Perfect Lattice Paths in the Plane

D. Yaqubi and A. Jafarzadeh

Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran
E-mail: daniel_yaqubi@yahoo.es, jafarzadeh@um.ac.ir

Abstract

Consider an \( m \times n \) table \( T \) and lattices paths \( \nu_1, \ldots, \nu_k \) in \( T \) such that each step \( \nu_{i+1} - \nu_i = (1, 1), (1, 0) \) or \( (1, -1) \). The number of paths from the \((1, i)\)-cell (resp. first column) to the \((s, t)\)-cell is denoted by \( D^1(s, t) \) (resp. \( D(s, t) \)). Also, the number of all paths form the first column to the last column is denoted by \( T_m(n) \). We give explicit formulas for the numbers \( D^1(s, t) \) and \( D(s, t) \).

Keywords: Lattice path, Dyck paths, Perfect lattice paths, Fibonacci numbers, Pell-Lucas numbers, Motzkin numbers.
Mathematics Subject Classification (2010): 05A15; 11B37; 11B39.

1 Introduction

A lattice path in \( \mathbb{Z}^2 \) is the drawing in \( \mathbb{Z}^2 \) of a sum of vectors from a fixed finite subset \( S \) of \( \mathbb{Z}^2 \), starting from a given point, say \((0, 0)\) of \( \mathbb{Z}^2 \). A typical problem in lattice paths is the enumeration of all \( S \)-lattice paths (lattice paths with respect to the set \( S \)) with a given initial and terminal point satisfying possibly some further constraints. A nontrivial simple case is the problem of finding the number of lattice paths starting from the origin \((0, 0)\) and ending at a point \((m, n)\) using only right step \((1, 0)\) and up step \((0, 1)\) (i.e., \( S = \{(1, 0), (0, 1)\} \)). The number of such paths is known to be the binomial coefficient \( \binom{m+n}{n} \). Yet another example, known as the ballot problem, is to find the number of lattice paths from \((1, 0)\) to \((m, n)\) with \( m > n \), using the same steps as above, that never touch the line \( y = x \). The number of such paths, known as ballot number, equals \( \frac{m-n}{m+n} \binom{m+n}{n} \). In the special case where \( m = n + 1 \), the ballot number is indeed the Catalan number \( C_n \).

Let \( T = T_{m,n} \) denote the \( m \times n \) table in the plane and \((x, y)\) be the cell in the columns \( x \) and row \( y \) (and refer to it as the \((x, y)\)-cell). The set of lattice paths from the \((i, j)\)-cell to the \((s, t)\)-cell, with steps belonging to a finite set \( S \), is denoted by \( L(i, j; s, t : S) \), and the number of those paths is denoted by \( l(i, j; s, t : S) \), where \( 1 \leq i, s \leq m \) and \( 1 \leq j, t \leq n \).
Throughout this paper, we set $S = \{(1, 1), (1, 0), (1, -1)\}$, and the corresponding lattice paths starting from the first column and ending at the last column are called perfect lattice paths. The number of all perfect lattice paths is denoted by $\mathcal{I}_m(n)$, that is,

$$\mathcal{I}_m(n) = \sum_{i,j=1}^{m} l(1, i; n, j; S).$$

Sometimes it is more convenient to name each step of lattice paths by a letter, and hence every lattice path will be encoded as a lattice word. We label the steps of the set $S = \{(1, 1), (1, 0), (1, -1)\}$ by letters $u = (1, 1)$, $x = (1, 0)$, and $d = (1, -1)$; also if $h$ is a letter of the word $W$, order or size of $h$ in $W$ is the number of times the letter $h$ appears in the word $W$ and it is denoted by $|h| = |h|_W$.

2 Main Results

Clearly, $\mathcal{I}_m(n)$ is the number of words $a_1 a_2 \ldots a_{n-1} a_n$ ($a_i \in \{1, \ldots, m\}$) such that $|a_{i+1} - a_i| \leq 1$ for all $i = 1, \ldots, n - 1$. In this section, we give formulas for the number $\mathcal{I}_m(n)$ in the cases where $n + 1 \leq m \leq 2n$ and $2n \leq m$. To achieve this goal, we must recall some further notations from [1]. The number of lattice paths from the $(1, i)$-cell to the $(s, t)$-cell is denoted by $\mathcal{D}^i(s, t)$. Indeed, $\mathcal{D}^i(s, t) = l(1, i; s, t : S)$. Also, the number of lattice paths from the first column to the $(s, t)$-cell is denoted by $\mathcal{D}_{m,n}(s, t)$, or $\mathcal{D}(s, t)$ if there is no confusion, that is,

$$\mathcal{D}_{m,n}(s, t) = \sum_{i=1}^{m} \mathcal{D}^i(s, t).$$

In what follows, the number of lattice paths from $(1, 1)$-cell to $(s, t)$-cell ($1 \leq s \leq n$ and $1 \leq t \leq m$), using just the two steps $(1, 1)$ and $(1, -1)$, is denoted by $\mathcal{A}(s, t)$. In other words, $\mathcal{A}(s, t) = l(1, 1; s, t : S')$, where $S' = \{(1, 1), (1, -1)\}$. Clearly $\mathcal{A}(s, t) = 0$ for $s \leq t$, and that $\mathcal{A}(s, t)$ is the number lattice paths from the $(1, 1)$-cell to $(s, t)$-cell not sliding above the line $y = x$. One observe that $\mathcal{A}(s, t) = 0$ if $s$, $t$ have distinct parities as the paths counted by $\mathcal{A}(s, t)$ begins from $(1, 1)$ and every step in $S'$ keeps the parities of entries so that such paths never meet $(s, t)$-cells with $(s, t)$ having distinct parities. Using the symbols $u$ and $d$, the number $\mathcal{A}(s, t)$ counts the words of length $s - 1$ on $\{u, d\}$ whose all initial subwords have more or equal $u$ than $d$. Analogous to $\mathcal{A}(s, t)$, the number $\mathcal{D}^i(s, t)$ counts the words $a_1 a_2 \ldots a_i$ with $1 \leq a_i \leq t$ such that $|a_{i+1} - a_i| \leq 1$ for all $1 \leq i \leq s - 1$. In other words, $\mathcal{D}^i(s, t)$ counts the number of words of length $s - 1$ on $\{u, d\}$ whose all initial subwords have more or equal $u$ than $d$.

Theorem 2.1. For all $1 \leq s, t \leq m$, we have

$$\mathcal{D}^i(s, t) = \sum_{i=0}^{\lfloor \frac{s-1}{2} \rfloor} \left( \binom{s-1}{s-t-2i} \right) \mathcal{A}(t + 2i, t).$$

Example 2.2. Using theorem 2.1, we can compute $\mathcal{D}^1(8, 4)$ as

$$\mathcal{D}^1(8, 4) = \sum_{i=0}^{\lfloor \frac{8-1}{2} \rfloor} \left( \binom{8-1}{8-4-2i} \right) \mathcal{A}(2i + 4, 4) = 133.$$
The numbers $A(s, t)$ are indeed computed as in the ballot problem were the paths can touch the $y = x$ line but never go above it. The number of such ballot paths from $(1, 0)$ to $(m, n)$ is \( \frac{m-n+1}{m+1} \binom{m+n}{m} \). Recall that $A(s, t)$ is the number of words $\mathcal{W}$ of length $s-1$ on $\{u, d\}$ with more or equal $u$ than $d$ in any initial subword, hence $A(s, t)$ is equal to the above number with $m := |u|_W$ and $n := |d|_W$. Now since $|u|_W + |d|_W = s - 1$ and $|u|_W - |d|_W = t - 1$, it follows that $m = (s + t)/2 - 1$ and $n = (s - t)/2$. Hence we obtain the following

**Lemma 2.3.** Inside the $n \times n$ table, we have

\[
A(s, t) = \frac{2t}{s + t} \left( \frac{s - 1}{s + t} \right).
\]

Hence

\[
D^1(s, t) = \sum_{i=0}^{\left\lfloor \frac{s-1}{2} \right\rfloor} \frac{t}{t + i} \binom{s - 1}{s - t - 2i} \binom{t + 2i - 1}{i}.
\]

for all $1 \leq s, t \leq n$.

In [1], we have computed the number $I_m(n)$ for all $m \geq 1$. In what follows, we shall give formulas for $I_m(n)$, where $n + 1 \leq m \leq 2n$. To achieve this goal, we use the numbers $\mathcal{H}(s, t)$ inside the $m \times n$ table defined as $\mathcal{H}(s, t) = \sum_{i=1}^{t} D^1(s, i)$, where $1 \leq s \leq n$ and $1 \leq t \leq m$.

**Lemma 2.4.** Inside the $m \times n$ table with $m \leq n \leq 2m$, we have

\[
\mathcal{H}(n, m) = D(n, n) - \sum_{i=m}^{n-1} 3^{n-i-1} D^1(i, m).
\]

**Example 2.5.** Using Lemma 2.4, we can calculate $\mathcal{H}(9, 5)$ as

\[
\mathcal{H}(9, 5) = D(9, 9) - \sum_{i=5}^{8} 3^{8-i} D^1(i, 5) = 1931.
\]

**Lemma 2.6.** Inside the $m \times n$ table, we have

\[
D^1(n, m) = \sum_{i=1}^{m} D^1(s, i) \times D^1(n - s + 1, m - i + 1).
\]

for all $1 \leq s \leq n$.

**Example 2.7.** Lemma 2.6 gives a way to compute $D^1(9, 5)$ in the following:

\[
D^1(9, 5) = \sum_{i=1}^{5} D^1(5, i) D^1(9 - 5 + 1, 5 - i + 1) = 195.
\]

**Lemma 2.8.** Inside the $m \times n$ table, we have

\[
D(s, t) = 3^{s-1} - \sum_{i=1}^{s-1} 3^{s-i-1} D^1(i, t) - \sum_{i=m+2-t}^{s-1} 3^{s-i-1} D^1(i, m + 1 - t)
\]

for all $s \leq n + 2$. 
Theorem 2.9. Inside the $m \times n$ table, we have

$$I_m(n) = \sum_{i=1}^{m} D(a,i)D(b,i)$$

for all $a, b \geq 1$ such that $a + b = n + 1$. In other words, the inner product of columns $a$ and $b$ equals $I_m(n)$. In particular, if $n = 2k - 1$ is odd, then

$$I_m(n) = \sum_{i=1}^{m} D_k^2.$$

Now, we calculate $S(x,y)$, the number of all perfect lattice paths from $(1,1)$-cell to $(x,y)$-cell in the whole space (not restricted to a table):

Theorem 2.10. The number $S(x,y)$ is given by

$$S(x,y) = \sum_{r=0}^{x-1} \binom{x-1}{r} \binom{x-r-1}{\frac{x-y-r}{2}} = \sum_{d=0}^{\left\lfloor \frac{x-y}{2} \right\rfloor} \binom{x-1}{d} (x-d-1) (x-y-2d).$$

Let $T = T_{m,n}$ and $S_{(a,b)}(x,y)$ denote the number of all perfect lattice path from $(a,b)$-cell to $(x,y)$-cell without leaving the table $T$. As before one can compute $S_{(a,b)}(x,y)$ by subtracting the number of all those paths starting from $(a,b)$-cell and ending at $(x,y)$-cell and leave the table from the total number of such paths. We have

Theorem 2.11. Inside the $m \times n$ table, for all $a \leq x$ and $b \leq y$, we have

$$S_{a,b}(x,y) = S(x-a+1,y-b+1) - \sum_{x'=a+b}^{x-y} D^1(x' - a, b) S(x-x' + 1, y + 1)$$

$$- \sum_{x'=m-a-b+1}^{x+y-m-1} D^1(x' - a, m-b+1) S(x-x' + 1, m-y).$$

Example 2.12. Utilizing Theorems 2.10 and 2.11, we see that, inside the $8 \times 8$ table,

$$S_{(2,1)}(7,3) = S(6,3) = 25.$$

References

Certificate

To whom it may concern:

This letter certifies that A. Jafarzadeh attended "10th Graph Theory and Algebraic Combinatorics Conference (GTACC10)" that was held on 17 and 18 January 2018 at Yazd University, Yazd, Iran, and gave a presentation (joint with D. Yaqubi) entitled "Perfect Lattice Paths in the Plane" at the conference.

Best regards,
Saeid Alikhani
Chair of Conference

Department of Mathematics, Faculty of Science, Yazd University, Yazd, Iran
Phone: +98 353-123-2715
confs.yazd.ac.ir/GTACC10