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A new and simple analytical approach to determining the natural frequencies of framed tube structures

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Abstract. This paper presents a new and simple solution for determining the natural frequencies of framed tube combined with shear-walls and tube-in-tube systems. The novelty of the presented approach is based on the bending moment function approximation instead of the mode shape function approximation. This novelty makes the presented solution very simpler and very shorter in the mathematical calculations process. The shear stiffness, flexural stiffness and mass per unit length of the structure are variable along the height. The effect of the structure weight on its natural frequencies is considered using a variable axial force. The effects of shear lag phenomena has been investigated on the natural frequencies of the structure. The whole structure is modeled by an equivalent non-prismatic shear-flexural cantilever beam under variable axial forces. The governing differential equation of motion is converted into a system of linear algebraic equations and the natural frequencies are calculated by determining a non-trivial solution for the system of equations. The accuracy of the proposed method is verified through several numerical examples and the results are compared with the literature.

Keywords: tall structure; batural frequency; shear-flexural deformation; axial force; weak form of integral equations; bending moment approximation; shear lag

1. Introduction

Choosing an appropriate lateral- resistant system is one of the most important parameters of an acceptable design of a tall structure. It should be noted that when buildings taller than a certain limit are to be constructed, common structural systems will no longer be suitable. This is because rigidity and stability criteria become more important than the strength criterion in the tall buildings (Malekinejad and Rahgozar 2012). Furthermore, the natural frequency of a tall structure is one of the most important parameters influencing the response of the lateral-resistant system to the earthquake excitation. Framed tube structures, as economical systems for high-rise buildings, act like a hollow boxed beam under lateral loads. The combination of the framed tube and other systems such as shear-walls is beneficial for reducing the shear lag effects on the structure. Many researchers have investigated the free vibration of tall structures through various approaches (Youlin 1984, Wang 1996a, Wang 1996b, Lashkari 1988, Wang 1989, Bozdogan 2009, Dym and Williams 2007, Lee 2007, Kwan 1994, Malekinejad and Rahgozar 2014, Rahgozar *et al*. 2011, Bozdogan and Ozturk 2009).

An analytical model for the dynamic analysis of tall buildings with a shear wall-frame structural system has been proposed (Park *et al.* 2014). It has been shown that the deformed shape of the shear wall-frame structural system is the combination of flexural mode and shear mode. A modified theory on the premise that a frame-wall system, deforming in shear and flexural modes, can be separated into two substructures that lie above and below the point of counter-flexure in the base story columns has been developed (Kazaz and Gülkan 2012). Mohammadnejad (2015) calculated the natural frequencies of flexural, axial and torsional vibration of the beams. He has converted the governing differential equations into corresponding weak form integral equations. A dynamic analysis of the combined system of framed tube and shear walls by Galerkin method using B-spline functions has been presented (Rahgozar *et al.* 2014). However, few studies have considered the effects of the tall structure weight on its vibrational characteristics. In a real tall structure, the stiffness and mass of the structure are variable along its height, with the weight of the structure being effective on its vibrational characteristics. Therefore, the modeling of tall structure by a cantilevered beam with variable stiffness and mass under effects of variable axial force caused by the structure weight may provide the realistic conditions for an accurate structural analysis. The first natural frequency of tall buildings with a combined system of framed tube, shear core, belt truss and an outrigger system with multiple jumped discontinuities in the cross section of the framed tube along with a shear core under axial force has been calculated (Kamgar and Saadatpour 2012). An analytical approach based on energy principles has been developed for computing the natural frequencies and mode shapes of multistory buildings constructed by framed tube, shear core

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and double belt trusses systems (Malekinejad and Rahgozar 2012, Malekinejad and Rahgozar 2013). The fundamental frequency of tall buildings that consist of framed tube, shear core, belt truss and outrigger systems in which the framed tube and shear core vary in size along the height of the structure has been determined (Jahanshahi and Rahgozar 2012). The free vibration analysis of asymmetric structures with shear wall-frames system using the modified finite element-transfer matrix methods have been proposed (Bozdogan 2013). Using the DQM method, the governing differential equation for free vibration of coupled shear walls has been solved (Bozdogan 2012). The effect of shear lag on braced tube and framed tube under wind load has been investigated (Mazinani *et al.* 2014). Free vibration analysis of tall structures with various lateral resisting systems and variable properties along the height has been investigated (Saffari and Mohammadnejad 2015). They have calculated the natural frequencies of tall structures using weak form integral equations. The weak form integral equations has been developed for free vibration analysis of non-prismatic beams (Mohammadnejad *et al.* 2014, Saffari *et al*. 2012).

2. Novelty of the presented method

In this paper, using the continuum approach, a framed tube combined with shear walls and tube-in-tube structures are replaced by an equivalent cantilever beam that has variable shear- flexural stiffness and mass along the height. Also, the weight of the structure is considered using a variable axial force. The governing differential equation of the motion is obtained and converted into a weak-form integral equation through repetitive integrations. The previous presented research works for conversion of the governing equation into its weak form is based on the mode shape function approximation which needs four times repetitive integration from the governing equation. But, novelty of the presented approach in this paper is the bending moment function approximation instead of the mode shape function which needs two repetitive integration. This novelty makes the solution very simpler and very shorter. By approximation of the bending moment function using a power series, the weak-form integral equation is converted into a system of linear algebraic equation. The natural frequencies of the structure are obtained by calculating a non-trivial solution for the resulting system of equations.

The dynamic response analysis of the tall buildings includes two general fields such as the finite elements approach and approximate analytical approach. The finite element approach is based on the discrete model and has to solve thousands of linear simultaneous equations to give quantitative results in detail. So it is a powerful tool for analysis and design at the detailed and final design stage of tall buildings. Presented method in this paper is an analytical approximate method that gives insight into characteristics of free vibration. It is simple and accurate enough that can be routinely used for the preliminary stage of building design. The advantages of both analytical and approximate methods of continuum modeling for tall building structures may not be replaced by the discrete modeling of finite element analysis.

In this paper, an equivalent beam is considered for the system of lateral resistance of the structure. The whole structure is idealized as a non-prismatic shear-flexural cantilever beam with hollow box cross-section under variable axial force due to the structure weight. Framed tube is replaced by a shear beam located at the center of shear rigidity and a flexural beam at the center of flexural rigidity (Kwan 1994). In addition, wall members are lumped and replaced by equivalent flexural and shear cantilever beams located at the flexure center and the shear rigidity of shear walls at the shear center (Malekinejad and Rahgozar 2014). Also, a tube-in-tube structure can be modeled as an assemblage of equivalent orthotropic plate panels. Consequently, a framed tube structure may be analyzed as a continuum (Kwan 1994, Malekinejad and Rahgozar 2014, Lee 2007). According to this modeling, the functions of shear stiffness, flexural stiffness, mass per unit length and axial force of the equivalent beam are calculated as follows:

$$\textit{for framed-tube with shear-wall}$$

$$\begin{cases} K_S(\mathrm{x}) = \left[K_S(\mathrm{x})\right]_F + \left[K_S(\mathrm{x})\right]_W \\ K_B(\mathrm{x}) = \left[K_B(\mathrm{x})\right]_F + \left[K_B(\mathrm{x})\right]_W \\ m(\mathrm{x}) = \left[m(\mathrm{x})\right]_F + \left[m(\mathrm{x})\right]_W \\ N(\mathrm{x}) = \int_x^H gm(x)dx \end{cases}$$

$$\textit{for tube-in-tube}$$

$$\begin{cases} K_S(\mathrm{x}) = \left[K_S(\mathrm{x})\right]_I + \left[K_S(\mathrm{x})\right]_O \\ K_B(\mathrm{x}) = \left[K_B(\mathrm{x})\right]_I + \left[K_B(\mathrm{x})\right]_O \\ m(\mathrm{x}) = \left[m(\mathrm{x})\right]_I + \left[m(\mathrm{x})\right]_O \\ N(\mathrm{x}) = \int_x^H gm(x)dx \end{cases}$$

Where g and H are gravity acceleration and structure height, and subscripts F, W, I and O denote the framed tube, shear wall, inner tube and outer tubes, respectively. Figs. 1 and 2 show the schematic modeling of both systems.

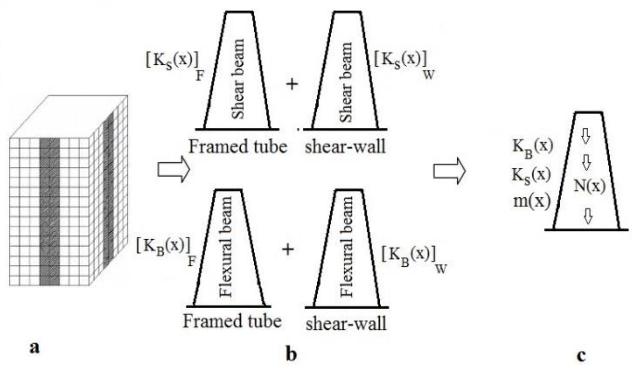

Fig. 1 Modeling of a tall structure, (a): combined system of the framed tube with shear-wall. (b): equivalent shear and flexural beams. (c): modeling of the whole structure by an equivalent beam with variable properties under variable axial forces

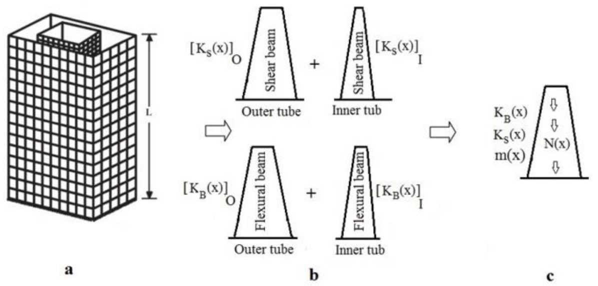

Fig. 2 Modeling of a tall structure, (a): a tube-in-tube tall structure, (b): equivalent shear and flexural beams, (c): modeling of the whole structure by an equivalent beam with variable properties under variable axial forces

3. Methodology: the weak form differential equations

The governing differential equation for free vibration of a beam with variable stiffness and mass is a partial differential equation with variable coefficients. Many mathematical techniques may be employed to determine the numerical solution or the approximate solution for this equation. The presented approach in this paper for conversion of the governing partial differential equation into solvable one is based on the conversion of the governing equation into its weak form. A differential equation includes a function and its derivatives. The weak form of the differential equation is obtained through the repetitive integration of the initial equation. The integration continues till the resulting integral equation, includes only the function itself after the last integration stage; derivatives of the function will have been eliminated due to the integration. The solution of the weak form of the differential equation instead of the initial equation has many applications in the finite elements analysis. Approximation of the weak form has less error in compared to the original form of differential equation (Reddy 1993).

4. Formulation and solution

4.1 Equivalent properties of the framed tube

Kwan (1994) proposed a model for analysis of framed tube structures. In his model, a number of assumptions are made in describing the framed tube system using equivalent orthotropic plates. Using assumptions proposed by Kwan (1994), the tall structure can be modeled as a cantilever beam with a variable cross section in height. It is normal practice to fix the value of Thickness of the membrane *t*such that the area of the membrane ($d.t$) is equal to the sectional area of the column (A_C).

$$t = \frac{A_C}{d} \tag{1}$$

In which d is center-to-center distance of the columns of the outer tube. Consider now the case of the frame unit subject to a lateral force Q. The lateral deflection may be computed as the sum of that due to bending Δ_b and due to shear Δ_S. The bending deflection Δ_b is given by

$$\frac{\Delta_b}{Q} = \frac{(h-H_b)^3}{12E_m I_C} + \left(\frac{h}{d}\right)^2 \frac{\left(d-H_c\right)^2}{12E_m I_b} \tag{2}$$

In which h, H_b, E_m, I_C, I_b, H_c are story height, height of beam, elastic modulus of the construction material, moments of inertia of the column, moments of inertia of the beam and height of the column, respectively. On the other hand, the shear deflection Δ_S is given by

$$\frac{\Delta_s}{Q} = \frac{(h-H_b)}{G_m A_{SC}} + \left(\frac{h}{d}\right)^2 \frac{\left(d-H_c\right)}{G_m A_{Sb}} \tag{3}$$

In which A_{Sb} and A_{SC} are effective shear areas of the beam and column, respectively, and G_m is shear modulus of the material. Equivalent shear modulus of the membrane is calculated as follows

$$G = \frac{\frac{h}{d.t}}{\frac{\Delta_b}{Q} + \frac{\Delta_s}{Q}} \tag{4}$$

Equivalent moment of inertia of the outer tube is calculated as follows

$$I_Z = \frac{1}{2} L_f L_W^2 t + \frac{1}{6} L_W^3 t \tag{5}$$

In which L_f, L_W are length of flange and web panels of the outer tube, respectively (Malekinejad and Rahgozar 2014).

4.2 Conversion of the governing differential equation into its weak form

Consider a tall structure with variable mass and stiffness along the height that is subjected to the action of the transverse loading, q, distributed along its height and axial force due to structure weight. Accounting for total potential energy of the system and applying Hamilton's principle, the governing equation of the motion for equivalent beam is given as follows (Kamgar and Saadatpour, 2012)

$$\begin{gathered} \frac{\partial^2}{\partial x^2}\left[k_B(x)\frac{\partial^2}{\partial x^2}\mathrm{W}(x,t)\right] - \frac{\partial}{\partial x}\left[k_S(x)\frac{\partial}{\partial x}\mathrm{W}(x,t)\right] - \\ \frac{\partial}{\partial x}\left[n(x)\frac{\partial}{\partial x}\mathrm{W}(x,t)\right] + m(x)\frac{\partial^2}{\partial t^2}\mathrm{W}(x,t) + \\ q(x,t) = 0, \\ 0 \le x \le H \end{gathered} \tag{6}$$

where $W(x,t)$, H, $m(x)$, $k_B(x)$, $k_S(x)$, $n(x)$ and $q(x,t)$ are the transverse displacement, structure height, mass per unit length, flexural stiffness, shear stiffness, axial force and lateral distributed applied load, respectively. For free vibration analysis, $q(x,t)$=0 is applied. By assumption of a harmonic vibration, the transverse displacement of the structure can be assumed as follows

$$\mathrm{W}\left(x,t\right) = w(x)e^{\mathrm{i}\Omega t} \tag{7}$$

Where $w(x)$ and $\Omega$are the mode shape function and natural frequency of the structure, respectively. By applying

$i^2=-1$ and substituting the relationship (7) into Eq. (6), a single-variable equation in terms of location is obtained as follows

$$\frac{d^2}{dx^2}\left[k_B(x)\frac{d^2}{dx^2}w(x)\right]-\frac{d}{dx}\left[k_S(x)\frac{d}{dx}w(x)\right]-\frac{d}{dx}\left[n(x)\frac{d}{dx}w(x)\right]-\Omega^2 m(x)w(x)=0,\quad 0\le x\le H \tag{8}$$

For further convenience, the following variables are introduced

$$\begin{cases}\xi=\dfrac{x}{H}\\ k_B(\xi)=EI_0K_B(\xi),\quad k_S(\xi)=GA_0K_S(\xi),\\ n(\xi)=N_0N(\xi),\quad m(\xi)=m_0\\ \beta^2=\dfrac{GA_0H^2}{EI_0},\quad \alpha^2=\dfrac{m_0\Omega^2H^4}{EI_0},\quad \gamma^2=\dfrac{N_0H^2}{EI_0}\end{cases} \tag{9}$$

In which parameters EI_0, GA_0, N_0 are: flexural stiffness, shear stiffness and axial force at the base of the structure. m_0 is average of the mass per unit length of the structure along the height. Also, β, α, γ are the non-dimensional parameters corresponding to stiffness, natural frequency and axial force of the structure, respectively. Substitution of the variables (9) into Eq. (8) results in the following differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[K_B(\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}w(\xi)\right]-\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\beta^2K_S(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}w(\xi)\right]-\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\gamma^2N(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}w(\xi)\right]-\alpha^2w(\xi)=0,\quad 0\le\xi\le 1 \tag{10}$$

Eq. (10) is, in fact, the free vibration equation of a tall structure based on the non-dimensional variable ξ. Previous presented solutions for conversion of the Eq. (10) into its weak form is based on the four times repetitive integration from both sides of this equation. But, it is the novelty of the presented approach in this paper that the weak form of the governing equation is obtained using two times repetitive integration. This novelty makes the solution very simpler and very shorter. For this purpose, the following relations are introduced

$$\frac{d^2w}{d\xi^2}=M(\xi),\quad \frac{dw}{d\xi}=\int_0^\xi M(s)ds,\quad w(\xi)=\int_0^\xi(\xi-s)M(s)ds \tag{11}$$

Substitution of the relations (11) into Eq. (10) results in the following differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[K_B(\xi)M(\xi)\right]-\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\beta^2K_S(\xi)\int_0^\xi M(s)ds\right]-\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\gamma^2N(\xi)\int_0^\xi M(s)ds\right]-\alpha^2\int_0^\xi(\xi-s)M(s)ds=0,\quad 0\le\xi\le 1 \tag{12}$$

In order to transform Eq. (12) into its weak form, both sides of this equation are integrated twice with respect to ξ within the range 0 to ξ. The results are the integral equations as follows

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_B(\xi)M(\xi)\right]-\beta^2K_S(\xi)\int_0^\xi M(s)ds-\gamma^2N(\xi)\int_0^\xi M(s)ds-\frac{\alpha^2}{2}\int_0^\xi(\xi-s)^2M(s)ds=C_1,\quad 0\le\xi\le 1 \tag{13}$$

$$K_B(\xi)M(\xi)+\int_0^\xi h_1(\xi,s)M(s)ds=C_1\xi+C_2 \tag{14}$$

In which

$$h_1(\xi,s)=\int_0^s\left[\beta^2K_S(s)+\gamma^2N(s)\right]ds-\int_0^\xi\left[\beta^2K_S(s)+\gamma^2N(s)\right]ds-\frac{\alpha^2}{6}(\xi-s)^3 \tag{15}$$

Eq. (14) is the weak form integral equation of Eq. (10). In Eq. (14), C_1, C_2 are the integration constants which are determined through boundary conditions of the tall building. Eqs. (13)-(14) are used to determine the integration constants. The following boundary conditions are introduced for the structure

$$\begin{cases}\xi=1\rightarrow K_B(\xi)\dfrac{d^2w}{d\xi^2}=0\rightarrow K_B(\xi)M(\xi)=0\\ \xi=1\rightarrow \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left[K_B(\xi)\dfrac{d^2w}{d\xi^2}\right]-\beta^2K_S(\xi)\dfrac{dw}{d\xi}-\gamma^2N(\xi)\dfrac{dw}{d\xi}=0\end{cases} \tag{16}$$

or

$$\left|\ \frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_B(\xi)M(\xi)\right]-\beta^2K_S(\xi)\int_0^\xi M(s)ds-\gamma^2N(\xi)\int_0^\xi M(s)ds=0\right. \tag{17}$$

Applying condition (16) to Eq. (14) and condition (17) to Eq. (13) yields

$$\begin{cases}-\dfrac{\alpha^2}{2}\displaystyle\int_0^1(1-s)^2M(s)ds=C_1\\ \displaystyle\int_0^1\left[h_1(1,s)+\frac{\alpha^2}{2}(1-s)^2\right]M(s)ds=C_2\end{cases} \tag{18}$$

Substituting the integration constants C_1, C_2 into Eq. (14) provides an integral equation in $M(\xi)$ as follows

$$\int_0^\xi h_1(\xi,s)M(\mathrm{s})\mathrm{ds}+\int_0^1 h_2(\xi,s)M(\mathrm{s})\mathrm{ds}+K_B(\xi)M(\xi)=0 \tag{19}$$

Where

$$h_2(\xi,s)=\frac{\alpha^2}{2}(1-s)^2(\xi-1)-h_1(1,s) \tag{20}$$

4.3 Solution of the resulting integral equation

In the preceding section, the governing equation was converted into the integral Eq. (19). At previous presented research works for conversion of the Eq. (19) into a solvable one, the mode shape function has been approximated using a power series. But, novelty of the presented approach in this paper is the approximation of the function $M(\xi)$ using a power series. $M(\xi)$ is corresponding to the bending moment function. The function $M(\xi)$ is the only unknown parameter in the integral Eq. (19). This function is approximated by the following power series

$$M(\xi) = \sum_{r=0}^{R} c_r \xi^r \tag{21}$$

Where C_r (r=0,1,…R) are unknown coefficients to be determined and R is a given positive integer, which is adopted such that the accuracy of the results are sustained. Introducing Eq. (21) into integral Eq. (19) yields

$$\sum_{r=0}^{R} \left[\int_0^{\xi} h_1(\xi,s)s^r \mathrm{d}s + \int_0^{1} h_2(\xi,s)s^r \mathrm{d}s + K_B(\xi)\xi^r \right] c_r = 0 \tag{22}$$

Both sides of Eq. (22) are multiplied by ξ^m and integrated subsequently with respect to ξ between 0 and 1. This results in a system of linear algebraic equations in C_r

$$\sum_{r=0}^{R} \left[H_1(m,r) + H_2(m,r) + G(m,r) \right] c_r = 0 \qquad m = 0,1,2,...,R \tag{23}$$

In which the functions H_1 (m,r), H_2 (m,r) and $G(m,r)$ are expressed as follows

$$\begin{cases} H_1(m,r) = \int_0^1 \int_0^{\xi} h_1(\xi,s) s^r \xi^m \mathrm{d}s\mathrm{d}\xi \\ H_2(m,r) = \int_0^1 \int_0^1 h_2(\xi,s) s^r \xi^m \mathrm{d}s\mathrm{d}\xi \\ G(m,r) = \int_0^1 \xi^{r+m} K_B(\xi)\mathrm{d}\xi \end{cases} \tag{24}$$

The system of linear algebraic Eq. (23) may be expressed in matrix notations as follows

$$[A]_{(R+1)\times(R+1)} [C]_{(R+1)\times 1} = [0]_{(R+1)\times 1} \tag{25}$$

In which $[A]$ and $[C]$ are the coefficients matrix and unknowns vector respectively. The only unknown parameter in the coefficients matrix $[A]$ is the non-dimensional natural frequency of the tall structure α. $[C]$=0 is a trivial solution for the resulting system of equations. The non-dimensional natural frequencies are determined by calculating a non-trivial solution for the resulting system of equations. To do so, the determinant of the coefficients matrix of the system has to be vanished. Accordingly, a frequency equation in α (which is a polynomial function of the order 2(R+1)) is introduced. The roots of the frequency equation are the non-dimensional natural frequencies of the tall structure. Given the fact that the function $M(\xi)$ is approximated by the power series of (21), the accuracy of results can be improved by taking into account a larger number of the series sentences (larger R).

5. Shear lag effects

5.1 Conversion of the governing differential equation into its weak form

The shear lag effect is an important parameter on the lateral stiffness of the framed tube. It can decreases the lateral stiffness of the structure. Therefore, it can decreases the natural frequencies of the framed tube. Coull and Bose (1975) have shown that a coefficient due to the shear lag effect, as an approximation, can be used in calculating the deformations of a framed tube structure when using the equivalent closed tube method. The governing differential equation is found to be as follows

$$\frac{\partial^4}{\partial x^4}\mathrm{W}(\mathrm{x},t) - \lambda^2 \frac{\partial^2}{\partial x^2}\mathrm{W}(\mathrm{x},t) + \frac{m}{K_{Bi}}\frac{\partial^2}{\partial t^2}\mathrm{W}(\mathrm{x},t) - \frac{m}{2K_B}\lambda^2 x^2 \frac{\partial^2}{\partial t^2}\mathrm{W}(\mathrm{x},t) = 0 \tag{26}$$

In which $W(x,t)$, K_{Bi}, K_B, m are the lateral displacement, the flexural stiffness of inner tube, the sum of the flexural stiffness of the outer and inner tubes and mass per unit length of the structure, respectively. λ is the parameter corresponding to the shear lag phenomena. This parameter is calculated as follows

$$\lambda^2 = K_f.h\left(\frac{1}{K_{Bo}} + \frac{1}{K_{Bi}}\right) \tag{27}$$

In which K_f, K_{Bo}, h are the equivalent story shearing rigidity of the outer tube, flexural stiffness of the outer tube and the story height, respectively. By assumption of a harmonic vibration, the transverse displacement of the structure is assumed as follows

$$\mathrm{W}(x,t) = \mathrm{w}(x)e^{\mathrm{i}\Omega t} \tag{28}$$

For more convenience, the following parameters are introduced

$$\begin{cases} \xi = \dfrac{x}{H} \\ g_1(\xi) = \dfrac{m\Omega^2\lambda^2H^6}{2K_B}\xi^2 - \dfrac{m\Omega^2H^4}{K_{Bi}} \end{cases} \tag{29}$$

By introducing the relations (28) and (29) into Eq. (26), the following differential equation is obtained

$$\frac{d^4}{d\xi^4}\mathrm{w}(\xi) - \lambda^2H^2\frac{d^2}{d\xi^2}\mathrm{w}(\xi) + g_1(\xi)\mathrm{w}(\xi) = 0 \tag{30}$$

In order to transform Eq. (30) into its weak form, both sides of this equation are integrated twice with respect to ξ within the range 0 to ξ. The results are the integral equations as follows

$$\frac{d^3}{d\xi^3}\mathrm{w}(\xi)-\lambda^2H^2\frac{d}{d\xi}\mathrm{w}(\xi)+\int_0^{\xi}g_1(\mathrm{s})\mathrm{w}(s)ds=C_1 \quad (31)$$

$$\frac{d^2}{d\xi^2}\mathrm{w}(\xi)-\lambda^2H^2\mathrm{w}(\xi)+\int_0^{\xi}\left(\xi-s\right)g_1(\mathrm{s})\mathrm{w}(s)ds= C_1\xi+C_2 \quad (32)$$

Further, integration from both sides of Eq. (32) twice with respect to ξ from 0 to ξ yields

$$\frac{d}{d\xi}\mathrm{w}(\xi)-\lambda^2H^2\int_0^{\xi}\mathrm{w}(s)ds+\int_0^{\xi}\frac{\left(\xi-s\right)^2}{2}g_1(\mathrm{s})\mathrm{w}(s)ds =\frac{C_1}{2}\xi^2+C_2\xi+C_3 \quad (33)$$

$$\mathrm{w}(\xi)+\int_0^{\xi}f_1(\xi,s)\mathrm{w}(s)ds=\frac{C_1}{6}\xi^3+\frac{C_2}{2}\xi^2+C_3\xi+C_4 \quad (34)$$

In which, the function f_1 (ξ, s) is calculated as follows

$$f_1(\xi,s)=-\lambda^2H^2\left(\xi-s\right)+\frac{\left(\xi-s\right)^3}{6}g_1(\mathrm{s}) \quad (35)$$

In Eq. (34) C_1, C_2, C_3 and C_4 are the integration constants which are determined through boundary conditions of both ends of the framed tube. Eqs. (31)-(34) are applicable for determination of the integration constants.

5.2 Boundary conditions

The following boundary conditions must be considered at the base of the structure

$$\xi=0 \longrightarrow \mathrm{w}(0)=0 \quad (36)$$

$$\xi=0\longrightarrow\frac{d\mathrm{w}}{d\xi}=0 \quad (37)$$

Also, the following boundary conditions are established at the roof of the framed tube

$$\xi=1\longrightarrow\frac{d^2}{d\xi^2}\mathrm{w}(1)=0 \quad (38)$$

$$\xi=1\longrightarrow\frac{d^3}{d\xi^3}\mathrm{w}(1)-\lambda^2H^2\frac{d}{d\xi}\mathrm{w}(1)=0 \quad (39)$$

Application of the condition (36) at Eq. (34) and condition (37) at Eq. (33) leads to:

$$C_3=C_4=0$$

Also, Application of the condition (38) at Eq. (32) and condition (39) at Eq. (31) leads to

$$\begin{cases}C_1=\int_0^1 g_1(s)\mathrm{w}(s)ds\\ C_2=\left(\dfrac{2}{2+\lambda^2H^2}\right)\int_0^1 g_2(s)\mathrm{w}(s)ds\end{cases} \quad (40)$$

In which the function $g_2(s)$ is calculated as follows

$$g_2(s)=-\frac{\lambda^2H^2}{6}g_1(s)-\lambda^2H^2f_1(1,s)-sg_1(s) \quad (41)$$

Substitution of the integration constants C_1, C_2, C_3 and C_4 into Eq. (34) results in an integral equation as follows

$$\mathrm{w}(\xi)+\int_0^{\xi}f_1(\xi,s)\mathrm{w}(\mathrm{s})\,\mathrm{ds}+\int_0^1 f_2(\xi,s)\mathrm{w}(\mathrm{s})\,\mathrm{ds}=0 \quad (42)$$

In which the function f_2 (ξ, s) is calculated as follows

$$f_2(\xi,s)=-\frac{\xi^3}{6}g_1(s)-\left(\frac{\xi^2}{2+\lambda^2H^2}\right)g_2(s) \quad (43)$$

At section 4, the governing differential equation was converted into corresponding weak form using two repetitive integration and the bending moment function was approximated using the power series (21). In this section, the weak form was calculated using four times repetitive integration and it requires that the mode shape function $w(\xi)$ is approximated using the power series. The mathematical calculation for conversion of the Eq. (42) into system of linear algebraic equation is exactly the same as what was stated in section 4.3.

6. Numerical examples

To verify the accuracy and efficiency of the proposed approximate method, three numerical examples are presented for determining the natural frequencies of the symmetric tall structure. All of which have been examined in the literature. Then, a comparison is presented between the results.

6.1 The tall structure with framed tube and shear walls

The first two natural frequencies of high-rise reinforced concrete buildings, which are 70-story, 80-story and 90-story high, as presented by Rahgozar *et al.* (2014), are analyzed based on the approach presented in this paper. The lateral load-resisting system of buildings is a framed tube combined with a shear wall. The structure has been modeled by an equivalent cantilever beam with hollow section. The structures properties are given in Table 1.

The first two natural frequencies of the structures are calculated and they are presented in the Table 2. The results have been compared with results of other references and results obtained using four times repetitive integration method.

The results of Table 2 present that the accuracy of results have been improved by taking into account a larger

Table 1 The properties of the framed tube combined with shear wall

	H (m)	K_B (KN−m^2)	K_S (KN)	m (kg.s^2/m^2)
70-story (plan a)	210	2.61×10^{13}	77.56×10^{8}	681408
70-story (plan b)	210	1.10×10^{13}	56.80×10^{8}	446492
80-story (plan a)	240	3.21×10^{13}	95.37×10^{8}	732480
80-story (plan b)	240	1.34×10^{13}	69.08×10^{8}	482972
90-story	270	3.86×10^{13}	143.32×10^{8}	817728

Table 2 The first two natural frequencies of the framed tube combined with shear wall (rad/sec)

	MODE	Presented Approach R=1	Presented Approach R=2	Presented Approach R=3	Four times repetitive integration method	Rahgozar *et al.* (2014) *BSF*	SAP-2000
70-story (plan a)	1	1.1490	1.1039	1.1038	1.1041	1.104	1.097
	2	6.2611	4.2826	4.2162	4.2	4.2	3.75
70-story (plan b)	1	1.1370	1.0833	1.0827	1.0827	1.08	1.05
	2	5.7060	3.9653	3.8761	3.8643	3.86	3.76
80-story (plan a)	1	1.0398	0.9947	0.9944	0.9943	0.995	0.99
	2	5.4279	3.7437	3.6740	3.6593	3.66	3.45
80-story (plan b)	1	1.0265	0.9751	0.9740	0.9736	0.973	0.92
	2	4.9975	3.4907	3.3985	3.3883	3.38	3.45
90-story	1	1.0194	0.9693	0.9684	0.9682	0.969	0.967
	2	5.0123	3.4957	3.4082	3.3980	3.402	3.44

number of the series sentences (larger R).

The main sources of error between the proposed approximate method and as compared to SAP2000 finite elements are as follows: (a) all closely spaced perimeter columns tied at each floor level by deep spandrel beams considered to form a tubular structure, (b) modeling the frame panels as equivalent orthotropic membranes, so it can be analyzed as a continuous structure, (c) the approximation to derive K_B and K_S parameters and (d) the effect of shear lag in the external and internal tubes has been neglected in assessing the global behavior of the tubular structures in approximate method (Malekinejad and Rahgozar 2014).

In order to investigate the effects of the structural parameters and axial force on the natural frequencies of the structure, a basis structure with structural properties as

$$\begin{aligned} &\left(K_B\right)_{bs} = 2.61\times10^{13}\ KN-m^2\ , \\ &\left(K_S\right)_{bs} = 77.56\times10^{8}\ KN \\ &\left(m\right)_{bs} = 681408\,\frac{kg.s^2}{m^2}\ , \quad \left(H\right)_{bs} = 210\,m \end{aligned} \tag{44}$$

is considered, in which "*bs*" denotes "basis structure". The first three natural frequencies of this structure have been calculated as:

$$\Omega_{1bs} = 1.1037\frac{rad}{\sec}, \quad \Omega_{2bs} = 4.1972\frac{rad}{\sec}, \quad \Omega_{3bs} = 9.7388\frac{rad}{\sec}$$

The ratios $\frac{\beta}{\beta_{bs}}$ and $\frac{\gamma}{\gamma_{bs}}$ are assumed to change between 0.25 through 2.5 and 1 through 3, respectively. The effects of these changes on the natural frequencies of the basis structure are calculated as follows: $\%\,\text{Diff} = \frac{\Omega-\Omega_{bs}}{\Omega_{bs}}$. The results are presented in the Table 3.

The results of Table 3 present that for the first mode of the vibration, with $\frac{\beta}{\beta_{bs}} = 0.25$, variation of $\frac{\gamma}{\gamma_{bs}}$ from 1 to 1.5 and 1.5 to 2 results in 21% and 60% decrease for the natural frequency, respectively. This decrease for second mode is 2.7% and 4% and for third mode is 1% and 1.4%, respectively. This presents that effect of axial force is significant for lower modes. For first mode, with $\frac{\beta}{\beta_{bs}} = 0.25$, variation of $\frac{\gamma}{\gamma_{bs}}$ from 1.5 to 2 results in 1.4% decrease for the natural frequency. In compared to results obtained for $\frac{\beta}{\beta_{bs}} = 0.25$, this presents that parameter βcan decrease effects of axial force.

Table 3 The effects of structural parameters and axial force on the natural frequencies of the basis structure with percent

	$\frac{\beta}{\beta_{bs}}$	$\frac{\gamma}{\gamma_{bs}}$			
		1→1.15	1.5→2	2→2.5	2.5→3
MODE 1	0.25	−21.0	−60.5	N/A	N/A
	0.5	−10.7	−19.7	−46.1	N/A
	1	−4.1	−6.3	−9.4	−14.4
	1.5	−2.2	−3.3	−4.5	−6.1
	2	−1.4	−2.0	−2.7	−3.5
	2.5	−1.0	−1.4	−1.8	−2.3
		1→1.15	1.5→2	2→2.5	2.5→3
MODE 2	0.25	−2.7	−4.0	−5.6	−7.8
	0.5	−2.3	−3.4	−4.8	−6.5
	1	−1.6	−2.3	−3.1	−4.0
	1.5	−1.0	−1.5	−2.0	−2.6
	2	−0.7	−1.1	−1.4	−1.8
	2.5	−0.6	−0.8	−1.0	−1.3
		1→1.15	1.5→2	2→2.5	2.5→3
MODE 3	0.25	−1.0	−1.4	−1.9	−2.4
	0.5	−0.9	−1.3	−1.8	−2.3
	1	−0.8	−1.1	−1.5	−1.9
	1.5	−0.6	−0.9	−1.2	−1.5
	2	−0.5	−0.7	−0.9	−1.2
	2.5	−0.4	−0.6	−0.8	−0.9

For quick calculation of the fundamental natural frequency of the tall structures, a simple formula can be more effective. Hence, with R=1 the following simple equation is obtained for quick calculation of the first non-dimensional natural frequency of the structure

Table 4 Variation of the equivalent properties of the 40-storey and 50-storey buildings along the height

	No. Storey	Height from the base of the structure (m)	Shear Stiffness K_S (kg)	Flexural Stiffness K_B (kg.m^2)	Mass per unit length m (kg/m)
40-STOREY	10	30	43.803×10^{8}	1.0548×10^{13}	378576
	21	63	43.803×10^{8}	1.0548×10^{13}	410432
	40	120	23.129×10^{8}	5.9091×10^{12}	343968
50-STOREY	10	30	1.5327×10^{10}	2.4084×10^{13}	482238.72
	20	60	8.4894×10^{9}	1.6589×10^{13}	444240
	30	90	4.3803×10^{9}	1.0548×10^{13}	380646.4
	40	120	2.3129×10^{9}	5.9091×10^{12}	327868.8
	50	150	1.7745×10^{9}	2.6340×10^{12}	290304

$$(\alpha_1)^2=\frac{-b-\sqrt{b^2-4a.c}}{2a} \tag{45}$$

In which

$$\begin{cases} a=\dfrac{1}{181440} \\ \text{b}=\dfrac{\gamma^2}{18900}-\dfrac{17}{2520}-\dfrac{53}{151200}\beta^2 \\ c=\dfrac{1}{12}+\dfrac{13}{360}\beta^2+\dfrac{\beta^4}{960}-\dfrac{\gamma^2}{90}-\dfrac{\beta^2\gamma^2}{1440}+\dfrac{\gamma^4}{14400} \\ {\alpha_1}^2=\dfrac{m_0{\Omega_1}^2H^4}{EI_0} \end{cases} \tag{46}$$

The weight of structures is considered as a variable axial force along the height of the equivalent beam. Using the data presented in Table 4, the functions of shear stiffness $K_B(\xi)$, flexural stiffness $K_B(\xi)$ and mass per unit length $m(\xi)$ are obtained as follows

for 40-storey buiding

$$\begin{cases} K_S(\xi)=\begin{bmatrix}1.0282\xi^4-2.0233\xi^3+\\1.1905\xi^2-0.2161\xi+0.0438\end{bmatrix}\times10^{11} \quad \text{kg} \\ \qquad (0\le\xi\le1) \\ K_B(\xi)=\begin{bmatrix}2.3072\xi^4-4.54\xi^3+\\2.6714\xi^2-0.485\xi+0.1055\end{bmatrix}\times10^{14} \quad \text{kg.m}^2 \\ \qquad (0\le\xi\le1) \\ m(\chi)\equiv m_{ave}=377658 \quad \text{kg/m} \end{cases} \tag{47}$$

for 50-storey buiding

$$\begin{cases} K_S(\xi)=\kappa\begin{bmatrix}-0.6811\xi^4+1.5203\xi^3-\\0.9296\xi^2-0.045\xi+\\0.1534\end{bmatrix}\times10^{11} \quad \text{kg} \\ \qquad (0\le\xi\le1) \\ K_B(\xi)=\begin{bmatrix}-0.4642\xi^4+1.2323\xi^3-\\0.9282\xi^2-0.0555\xi+0.2415\end{bmatrix}\times10^{14} \quad \text{kg.m}^2 \\ \qquad (0\le\xi\le1) \\ m(\chi)\equiv m_{ave}=385059 \quad \text{kg/m} \end{cases} \tag{48}$$

In which $\kappa$is the shear correction factor equal to 0.86623. The structure is 120 m and 150 m high for 40-storey and 50-storey buildings, respectively. The axial force function $N(\xi)$ caused by the weight of the structure is calculated as follows

$$\begin{aligned} &N(\text{x})=\int_x^H gm(x)dx \;\rightarrow \\ &N(\xi)=Hgm_{ave}\int_\xi^1 d\xi=\left(Hgm_{ave}\right)(1-\xi) \end{aligned} \tag{49}$$

The first natural frequency of the structure is calculated and compared with the result of Kamgar and Saadatpour (2012) and the analysis of SAP-2000. The results are presented in Table 5.

The first natural frequency of the structures in the absence of the axial force has been obtained as follows: $\Omega_1=1.8641\,\dfrac{\text{rad}}{\text{sec}}$ for 40-storey building and $\Omega_1=1.7227\,\dfrac{\text{rad}}{\text{sec}}$ for 50-storey building.

Table 5 The first natural frequency of the 40-storey and 50-storey structures with combined system and variable properties (rad/sec)

	Presented approach (two repetitive integration)	Four times repetitive integration method	Kamgar and Saadatpour (2012)	SAP -2000
40-storey	1.8298	1.9393	1.855	1.8034
50-storey	1.6919	1.7208	1.551	1.6175

Table 6 The first three natural frequencies of the structure with effects of shear lag

	Presented approach	Youlin (1984)	Wang (1996)	Lee (2007)
Ω_1	3.5259	3.2784	3.462	3.5180
Ω_2	19.2276	17.9212	21.525	20.763
Ω_3	52.4792	49.2027	_	_

6.3 Effects of shear log on the natural frequencies of framed tube

In order to investigate the effects of shear lag on the natural frequencies of the framed tube, the first three natural frequencies of a tube-in-tube structure that was examined by previous researchers have been calculated. The properties of the structure are as follows:

$$K_{BO}=35.2872\times10^8\,ton-m^2,\; K_{Bi}=7.5538\times10^8\,ton-m^2,$$

$$K_f.h=1.11678\times10^5\;ton,\; H\text{=75.9 m},\; h\text{=3.6 m},$$

$$m=325.828\,\frac{ton}{m},\quad \lambda=0.01339745\,.$$

The first three natural frequencies of the structure were calculated and the results were presented in the Table 6.

7. Quick design chart

For the preliminary analysis, a design chart can help to determine the natural frequencies quickly. The system of linear Eq. (23) is solved for values β=0−15. The resulting values of α_i (i=1,2,…,4) are given in Fig. 3. It is obvious

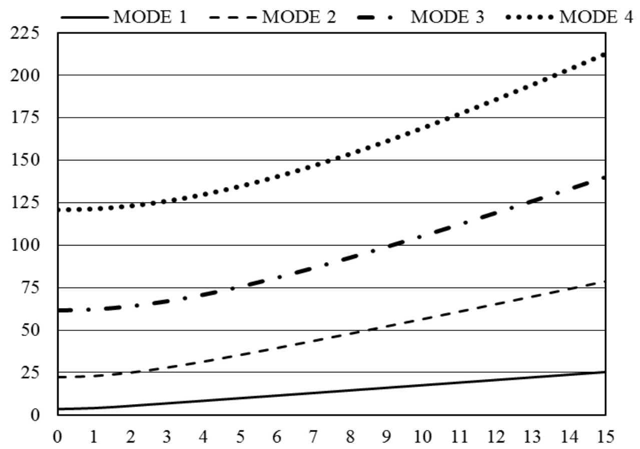

Fig. 3 Variations of α_i (i=1,2,…,4) with β $\alpha_i^2 = \frac{m_0 \Omega_i^2 H^4}{EI_0}$, $\beta^2 = \frac{GA_0 H^2}{EI_0}$

that for a given value of parameter β, the corresponding values of α_i (i=1−4) can be determined quickly.

8. Conclusions

In this study, the application of the weak form integral equations for determining the natural frequencies of tall structures with shear-flexural deformation has been presented. The novelty of the presented approach is based on the bending moment function approximation instead of the mode shape function approximation. This novelty makes the presented solution very simpler and very shorter in the mathematical calculations process. Because, at the previous research works it is needed four times repetitive integration for conversion of the governing equation into its weak form. But, the presented approach in this paper needs two repetitive integration. The governing partial differential equation of motion was converted into its weak form integral equation. To solve the resulting integral equation, the bending moment function of the vibration was approximated by a power series. Substitution of the power series into the weak form integral equation results in a system of linear algebraic equations. The natural frequencies of the tall structure have been calculated by determining a non-trivial solution for the system of linear algebraic equations. A design chart and a simple formula have been proposed for quick calculation of the fundamental natural frequency of shear-flexural structures. The effects of shear lag phenomena on the governing differential equation and the natural frequencies of the structure have been investigated. The results of the paper present that effect of axial force is significant for lower modes of the vibration. The results of the Table 2 present that presented approach has a quick convergence rate. Since with R=2, the results obtained has a good agreement in compared to results of SAP-2000 and other references. The accuracy, simplicity and reliability of the proposed method were verified thorough several numerical examples. Differences between natural frequencies of proposed method and the ones obtained in the literature were in acceptable ranges.

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