sum_{j=1}^{k} \pm\lambda_j [[m'_j, l_n], l'_{j_1}, \ldots, l'_{j_{n-1}}],$$

where $m'_j \in M$ and $l'_{j_k} \in \{l_{i_1}, \ldots, l_{i_{n-1}}\}$. Thus $[M,_n L]$ is finite dimension.

Conversely, let $\{x_1, \ldots, x_t\}$ be a finite set of generators of L. Then define the map $\eta_i : C_L([M,_n L]) \cap [M,_{n-1} L] \to [M,_n L]$ given by $\eta_i(x) = [x, x_i]$, for all $x \in C_L([M,_n L]) \cap [M,_{n-1} L]$ and $1 \leq i \leq t$. Clearly, η_i is a Lie algebra homomorphism and

$$\begin{aligned} Ker(\eta_i) &= \{x \in C_L([M,_n L]) \cap [M,_{n-1} L];\ \ \eta_i(x) = [x, x_i] = 0\} \\ &= C_L(x_i) \cap C_L([M,_n L]) \cap [M,_{n-1} L]. \end{aligned}$$

Therefore, $D_i = \frac{C_L([M,_n L]) \cap [M,_{n-1} L]}{ker(\eta_i)}$ is isomorphic with a subalgebra of $[M,_n L]$ and hence it is of finite dimension. Now, since M is an ideal of L, the following equalities are held,

$$\begin{aligned} \cap_{i=1}^{t} Ker(\eta_i) &= \cap_{i=1}^{t} C_L(x_i) \cap C_L([M,_n L]) \cap [M,_{n-1} L] \\ &= Z(L) \cap C_L([M,_n L]) \cap [M,_{n-1} L] \\ &= Z(L) \cap [M,_{n-1} L] \cap M \\ &= Z(M,L) \cap [M,_{n-1} L]. \end{aligned}$$

Thus $C_1 = \frac{C_L([M,_n L]) \cap [M,_{n-1} L]}{Z(M,L) \cap [M,_{n-1} L]} \leq \oplus_{i=1}^{t} D_i$, and hence it is of finite dimension. Now, define the map $f : [M,_{n-1} L] \longrightarrow Der[M,_n L]$ given by $f(x) = adx$, which is a homomorphism and

$$Ker(f) = \{x \in [M,_{n-1} L];\ \ adx(y) = [x, y] = 0,\ \forall y \in [M,_n L]\}.$$

Clearly, $Ker(f) \subseteq C_L([M,_n L])$. Thus $Ker(f) = C_L([M,_n L]) \cap [M,_{n-1} L]$ and so $C_2 = \frac{[M,_{n-1} L]}{C_L([M,_n L]) \cap [M,_{n-1} L]}$ is also of finite dimension. Now, the result follows as $\frac{C_2}{C_1} \cong \frac{[M,_{n-1} L]}{Z(M,L) \cap [M,_{n-1} L]}$. □

THEOREM 3.6. *For every pair of Lie algebras* (M_1, L_1), *there exists a pair* (M_2, L_2) *such that* $(M_1, L_1) \sim_n (M_2, L_2)$ *and*

1. $Z(M_2, L_2) \cap [M_2,_{n-1} L_2] \subseteq [M_2,_n L_2]$;
2. *If* $\frac{[M_1,_{n-1} L_1]}{Z(M_1, L_1) \cap [M_1,_{n-1} L_1]}$ *is of finite dimension, then so is* $[M_2,_{n-1} L_2]$;
3. *If* L_1 *is finitely generated, then so is* L_2.

Proof. (1) Let T be the complement of $Z(M_1, L_1) \cap [M_{1,n} L_1]$ in $Z(M_1, L_1) \cap [M_{1,n-1} L_1]$. Then it is clear that T is an ideal of L_1 and $T \cap [M_{1,n} L_1] = 0$. Now, put $L_2 = \frac{L_1}{T}$ and $M_2 = \frac{M_1}{T}$, then $(M_1, L_1) \sim_n (M_2, L_2)$ by Corollary 2.10. Let $\bar{l} = l + T \in Z(M_2, L_2) \cap [M_{2,n-1} L_2]$, for some $l \in [M_{1,n-1} L_1]$. It is clear that for every $l_1 \in L_1$, $[l, l_1] \in [M_{1,n} L_1]$ and $[l, l_1] \in T$, since $l + T \in Z(M_2, L_2)$ and $l_1 + T \in L_2$. Therefore $[l, l_1] \in T \cap [M_{1,n} L_1] = 0$ and hence $l \in M_1 \cap Z(L_1) = Z(M_1, L_1)$. Now, by the above discussion

$$l \in [M_{1,n-1} L_1] \cap Z(M_1, L_1) = [M_{1,n} L_1] \cap Z(M_1, L_1) + T.$$

This implies that $l + T \in [M_{1,n} L_1] \cap Z(M_1, L_1) + T \subseteq [M_{2,n} L_2]$.

(2) If $A_1 = \frac{[M_{1,n-1} L_1]}{Z(M_1,L_1) \cap [M_{1,n-1} L_1]}$ is finite dimension then by Lemma 3.5, $[M_{1,n} L_1]$ is also of finite dimension. On the other hand, $A_2 = \frac{Z(M_1,L_1) \cap [M_{1,n-1} L_1]}{T}$ is isomorphic with a subalgebra of $[M_{1,n} L_1]$, which implies that A_2 is a finite dimension. Therefore

$$\frac{A_1}{A_2} \cong \frac{[M_{1,n-1} L_1]}{T} \cong [M_{2,n-1} L_2]$$

is also of finite dimension.

(3) It is clear, as $L_2 = \frac{L_1}{T}$. □

THEOREM 3.7. *Let (M_1, L_1) be a pair of Lie algebras such that $dim([M_{1,n-1} L_1])$ is finite. Then $Z(M_1, L_1) \cap [M_{1,n-1} L_1]$ is a subalgebra of $[M_{1,n} L_1]$ if and only if, for every pair of Lie algebras (M_2, L_2) so that $(M_1, L_1) \sim_n (M_2, L_2)$, then $dim([M_{1,n-1} L_1]) \leq dim([M_{2,n-1} L_2])$.*

Proof. First suppose that $Z(M_1, L_1) \cap [M_{1,n-1} L_1]$ is a subalgebra of $[M_{1,n} L_1]$, then consider the pair (M_2, L_2) for which $(M_1, L_1) \sim_n (M_2, L_2)$. Using Corollary 2.4, we have

$$(\frac{M_1}{Z(M_1, L_1)}, \frac{L_1}{Z(M_1, L_1)}) \sim_{n-1} (\frac{M_2}{Z(M_2, L_2)}, \frac{L_2}{Z(M_2, L_2)}).$$

Put $\widehat{L_i} = \frac{L_i}{Z(M_i,L_i)}$ and $\widehat{M_i} = \frac{M_i}{Z(M_i,L_i)}$, $i = 1, 2$, then

$$dim([\widehat{M}_{1,n-1} \widehat{L_1}]) = dim([\widehat{M}_{2,n-1} \widehat{L_2}]).$$

Therefore

$$dim([M_{1,n-1} L_1]) - dim(Z(M_1, L_1) \cap [M_{1,n-1} L_1]) =$$
$$dim([M_{2,n-1} L_2]) - dim(Z(M_2, L_2) \cap [M_{2,n-1} L_2]),$$

which gives

$$dim([M_{1,n-1} L_1]) = dim([M_{2,n-1} L_2]) + dim(Z(M_1, L_1) \cap [M_{1,n-1} L_1]) -$$
$$dim(Z(M_2, L_2) \cap [M_{2,n-1} L_2]) \geq$$
$$dim([M_{2,n-1} L_2]) + dim(Z(M_1, L_1) \cap [M_{1,n-1} L_1]) - dim(Z(M_2, L_2) \cap [M_{2,n} L_2]).$$

The inequality follows, by using Lemma 2.1(3).

Conversely, by Theorem 3.6, there exists a pair of Lie algebras such that $(H,K) \sim_n (M_1,L_1)$ and

$$Z(H,K) \cap [H,_{n-1} K] \subseteq [H,_n K].$$

Now, by similar discussion as above we have

$$dim([M_1,_{n-1} L_1]) \geq dim([H,_{n-1} K]) + dim(Z(M_1,L_1) \cap [M_1,_{n-1} L_1]) - dim(Z(M_1,L_1) \cap [M_1,_n L_1]).$$

Using the assumption

$$dim(Z(M_1,L_1) \cap [M_1,_n L_1]) \geq dim(Z(M_1,L_1) \cap [M_1,_{n-1} L_1]).$$

Therefore

$$dim(Z(M_1,L_1) \cap [M_1,_n L_1]) = dim(Z(M_1,L_1) \cap [M_1,_{n-1} L_1]),$$

and hence

$$Z(M_1,L_1) \cap [M_1,_{n-1} L_1] \subseteq [M_1,_n L_1].$$

This completes the proof. □

Now, we are able to prove our main result in this section.

Theorem 3.8. *Let $\mathcal{C}$ be a relative n-isoclinism family of Lie algebras. If (M,L) is a pair of Lie algebras in $\mathcal{C}$ such that L is finitely generated and $[M,_n L]$ is of finite dimension, then there exists an n-stem pair of Lie algebras $(R,S) \in \mathcal{C}$ such that $dim([R,_{n-1} S] = min\{dim([M,_{n-1} L]);\ \ for\ all\ (M,L) \in \mathcal{C}\}$.*

Proof. Lemma 3.5 gives that $\frac{[M,_{n-1}L]}{Z(M,L)\cap[M,_{n-1}L]}$ is finite dimension. Using Theorem 3.6, there exists a pair (R,S) such that

$$Z(R,S) \cap [R,_{n-1} S] \subseteq [R,_n S],$$

and hence $[R,_{n-1} S]$ is of finite dimension. By Theorem 3.7 and the above inclusion, we obtain

$$dim([R,_{n-1} S]) \leq dim([M,_{n-1} L]),$$

which is to say that

$$dim([R,_{n-1} S] = min\{dim([M,_{n-1} L]);\ \ (M,L) \in \mathcal{C}\}.\ \ (*)$$

To complete the prove it is enough to show that any pair of Lie algebras with the above property is an n-stem pair. We proceed by induction, for $n = 1$ we have $dim(R) = \min\{dim(M);\ \ (M,L) \in \mathcal{C}\}$. To prove our goal, suppose that J is a complement of $Z(R,S) \cap [R,S]$ in $Z(R,S)$. It is clear that $J \cap [R,S] = 0$ and by Corollary 2.10, $(\frac{R}{J},\frac{S}{J}) \sim (R,S)$. Now, by the minimality of $dim(R)$, we have

$J = 0$. Therefore $Z(R,S) \subseteq [R,S]$, and hence (R,S) is a 1-stem pair. For (M,L), assume there exists an $(n-1)$-stem pair (R,S) such that $(R,S) \sim_{n-1} (M,L)$. By Corollary 2.3, we have $(R,S) \sim_n (M,L)$ and using (*) and Theorem 3.7,

$$Z(R,S) = Z(R,S) \cap [R,_{n-1} S] \subseteq [R,_n S].$$

Therefore, (R,S) is an n-stem pair and the proof is complete. □

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Received 2 August, 2012.