On wavelet multipliers and Landau–Pollak–Slepian operators on locally compact abelian topological groups

M. Kh. Abdullah, R. A. Kamyabi Gol & M. Janfada

Journal of Pseudo-Differential Operators and Applications

ISSN 1662-9981

J. Pseudo-Differ. Oper. Appl.
DOI 10.1007/s11868-019-00286-2

ONLINE FIRST

Springer

J. Pseudo-Differ. Oper. Appl.
https://doi.org/10.1007/s11868-019-00286-2

On wavelet multipliers and Landau–Pollak–Slepian operators on locally compact abelian topological groups

M. Kh. Abdullah[1] · R. A. Kamyabi Gol[1,2] · M. Janfada[1]

Received: 18 November 2018 / Revised: 20 February 2019 / Accepted: 22 February 2019

Abstract
In this paper, we define the wavelet multiplier and Landau–Pollak–Slepian (L.P.S) operators on the Hilbert space $L^2(G)$, where G is a locally compact abelian topological group and investigate some of their properties. In particular, we show that they are bounded linear operators, and are in Schatten p-class spaces, $1 \leq p \leq \infty$, and we determine their trace class.

Keywords Locally compact abelian group · Dual group · Wavelet multiplier operator · Landau–Pollak–Slepian operator · Admissible wavelets · Unitary representation

Mathematics Subject Classification Primary 43A15; Secondary 43A25 · 42C15

1 Introduction

In applied mathematics, it is true to say that nowadays wavelet theory is an essential area. From 30 years ago, wavelets have established themselves as a key methodology for efficiently representing signals or operators with applications ranging from more theoretical tasks such as adaptive schemes for solving elliptic partial differential equations to more practical tasks such as data compression.

✉ R. A. Kamyabi Gol
kamyabi@um.ac.ir

M. Kh. Abdullah
Mohammad1974Math1974@gmail.com

M. Janfada
janfada@um.ac.ir

1 Department of Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Islamic Republic of Iran

2 Department of Mathematics, Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Mashhad, Islamic Republic of Iran

Published online: 06 March 2019

Throughout this paper, G denotes a locally compact abelian topological group with the Haar measure [3,4,10] dx and $\hat{G}$ is the dual group of G with the Haar measure $d\xi$ such that $d\xi$ is the dual measure of dx , the elements of G are denoted by x, y and so on, while the elements of $\hat{G}$ are denoted by ξ, ω and so forth. For $\sigma \in L^\infty(\hat{G})$ we define the linear operator $T_\sigma : L^2(G) \to L^2(G)$ by $T_\sigma u = \mathcal{F}^{-1}\sigma\mathcal{F}u$, $u \in L^2(G)$ where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Plancherel and inverse Plancherel transforms, respectively. We recall that Fourier transform of f defined by $\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_G \overline{\omega(x)} f(x)dx$ and the inverse Fourier transform is defined by $\mathcal{F}^{-1}(f)(x) = \check{f}(x) = \int_{\hat{G}} \omega(x) f(\omega)d\omega$ for f (if exist) [5]. The Fourier transform on $L^1(G) \cap L^2(G)$ can be extended uniquely to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$ known as Plancherel Theorem [3,4]. The translation operator L_y is defined by $L_y f(x) = f(y^{-1}x)$ for any function f. The convolution [3,5] of f and g is the function defined by $(f * g)(x) = \int_G f(y)g(y^{-1}x)dy$.

In 1999, He and Wong [7] discussed wavelet multipliers and L.P.S. operators on $\mathbb{R}^n$. Our aim in this paper is to give a generalization of wavelet multiplier and L.P.S. operators on $L^2(G)$ where G is a locally compact abelian topological group. For this, we will define a unitary representation on the Hilbert space $L^2(G)$ by using properties of dual groups [3–5], and we find, among other things, the set of all admissible wavelets [1,6,8,10] for this unitary representation.

This paper is organized as follows: Sect. 2, starts with the definition of a unitary representation. Then we calculate the admissible wavelet for this unitary representation, then we show the operator $P_{\sigma,\varphi} : L^2(G) \to L^2(G)$, is unitarily equivalent to the wavelet operator $\varphi T_\sigma \overline{\varphi} : L^2(G) \to L^2(G)$ and state some preliminaries and related notations of these operators. In Sect. 3, we will discuss the boundedness of wavelet operator on two stages, first for $\sigma \in L^1(\hat{G})$, and second for $\sigma \in L^p(\hat{G})$, $1 < p \leq \infty$ by using The Riesz–Thorn Theorem [10]. This section also shows that the wavelet multiplier operators are in the Schatten p-class spaces [10,11] and then we will find the trace of these operators. In the end, in Sect. 4, we will give the definition of L.P.S. operator $Q_C P_\Omega Q_C : L^2(G) \to L^2(G)$, investigate some of its properties including the relationship between wavelet multiplier and L.P.S operators in special case, and finally evaluate the trace of this operator.

2 Wavelet multiplier operator on $L^2(G)$

In this section we introduce the wavelet multiplier operator $\varphi T_\sigma \overline{\varphi}$ where $T_\sigma \in B(L^2(G))$ is defined by $T_\sigma = \mathcal{F}^{-1}\sigma\mathcal{F}$ and $\varphi \in L^p(G)$, $1 \leq p \leq \infty$ and establish some of its properties. Let $\pi : \hat{G} \to U(L^2(G))$ be the unitary representation of the group $\hat{G}$ on the Hilbert space $L^2(G)$, denoted by $\{\pi, L^2(G)\}$, defined by

$$(\pi(\omega)u)(x) = \langle x, \omega\rangle u(x), \quad \omega \in \hat{G}, x \in G.$$

The nonzero element $\varphi \in L^2(G)$ is called an admissible wavelet for the unitary representation $\{\pi, L^2(G)\}$ if

$$\int_{\hat{G}} |\langle \varphi, \pi(\omega)\varphi\rangle|^2 d\omega < \infty.$$

In this case the value of the above integral is called the wavelet constant associated with the admissible φ and denoted by c_φ , and $\{\pi, L^2(G)\}$ is called square integrable representation [1,6,9,10].

The following facts will be used frequently.

$$(\pi(\eta) f)^\wedge(\xi) = L_\eta \hat{f}(\xi) \tag{2.1}$$

$$\langle u, \pi(\xi)\varphi\rangle = (\hat{u} * \hat{\bar{\varphi}})(\xi) = (u\bar{\varphi})^\wedge(\xi). \tag{2.2}$$

The following proposition characterizes the admissible vectors for the unitary representation $\{\pi, L^2(G)\}$.

Proposition 2.1 *The admissible wavelet for the unitary representation* $\{\pi, L^2(G)\}$ *defined on* $\hat{G}$ *consists of all function* $\varphi \in L^2(G) \cap L^4(G) \cap L^\infty(G)$ *for which* $\|\varphi\|_2 = 1$.

Proof Using Plancherel Theorem and (2.1), (2.2) we have

$$\begin{aligned} c_\varphi &= \int_{\hat{G}} |\langle \varphi, \pi(\xi)\varphi\rangle|^2 d\xi \\ &= \int_{\hat{G}} |(\hat{\varphi} * \hat{\bar{\varphi}})(\xi)|^2 d\xi \\ &= \int_{\hat{G}} |(\varphi\bar{\varphi})^\wedge(\xi)|^2 d\xi = \|(\varphi\bar{\varphi})^\wedge\|_2^2 \\ &= \|\varphi\bar{\varphi}\|_2^2 = \||\varphi|^2\|_2^2 = \|\varphi\|_4^4. \end{aligned}$$

□

Now by using (2.1), (2.2) and Plancherel Theorem we can prove the following proposition.

Proposition 2.2 *Let* $\varphi \in L^2(G) \cap L^\infty(G)$*, then for any* $u, v \in L^2(G)$*,*

$$\int_{\hat{G}} \langle u, \pi(\xi)\varphi\rangle_{L^2(G)} \langle \pi(\xi)\varphi, v\rangle_{L^2(G)} d\xi = \langle \varphi u, \varphi v\rangle_{L^2(G)}.$$

Proof By Plancherel Theorem, and (2.1), (2.2) we get

$$\begin{aligned} &\int_{\hat{G}} \langle u, \pi(\xi)\varphi\rangle_{L^2(G)} \langle \pi(\xi)\varphi, v\rangle_{L^2(G)}\, d\xi \\ &\quad = \int_{\hat{G}} (\hat{u} * \hat{\bar{\varphi}})(\xi)\overline{(v * \hat{\bar{\varphi}})(\xi)}\, d\xi \\ &\quad = \int_{\hat{G}} (u\bar{\varphi})^\wedge(\xi)\overline{(v\bar{\varphi})^\wedge(\xi)}\, d\xi \\ &\quad = \int_G (u\bar{\varphi})(x)\overline{(v\bar{\varphi})(x)}\, dx \end{aligned}$$

$$= \int_G (\varphi u)(x)\overline{(\varphi v)(x)}\, dx$$
$$= \langle \varphi u, \varphi v\rangle_{L^2(G)}.$$

□

Now for $\sigma \in L^\infty(\hat{G})$ and $\varphi \in L^2(G) \cap L^\infty(G)$, we define $P_{\sigma,\varphi} : L^2(G) \to L^2(G)$ by

$$\langle P_{\sigma,\varphi} u, v\rangle = \int_{\hat{G}} \sigma(\xi)\langle u, \pi(\xi)\varphi\rangle_{L^2(G)}\langle \pi(\xi)\varphi, v\rangle_{L^2(G)} d\xi. \tag{2.3}$$

With the above notations we have $\langle P_{\sigma,\varphi} u, v\rangle = \langle \varphi T_\sigma \bar{\varphi} u, v\rangle$, for all $u, v \in L^2(G)$. Indeed

$$\begin{aligned}\langle P_{\sigma,\varphi} u, v\rangle &= \int_{\hat{G}} \sigma(\xi)\langle u, \pi(\xi)\varphi\rangle_{L^2(G)}\langle \pi(\xi)\varphi, v\rangle_{L^2(G)} d\xi\\ &= \int_{\hat{G}} \sigma(\xi)\langle \hat{u}, \widehat{\pi(\xi)\varphi}\rangle_{L^2(G)}\langle \widehat{\pi(\xi)\varphi}, \hat{v}\rangle_{L^2(G)} d\xi\\ &= \int_{\hat{G}} \sigma(\xi)(\hat{u} * \hat{\bar{\varphi}})(\xi)\overline{(\hat{v} * \hat{\bar{\varphi}})(\xi)} d\xi\\ &= \int_{\hat{G}} \sigma(\xi)(\bar{\varphi}u)^\wedge(\xi)\overline{(\bar{\varphi}v)^\wedge(\xi)} d\xi = \int_{\hat{G}} (\sigma(\bar{\varphi}u)^\wedge)(\xi)\overline{(\bar{\varphi}v)^\wedge(\xi)} d\xi\\ &= \int_G (\sigma(\bar{\varphi}u)^\wedge)^\vee(x)\overline{(\bar{\varphi}v)(x)} dx = \int_G \varphi(\sigma(\bar{\varphi}u)^\wedge)^\vee(x)\overline{v(x)} dx\\ &= \langle \varphi T_\sigma \bar{\varphi} u, v\rangle.\end{aligned}$$

Now, we aim to show that the linear operators $P_{\sigma,\varphi} : L^2(G) \to L^2(G)$ for $\sigma \in L^p(\hat{G})$, $1 \le p \le \infty$ are bounded linear operators [2,11]. For the case $\sigma \in L^1(\hat{G})$, this is shown in the following proposition.

Proposition 2.3 *Let $\sigma \in L^1(\hat{G})$ and let $\varphi \in L^2(G) \cap L^\infty(G)$ such that $\|\varphi\|_2 = 1$. Then $P_{\sigma,\varphi} : L^2(G) \to L^2(G)$ is a bounded linear operator and $\|P_{\sigma,\varphi}\|_{B(L^2(G))} \le \|\sigma\|_{L^1(\hat{G})}$.*

Proof Let $\sigma \in L^1(\hat{G})$, $\varphi \in L^2(G) \cap L^\infty(G)$ with $\|\varphi\|_2 = 1$; Then

$$\begin{aligned}|\langle P_{\sigma,\varphi} u, v\rangle| &= \Big| \int_{\hat{G}} \sigma\big(\xi\big)\langle u, \pi(\xi)\varphi\rangle_{L^2(G)}\langle \pi(\xi)\varphi, v\rangle_{L^2(G)} d\xi \Big|\\ &\le \int_{\hat{G}} \big|\sigma\big(\xi\big)\big|\big|\langle u, \pi(\xi)\varphi\rangle_{L^2(G)}\big|\big|\langle \pi(\xi)\varphi, v\rangle_{L^2(G)}\big| d\xi\\ &\le \int_{\hat{G}} \big|\sigma\big(\xi\big)\big|\|u\|_2\|\pi(\xi)\varphi\|_2^2\|v\|_2 d\xi\\ &= \int_{\hat{G}} \big|\sigma\big(\xi\big)\big|\|u\|_2\|\varphi\|_2^2\|v\|_2 d\xi\end{aligned}$$

$$= \|u\|_2\|v\|_2 \int_{\hat{G}} \big|\sigma(\xi)\big| d\xi$$
$$= \|u\|_2\|v\|_2\|\sigma\|_{L^1(\hat{G})}.$$

So that $\|P_{\sigma,\varphi}\|_{B(L^2(G))} \leq \|\sigma\|_{L^1(\hat{G})}$. □

To prove the boundedness of $P_{\sigma,\varphi} : L^2(G) \to L^2(G)$ for $\sigma \in L^p(\hat{G})$, $1 < p \leq \infty$, we use the Riesz–Thorn Theorem, which for readers' convenience, we shall include the version we shall use [10].

Let (X, μ) be a measure space and (Y, ν) be a σ-finite measure space. Let T be a linear transformation with domain $\mathcal{D}$ consisting of all simple functions f on X such that

$$\mu(\{s \in X : f(s) \neq 0\}) < \infty,$$

and such that the range of T is contained in the set of all measurable functions on Y. Suppose that $\alpha_1, \alpha_2, \beta_1$ and β_2 are numbers in the interval $[0, 1]$ and there exist positive constants M_1 and M_2 such that

$$\|Tf\|_{L^{\frac{1}{\beta_j}}(Y)} \leq M_j \|f\|_{L^{\frac{1}{\alpha_j}}(Y)}, \ \ f \in \mathcal{D}, \ \ j = 1, 2.$$

Then $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$ and $\beta = (1-\theta)\beta_1 + \theta\beta_2$, we have

$$\|Tf\|_{L^{\frac{1}{\beta}}(Y)} \leq M_1^{1-\theta} M_2^{\theta} \|f\|_{L^{\frac{1}{\alpha}}(Y)}, \ \ f \in \mathcal{D}.$$

Theorem 2.4 *Let $\sigma \in L^p(\hat{G})$, $1 < p \leq \infty$ and let $\varphi \in L^2(G) \cap L^\infty(G)$ be such that $\|\varphi\|_2 = 1$. Then there exists a unique bounded linear operator $P_{\sigma,\varphi} : L^2(G) \to L^2(G)$ such that $\|P_{\sigma,\varphi}\|_{B(L^2(G))} \leq \|\varphi\|_{L^\infty(G)}^{\frac{2}{q}} \|\sigma\|_{L^p(\hat{G})}$ and for all $u, v \in L^2(G)$, $\langle P_{\sigma,\varphi}u, v\rangle_{L^2(G)}$ is given in (2.3) for all simple functions σ on $\hat{G}$ for which the Haar measure of the set $\{\xi \in \hat{G} : \sigma(\xi) \neq 0\}$ is finite.*

Proof For $\sigma \in L^\infty(\hat{G})$, we get

$$\begin{aligned}
\big|\langle P_{\sigma,\varphi}u, v\rangle_{L^2(G)}\big| &= \Big| \int_{\hat{G}} \sigma\big(\xi\big)\langle u, \pi(\xi)\varphi\rangle_{L^2(G)}\langle \pi(\xi)\varphi, v\rangle_{L^2(G)} d\xi \Big| \\
&\leq \int_{\hat{G}} \big|\sigma\big(\xi\big)\big|\big|\langle u, \pi(\xi)\varphi\rangle_{L^2(G)}\big|\big|\langle \pi(\xi)\varphi, v\rangle_{L^2(G)}\big| d\xi \\
&\leq \|\sigma\|_{L^\infty(\hat{G})}\Big[\int_{\hat{G}} \big|\langle u, \pi(\xi)\varphi\rangle_{L^2(G)}\big|^2 d\xi\Big]^{\frac{1}{2}}\Big[\int_{\hat{G}} \langle \pi(\xi)\varphi, v\rangle_{L^2(G)}\big|^2 d\xi\Big]^{\frac{1}{2}} \\
&= \|\sigma\|_{L^\infty(\hat{G})}\Big[\int_{\hat{G}} \big|(\hat{u} * \hat{\bar{\varphi}})\big(\xi\big)\big|^2 d\xi\Big]^{\frac{1}{2}}\Big[\int_{\hat{G}} \big|\overline{\big(v * \hat{\bar{\varphi}}\big)\big(\xi\big)}\big|^2 d\xi\Big]^{\frac{1}{2}} \\
&= \|\sigma\|_{L^\infty(\hat{G})}\|\hat{u} * \hat{\bar{\varphi}}\|_2\|\overline{\hat{u} * \hat{\bar{\varphi}}}\|_2 = \|\sigma\|_{L^\infty(\hat{G})}\|(\bar{\varphi}u)^\wedge\|_2\|(\bar{\varphi}v)^\wedge\|_2
\end{aligned}$$

$$= \|\sigma\|_{L^\infty(G)}\|\bar{\varphi}u\|_2\|\bar{\varphi}v\|_2 \le \|\sigma\|_{L^\infty(\hat{G})}\|\bar{\varphi}\|^2_{L^\infty(G)}\|u\|_2\|v\|_2$$
$$= \|\sigma\|_{L^\infty(\hat{G})}\|\varphi\|^2_{L^\infty(G)}\|u\|_2\|v\|_2,$$

thus

$$\|P_{\sigma,\varphi}\|_{B(L^2(G)} \le \|\sigma\|_{L^\infty(\hat{G})}\|\varphi\|^2_{L^\infty(G)}.$$

For $1 < p < \infty$, the Riesz–Thorn Theorem completes the proof. □

Now Proposition 2.3 and Theorem 2.4 allow us to define the wavelet multiplier operator $\varphi T_\sigma \bar{\varphi} : L^2(G) \to L^2(G)$ for all $\sigma \in L^p(\hat{G})$, $1 \le p \le \infty$ and all $\varphi \in L^2(G) \cap L^\infty(G)$ with $\|\varphi\|_2 = 1$ which is the same as the bounded linear operator $P_{\sigma,\varphi} : L^2(G) \to L^2(G)$.

Remark 2.5 Let φ be an admissible wavelet for the square integralable representation $\{\pi, L^2(G)\}$, then the linear operator $L_{\sigma,\varphi} : L^2(G) \to L^2(G)$ which is defined as $\langle L_{\sigma,\varphi}u, v\rangle = \dfrac{1}{c_\varphi}\int_{\hat{G}} \sigma\big(\xi\big)\langle u, \pi(\xi)\varphi\rangle_{L^2(G)}\langle \pi(\xi)\varphi, v\rangle_{L^2(G)}d\xi$ is called localization operator associated to the symbol σ and admissible wavelet φ , hence from Proposition 2.1, we have $c_\varphi = \|\varphi\|_4^4$ and from (2.3) we get that $P_{\sigma,\varphi} = \|\varphi\|_4^4 L_{\sigma,\varphi}$ also $L_{\sigma,\varphi} \in S_1$ with $\|L_{\sigma,\varphi}\|_{S_1} \le \frac{1}{c_\varphi}\|\sigma\|_{L^1(\hat{G})}$ for more details see [8,9].

3 The Schatten–von Neumann property

We recall that an operator T on a Hilbert space $\mathcal{H}$ is called a compact operator [2,4,9,11] (or completely continuous operator) if, for every bounded sequence $\{x_n\}$ in $\mathcal{H}$, the sequence $\{Tx_n\}$ contains a convergent subsequence. Now if T is a compact operator on a separable Hilbert space $\mathcal{H}$, then there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in $\mathcal{H}$ such that

$$T(x) = \sum_n \lambda_n\langle x, e_n\rangle\sigma_n, \quad x \in \mathcal{H},$$

where λ_n is the n-th singular value of T [2,9,11]. Given $0 < p < \infty$, we define the Schatten p-class of $\mathcal{H}$, denoted by $\mathcal{S}_p(\mathcal{H})$ or simply $\mathcal{S}_p$, to be the space of all compact operators T on $\mathcal{H}$ such that its singular value sequence $\{\lambda_n\}$ belonging to ℓ_p (the p-summable sequence space) [9,11]. We will be mainly concerned with the range $1 \le p < \infty$. In this case, $\mathcal{S}_p$ is a Banach space with the norm $\|T\|_p$ defined by

$$\|T\|_p = \big[\sum_n |\lambda_n|^p\big]^{\frac{1}{p}},$$

$\mathcal{S}_1$ is also called the trace class, and $\mathcal{S}_2$ is usually called the Hilbert- Schmidt class.

The following theorem contains sufficiently conditions for the wavelet multiplier operator is in trace class.

Theorem 3.1 *Let $\sigma \in L^1(\hat{G})$ and $\varphi \in L^2(G) \cap L^4(G) \cap L^\infty(G)$ such that $\|\varphi\|_2 = 1$. Then the wavelet multiplier operator $\varphi T_\sigma \bar{\varphi} : L^2(G) \to L^2(G)$ is in $\mathcal{S}_1$ and $\|\varphi T_\sigma \bar{\varphi}\|_{\mathcal{S}_1} \leq \|\sigma\|_{L^1(\hat{G})}$.*

Proof By Remark 2.5 the proof is clear. □

Now we are going to show that the wavelets multipliers operators $\varphi T_\sigma \bar{\varphi}$ is in $\mathcal{S}_p$ for $1 \leq p \leq \infty$, where $\sigma \in L^p(\hat{G})$. To do this, we need to recall some notations and terminologies.

Let B_0 and B_1 be two complex Banach spaces, we called B_0 and B_1 compatible if we have $B_k \subseteq V$, $k = 0, 1$ for some complex vector space V. Suppose that $\mathcal{S} = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1\}$ and let B be any complex Banach space, a function $f : \mathcal{S} \to B$ is called analytic on $\mathcal{S}$ if for every g (bounded linear functional on B) we have the decomposition $g \circ f : \mathcal{S} \to \mathbb{C}$ analytic on $\mathcal{S}$. Now let $\mathcal{F}(B_0, B_1)$ (B_0 and B_1 are compatible Banach spaces), be the set of all bounded and continuous functions f from $\overline{\mathcal{S}}$ into $B_0 + B_1$ such that f is analytic on $\mathcal{S}$ and the mappings

$$\mathbb{R} \ni y \to f(k + iy) \in B_k, \ k = 0, 1,$$

are continuous from $\mathbb{R}$ into B_k, $k = 0, 1$. Now one can show that $\mathcal{F}(B_0, B_1)$ is a complex Banach space with the norm $\| \ \|_{\mathcal{F}}$ defined as

$$\|f\|_{\mathcal{F}} = \max_{k=0,1} \sup_{y \in \mathbb{R}} \|f(k + iy)\|_{B_{k'}}, \quad f \in \mathcal{F}(B_0, B_1).$$

For any θ in the interval $[0, 1]$, B_θ is the subspace of $B_0 + B_1$ consisting of all elements b in $B_0 + B_1$ such that $b = f(\theta)$ for some f in $\mathcal{F}(B_0, B_1)$, then B_θ is a complex Banach space with respect to the norm $\| \ \|_\theta$ defined as

$$\|b\|_\theta = \inf_{b = f(\theta)} \|f\|_{\mathcal{F}}, \quad b \in B_\theta,$$

and the interpolation space between the spaces B_0 and B_1 is B_θ, which denoted by $[B_0, B_1]$.

Suppose that we have two pairs of compatible Banach spaces, like B_0, B_1 and $\tilde{B_0}, \tilde{B_1}$, and let T be any bounded linear operator from $B_0 + B_1$ into $\tilde{B_0} + \tilde{B_1}$, so as, T is a bounded linear operator from B_k into $\tilde{B_k}$ with norm less than or equal to M_k, $k = 0, 1$. Then for any real number θ in the interval $(0, 1)$, T is a bounded linear operator from $[B_0, B_1]_\theta$ into $[\tilde{B_0}, \tilde{B_1}]_\theta$ with norm not bigger than $M_0^{1-\theta} M_1^\theta$.

In particular for $1 \leq p \leq \infty$,

$$[L^1(X, \mu), L^\infty(X, \mu)]_{\frac{1}{q}} = L^p(X, \mu),$$

and

$$[\mathcal{S}_1, \mathcal{S}_\infty]_{\frac{1}{q}} = \mathcal{S}_p,$$

where (X, μ) is a measure space and q is the conjugate index of p. See [10,11] for more details.

Theorem 3.2 *Let $\sigma \in L^p(\hat{G})$, $1 \leq p \leq \infty$ and $\varphi \in L^2(G) \cap L^4(G) \cap L^\infty(G)$ with $\|\varphi\|_2 = 1$. Then the wavelet multiplier operator $\varphi T_\sigma \bar{\varphi} : L^2(G) \to L^2(G)$ is in $\mathcal{S}_p$ and $\|\varphi T_\sigma \bar{\varphi}\|_{\mathcal{S}_p} \leq \|\varphi\|^{\frac{2}{q}}_{L^\infty(G)} \|\sigma\|_{L^p(\hat{G})}$.*

Proof For $p = 1$ the proof follows from Theorem 3.1; and for $p = \infty$ the proof follows from Theorem 2.4, thus for $1 < p < \infty$ the interpolation Theorem as mentioned above complete the proof. □

In the following theorem, we investigate the trace of the wavelet multiplier operator.

Theorem 3.3 *Let $\sigma \in L^1(\hat{G})$ and $\varphi \in L^2(G) \cap L^4(G) \cap L^\infty(G)$ be such that $\|\varphi\|_2 = 1$. Then $\mathrm{tr}(\varphi T_\sigma \bar{\varphi}) = \int_{\hat{G}} \sigma(\xi) d\xi$.*

Proof Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis for $L^2(G)$. We get

$$\begin{aligned}
\mathrm{tr}(\varphi T_\sigma \bar{\varphi}) = \mathrm{tr}(P_{\sigma,\varphi}) &= \sum_{k=1}^\infty \langle P_{\sigma,\varphi}\varphi_k, \varphi_k\rangle = \sum_{k=1}^\infty \int_{\hat{G}} \sigma(\xi) |\langle \varphi_k, \pi(\xi)\varphi\rangle|^2 d\xi \\
&= \int_{\hat{G}} \sigma(\xi) \sum_{k=1}^\infty |\langle \varphi_k, \pi(\xi)\varphi\rangle|^2 d\xi = \|\pi(\xi)\varphi\|_2^2 \int_{\hat{G}} \sigma(\xi) d\xi \\
&= \|\varphi\|_2^2 \int_{\hat{G}} \sigma(\xi) d\xi = \int_{\hat{G}} \sigma(\xi) d\xi.
\end{aligned}$$

□

4 The Landau–Pollak–Slepian opearator

Here we will give the Landau–Pollak–Slepian (L.P.S) operator $Q_C P_\Omega Q_C : L^2(G) \to L^2(G)$ where C and Ω are a compact neighborhoods of identity elements of G and $\hat{G}$, respectively, and investigate some important properties of L.P.S operator and finally we consider the trace of this operator.

At first, let us define the linear operators $P_\Omega : L^2(G) \to L^2(G)$ and $Q_C : L^2(G) \to L^2(G)$ by $(P_\Omega f)^\wedge(\xi) = (\check{\chi}_\Omega * f)^\wedge(\xi)$ and $(Q_C f)(x) = (\chi_C f)(x)$, for all $f \in L^2(G)$, which are in fact orthogonal projections, as the following proposition shows.

Proposition 4.1 *With the notations as above, $P_\Omega : L^2(G) \to L^2(G)$ and $Q_C : L^2(G) \to L^2(G)$ are orthogonal projections.*

Proof Note that

$$\begin{aligned}\langle P_\Omega f, g\rangle &= \langle (P_\Omega f)^\wedge, \hat{g}\rangle = \int_{\hat{G}} (\check{\chi}_\Omega * f)^\wedge(\xi)\overline{\hat{g}(\xi)}d\xi = \int_{\hat{G}} \hat{f}(\xi)\chi_\Omega(\xi)\overline{\hat{g}(\xi)}d\xi \\ &= \int_{\hat{G}} \hat{f}(\xi)\overline{(\check{\chi}_\Omega * g)^\wedge(\xi)}d\xi = \int_{\hat{G}} \hat{f}(\xi)\overline{(P_\Omega g)^\wedge(\xi)}d\xi \\ &= \langle \hat{f}, (P_\Omega g)^\wedge(\xi)\rangle = \langle f, P_\Omega g\rangle.\end{aligned}$$

Therefore $P_\Omega : L^2(G) \to L^2(G)$ is self-adjoint. Also

$$\begin{aligned}\langle Q_C f, g\rangle &= \int_G (Q_C f)(x)\overline{g(x)}dx = \int_C f(x)\overline{g(x)}dx \\ &= \int_G f(x)\overline{(Q_C g)(x)}dx = \langle f, Q_C g\rangle.\end{aligned}$$

Therefore $Q_C : L^2(G) \to L^2(G)$ is self-adjoint. On the other hand, we have

$$\begin{aligned}\langle P_\Omega^2 f, g\rangle &= \langle P_\Omega f, P_\Omega g\rangle = \langle (P_\Omega f)^\wedge, (P_\Omega g)^\wedge\rangle = \int_{\hat{G}} (P_\Omega f)^\wedge(\xi)\overline{(P_\Omega g)^\wedge(\xi)}d\xi \\ &= \int_{\hat{G}} (\check{\chi}_\Omega * f)^\wedge(\xi)\overline{(\check{\chi}_\Omega * g)^\wedge(\xi)}d\xi = \int_{\hat{G}} \hat{f}(\xi)\chi_\Omega(\xi)\overline{\hat{g}(\xi)\chi_\Omega(\xi)}d\xi \\ &= \int_{\hat{G}} (\check{\chi}_\Omega * f)^\wedge(\xi)\overline{\hat{g}(\xi)}d\xi = \int_{\hat{G}} (P_\Omega f)^\wedge(\xi)\overline{\hat{g}(\xi)}d\xi \\ &= \langle (P_\Omega f)^\wedge, \hat{g}\rangle = \langle P_\Omega f, g\rangle.\end{aligned}$$

Thus $P_\Omega^2 = P_\Omega$ and hence $P_\Omega : L^2(G) \to L^2(G)$ is an orthogonal projection. Also

$$\begin{aligned}\langle Q_C^2 f, g\rangle &= \langle Q_C f, Q_C g\rangle = \int_G (Q_C f)(x)\overline{(Q_C g)(x)}dx = \int_C f(x)\overline{g(x)}dx \\ &= \int_G (Q_C f)(x)\overline{g(x)}dx = \langle Q_C f, g\rangle.\end{aligned}$$

Thus $Q_C^2 = Q_C$ and hence $Q_C : L^2(G) \to L^2(G)$ is an orthogonal projection. □

Using the fact that P_Ω and Q_C are orthogonal projections, we get

$$\begin{aligned}&\sup\left\{\frac{\|P_\Omega Q_C f\|_2^2}{\|f\|_2^2} : f \in L^2(G), \|f\|_2 \neq 0\right\} \\ &\quad = \sup\left\{\frac{\langle P_\Omega Q_C f, P_\Omega Q_C f\rangle}{\|f\|_2^2} : f \in L^2(G), \|f\|_2 \neq 0\right\} \\ &\quad = \sup\left\{\frac{\langle P_\Omega^2 Q_C f, Q_C f\rangle}{\|f\|_2^2} : f \in L^2(G), \|f\|_2 \neq 0\right\}\end{aligned}$$

$$= \sup \left\{ \frac{\langle P_\Omega Q_C f, Q_C f \rangle}{\|f\|_2^2} : f \in L^2(G), \|f\|_2 \neq 0 \right\}$$
$$= \sup \left\{ \frac{\langle Q_C P_\Omega Q_C f, f \rangle}{\|f\|_2^2} : f \in L^2(G), \|f\|_2 \neq 0 \right\}$$
$$= \sup\{\langle Q_C P_\Omega Q_C f, f \rangle : f \in L^2(G), \|f\|_2 = 1\}.$$

Since $Q_C P_\Omega Q_C : L^2(G) \to L^2(G)$ is self-adjoint, it follows from the above that

$$\sup \left\{ \frac{\|P_\Omega Q_C f\|_2^2}{\|f\|_2^2} : f \in L^2(G), \|f\|_2 \neq 0 \right\} = \|Q_C P_\Omega Q_C\|_{B(L^2(G))}.$$

Theorem 4.2 *Let φ be the function on G defined by $\varphi(x) = \frac{1}{|C|^{\frac{1}{2}}} \chi_C(x)$, where $|C|$ denotes the Haar measure of C, and let σ be the function on $\hat{G}$ defined by $\sigma(\xi) = \chi_\Omega(\xi)$. Then the operator $Q_C P_\Omega Q_C : L^2(G) \to L^2(G)$ is unitarily equivalent to scalar multiple of the wavelet multiplier $\varphi T_\sigma \varphi : L^2(G) \to L^2(G)$. In fact $Q_C P_\Omega Q_C = |C|(\varphi T_\sigma \varphi)$.*

Proof From the definition of φ, we get that $\varphi \in L^2(G) \cap L^\infty(G)$ with $\|\varphi\|_2^2 = \int_G |\varphi(x)|^2 dx = \frac{1}{|C|} \int_C dx = 1$, so by Proposition 2.3 we have,

$$\langle \varphi T_\sigma \varphi u, v \rangle = \int_{\hat{G}} \sigma(\xi) \langle u, \pi(\xi)\varphi \rangle \langle \pi(\xi)\varphi, v \rangle d\xi, \quad u, v \in C_C(G),$$

and

$$\langle u, \pi(\xi)\varphi \rangle = \int_G u(x) \overline{\pi(\xi)\varphi(x)} dx$$
$$= \int_G u(x) \overline{\langle x, \xi \rangle \varphi(x)} dx$$
$$= \int_G u(x) \overline{\langle x, \xi \rangle} \varphi(x) dx$$
$$= \frac{1}{|C|^{\frac{1}{2}}} \int_G (\chi_C u)(x) \overline{\langle x, \xi \rangle} dx$$
$$= \frac{1}{|C|^{\frac{1}{2}}} \int_G (Q_C u)(x) \overline{\langle x, \xi \rangle} dx$$
$$= \frac{1}{|C|^{\frac{1}{2}}} (Q_C u)^\wedge(\xi).$$

So

$$\langle u, \pi(\xi)\varphi \rangle = \frac{1}{|C|^{\frac{1}{2}}} (Q_C u)^\wedge(\xi) \text{ and } \langle \pi(\xi)\varphi, v \rangle = \overline{\frac{1}{|C|^{\frac{1}{2}}} (Q_C v)^\wedge(\xi)}.$$

Now

$$
\begin{aligned}
\langle \varphi T_\sigma \varphi u, v\rangle &= \int_{\hat{G}} \sigma(\xi)\langle u, \pi(\xi)\varphi\rangle\langle \pi(\xi)\varphi, v\rangle d\xi \\
&= \frac{1}{|C|}\int_{\hat{G}} \sigma(\xi)(Q_C u)^\wedge(\xi)\overline{(Q_C v)^\wedge(\xi)} d\xi \\
&= \frac{1}{|C|}\int_{\hat{G}} \chi_\Omega(\xi)(Q_C u)^\wedge(\xi)\overline{(Q_C v)^\wedge(\xi)} d\xi \\
&= \frac{1}{|C|}\int_{\hat{G}} (\chi_\Omega(Q_C u)^\wedge)(\xi)\overline{(Q_C v)^\wedge(\xi)} d\xi \\
&= \frac{1}{|C|}\int_{\hat{G}} (\check{\chi}_\Omega * (Q_C u))^\wedge(\xi)\overline{(Q_C v)^\wedge(\xi)} d\xi \\
&= \frac{1}{|C|}\int_{C} (P_\Omega(Q_C u))^\wedge(\xi)\overline{(Q_C v)^\wedge(\xi)} d\xi \\
&= \frac{1}{|C|}\langle (P_\Omega(Q_C u))^\wedge, (Q_C v)^\wedge\rangle = \frac{1}{|C|}\langle P_\Omega Q_C u, Q_C v\rangle \\
&= \frac{1}{|C|}\langle Q_C P_\Omega Q_C u, v\rangle \quad \text{for all functions } u, v \in C_C(G).
\end{aligned}
$$

So

$$Q_C P_\Omega Q_C = |C|(\varphi T_\sigma \varphi).$$

□

Theorem 4.3 *With the above notations* $\mathrm{tr}(Q_C P_\Omega Q_C) = |C||\Omega|$.

Proof Theorem 4.3 is an immediate consequence of Theorems 4.2 and 3.3. □

References

1. Arefijamal, A.A., Kamyabi Gol, R.A.: A characterization of square integrable representations associatd with CWT. J. Sci. Islam. Repub. Iran **18**, 159–166 (2007)
2. Debnath, L., Mikusinski, P.: Introduction to Hilbert Spaces with Applications, 3rd edn. Elsevier Academic Press, Amsterdam (2005)
3. Deitmar, A.: A First Course in Harmonic Analysis, 2nd edn. Springer, Berlin (2000)
4. Deitmar, A., Echterhoff, S.: Principles of Harmonic Analysis. Springer, Berlin (2009)
5. Folland, G.B.: A Course in Abstract Harmonic Analysis, 2nd edn. CRC Press, Boca Raton (2015)
6. Führ, H.: Abstract Harmonic Analysis of Continuous Wavelet Transform. Springer Lecture Notes in Mathematics, Nr. 1863. Springer, Berlin (2005)
7. He, Z., Wong, M.W.: Wavelet multipliers and signals. J. Aust. Math. Soc. Ser. B **40**, 437–446 (1999)
8. Kamyabi Gol, R.A., Esmaeelzadeh, F., Raisi Tousi, R.: Localization operator on homogeneous spaces. Bull. Iran. Math. Sci. **39**(3), 455–467 (2013)
9. Li, J., Wong, M.W.: Localization operators for ridgelet transforms. Math. Model. Nat. Phenom. **9**(5), 194–203 (2014)
10. Wong, M.W.: Wavelet Transform and Localization Operators. Birkhauser Verlag, Basel (2002)
11. Zhu, K.: Operator Theory in Function Spaces. Marcel Dekker, New York (1990)