On c-capability and n-isoclinic families of a specific class of groups

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Abstract.   Let χ denote the class of all groups G such that $\Phi(G) \cap Z(G) = 1$. In this paper, it is shown that the converse of Baer's theorem holds for the groups in χ. Then we prove that the existence of the isomorphism between the center factors of the groups in χ suffices for those groups to be isoclinic. We also prove that the isoclinism coincides with the n-isoclinism in χ. Finally, we obtain a criterion for c-capability of finite groups in χ.

Keywords. c-Capable group; n-isoclinic; Baer's theorem; c-nilpotent multiplier.

2010 Mathematics Subject Classification.   Primary: 20D25; Secondary: 19C09, 20C25.

1. Motivation and preliminaries

There are numerous finiteness properties which restrict in some way a set of conjugates or a set of commutators in a group. Sometimes these restrictions are strong enough to impose a recognizable pattern on the group or even delineate the structure. Some finiteness properties of the upper and lower central series are studied by Schur [17], Baer [3] and Hall [8]. A consequence of Schur's work on projective representations [17] is that G' is finite if $Z(G)$ has finite index in G. This result is called as Schur's theorem. Baer [3] provided a generalization of Schur's theorem to higher terms of the upper and lower central series.

Baer's theorem. *If G is a group such that $G/Z_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite.*

It is known that the converse of theorems of Schur and Baer are not true in general (for instance, infinite extra special p-groups and the groups constructed by Hall [8] are some related counterexamples). Despite this, some authors tried to determine the conditions under which the converse of these theorems are true, or sometimes they tried to find some families of groups that satisfy these converses.

Hall [8] proved a partial converse of Baer's theorem. He actually showed that for a group G, the finiteness of $\gamma_{n+1}(G)$ implies the finiteness of $G/Z_{2n}(G)$.

In this article, we introduce a class χ of groups which satisfy the converse of Baer's theorem. The class χ is defined to be all groups G such that $\Phi(G) \cap Z(G) = 1$. One can easily see that the trivial group is in χ and if $G_1 \in \chi$ and $G_1 \cong G_2$, then $G_2 \in \chi$. Thus χ is in fact a class of groups. Moreover, for center factors of groups G in χ, we will give some bounds in terms of the order of $\gamma_{n+1}(G)$ (Corollaries 5 and 6). Also, we will focus on the n-isoclinic families and the c-capability of groups in χ. It is shown that each n-isoclinic family of groups in χ has a unique n-stem group and also each $(n+1)$-isoclinism induces an n-isoclinism. Moreover, a remarkable fact states that the c-capability of each finite group in χ depends only on the structure of its center. Finally, we show that each group in χ is n-isoclinic to at least one c-capable group.

The following proposition [11, Theorem 2.3 and Lemma 2.5] is vital in our main results. Recall that $\Phi(G)$, $Z_n(G)$ and $\gamma_{n+1}(G)$ are the Frattini subgroup, $(n+1)$-th terms of the upper and lower central series of a group G, respectively.

PROPOSITION 1

Let G be a group and N be a normal subgroup of G.

(1) *If $N \cap Z(G) = 1$, then $N \cap Z_n(G) = 1$ for all $n \geq 1$.*
(2) *If $N \cap \gamma_{n+1}(G) = 1$ for some n, then $N \subseteq Z_n(G)$.*

Recall that the upper central series of a group G need not reach G. The terminal subgroup of this series is called the hypercenter of G.

Lemma 2. *Let G be a group such that $\Phi(G) \cap Z(G) = 1$. Then $Z(G)$ is the hypercenter of G.*

Proof. Since $G' \cap Z(G) \subseteq \Phi(G) \cap Z(G)$, we have $G' \cap Z_n(G) = 1$ for each $n > 0$ by Proposition 1 and therefore $Z(G) = Z_n(G)$. Hence $Z(G)$ is the hypercenter of G. $\square$

Hall [8] proved that for a group G, the finiteness of $\gamma_{n+1}(G)$ implies the finiteness of $G/Z_{2n}(G)$. Combining Proposition 1 and Hall's statement show that the converse of Baer's theorem is true for groups in χ.

COROLLARY 3

Let G be a group such that $\Phi(G) \cap Z(G) = 1$ and $\gamma_{n+1}(G)$ be finite for some positive integer n. Then $G/Z(G)$ is finite.

We can also obtain an upper bound for the order of the center factor of groups in χ in terms of the order $\gamma_{n+1}(G)$, using the following theorem of [10].

Theorem 4 [10]. *If G is a group, $\gamma_{n+1}(G)$ is finite and $G/Z_n(G)$ is finitely generated. Then $|G/Z_n(G)| \leq |\gamma_{n+1}(G)|^{d(G/Z_n(G))^n}$, where $d(X)$ is the minimal number of generators of the group X.*

COROLLARY 5

Let G be a group such that $\Phi(G) \cap Z(G) = 1$ and $\gamma_{n+1}(G)$ be finite for some positive integer n. Then $|G/Z(G)| \leq |\gamma_{n+1}(G)|^{d(G/Z(G))^n}$.

We know from [15] that if κ is an infinite cardinal and G is a group such that $|G'/Z(G) \cap G'| = \kappa$, then $|G/Z_2(G)| \leq 2^\kappa$. Now since $Z_2(G) = Z(G)$ for each group G in χ, one can obtain a bound for the index of the center in G by a function of the order of the derived subgroup G'.

COROLLARY 6

Let G be an infinite group such that $\Phi(G) \cap Z(G) = 1$, and $|G'| = \kappa$ (κ is an infinite cardinal). Then $|G/Z(G)| \leq 2^\kappa$ and this estimate is sharp.

2. *n*-Isoclinic family of groups in χ

In 1940, Hall [7] introduced a principle of classification for solvable groups (and, of course, for prime-power groups) that is based on some special equivalence relation on groups, called isoclinism. This relation yields a classification of all groups into mutually exclusive classes of groups, such that all abelian groups collapse; all abelian groups are equivalent to 1. Hall first replaced the relation of isomorphism by isoclinism, which is an equivalence relation weaker than isomorphism, and divided the class of all groups into families. Then he gave some of the internal structural properties of these families. Hekster [11] generalized the concept of isoclinism as follows.

DEFINITION 7

Two groups G and H are *n*-isoclinic if there exist isomorphisms

$$\alpha : \frac{G}{Z_n(G)} \to \frac{H}{Z_n(H)} \qquad \text{and} \qquad \beta : \gamma_{n+1}(G) \to \gamma_{n+1}(H),$$

such that $\beta([g_1, g_2, \ldots, g_{n+1}]) = [h_1, h_2, \ldots, h_{n+1}]$, where $g_i \in G$ and $h_i Z_n(H) = \alpha(g_i Z_n(G))$, for each $1 \leq i \leq n + 1$. In this case, we write $G \sim_n H$. The 1-isoclinic groups G and H are briefly called isoclinic and shown by $G \sim H$.

Observe that the concept of *n*-isoclinism yields an equivalence relation on the set of all groups, which becomes more and more weak as n increases. Moreover, it is not difficult to deduce from the definition that an *n*-isoclinism induces an $(n + 1)$-isoclinism.

Now, we recall that a group S is an *n*-stem group if it satisfies $Z(S) \subseteq \gamma_{n+1}(S)$. A stem group is a 1-stem group. The existence of at least one stem group in each isoclinic family was proved by Hall.

Lemma 8. *Let G be a group such that $G' \cap Z(G) = 1$ and S be a stem group isoclinic to G. Then $Z_n(S)$ is trivial for each n.*

Proof. Since $G \sim S$, we have $Z(G) \cap G' \cong S' \cap Z(S)$. Using Proposition 1, $S' \cap Z_n(S) = 1$ and $Z_n(S) \subseteq Z(S)$. Therefore $Z(S)$ is the hypercenter of S. This completes the proof. $\square$

Let ψ be an isoclinic family of groups G such that $Z(G) \cap G' = 1$. Since the center of any stem group in ψ is trivial, the family ψ has the unique stem group $G/Z(G)$. Therefore, we can obtain the following theorem.

Theorem 9. *Let G_1 and G_2 be two groups in χ. Then $G_1 \sim G_2$ if and only if $G_1/Z(G_1) \cong G_2/Z(G_2)$.*

Theorem 9 asserts that the non isomorphic structures of center factors of the groups in χ classify the set of all groups in χ up to isoclinism.

In the sequel, it is shown that each n-isoclinic family of groups in χ is contained in an isoclinic family when $n \geq 1$. But we first show that the existence of isomorphism α in Definition 7 is a sufficient condition for groups in χ to be n-isoclinic.

Theorem 10. *Let G_1 and G_2 be two groups in χ such that $G_1/Z_n(G_1) \cong G_2/Z_n(G_2)$. Then $G_1 \sim G_2$.*

Proof. By Lemma 2, we have $G_1/Z(G_1) \cong G_2/Z(G_2)$. Now, the result follows using Theorem 9. $\square$

COROLLARY 11

Let G_1 and G_2 be two groups in χ. Then $G_1 \sim G_2$ if and only if $G_1 \sim_n G_2$.

Using Theorem 9 and Corollary 11, one can see that a full classification of n-isoclinism families of groups in χ is achieved if and only if the exact structures of the center factor groups in χ are given.

Let G be a group and $H \leq G$. Hekster [11] proved that $H \sim_n HZ_n(G)$. In particular, if $G = HZ_n(G)$, then $G \sim_n H$. Conversely, if $G/Z_n(G)$ is finite and $G \sim_n H$, then $G = HZ_n(G)$.

COROLLARY 12

Let G be a group such that $G' \cap Z(G) = 1$, $H \leq G$ and G' be finite. Then $G \sim H$ if and only if $G \sim_n H$.

Proof. Suppose that $G \sim_n H$. The finiteness of G' implies the finiteness of $G/Z_n(G)$, and therefore, $G = HZ_n(G)$. Since $Z(G)$ is the hypercenter of G, so $G = HZ(G)$ as required. $\square$

In the following, we try to give some structural properties of the center factor groups of finite groups in χ. But first we need the following definition.

DEFINITION 13

A group G is said to be elementary if each subgroup of G has trivial Frattini subgroup.

A characterization of a finite group G with the property $\Phi(G) \cap Z(G) = 1$ is given as follows.

Theorem 14. *Let G be a finite group such that $\Phi(G) \cap Z(G) = 1$. Then $G = H \times Z(G)$, where $Z(G)$ is elementary abelian, $Z(H) = 1$ and $\Phi(H) = \Phi(G)$.*

Proof. Since $Z(G)$ is a normal abelian subgroup of G such that $Z(G) \cap \Phi(G) = 1$, then $Z(G)$ has a complement H in G, using Theorem 5.2.13 in [16]. Therefore $G' = H'$ and $G = H \times Z(G)$. Moreover, $\Phi(Z(G)) = 1$ because of $\Phi(G) = \Phi(H) \times \Phi(Z(G))$. Consequently, $Z(G)$ is an elementary abelian group and the proof is completed. $\square$

COROLLARY 15

Let G be a finite group such that $\Phi(G) \cap Z(G) = 1$. Then there is a subgroup H of G such that $H \sim G$ and H is an n-stem group.

3. c-Capability of groups in χ

In 1938, Baer [1] initiated a systematic investigation for finding the conditions under which a group G must fulfill in order to be the group of inner automorphisms of some group E, i.e., $G \cong E/Z(E)$. Following Hall and Senior [9], such a group G is called capable. The notion of c-capability of groups was introduced by Burns and Ellis [5] and also by Moghaddam and Kayvanfar [14] simultaneously.

DEFINITION 16

A group G is said to be c-capable if there exists a group E such that $G \cong E/Z_c(E)$.

Obviously, 1-capability implies capability and also c-capability implies 1-capability for a group. The capability of abelian groups has been investigated by various authors. Baer [1] described all capable abelian groups which are direct sums of cyclic groups. The next lemma explains Baer's result for finitely generated abelian groups.

Lemma 17. *Let G be a finitely generated abelian group written as $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ such that each integer n_{i+1} is divisible by n_i (with the convention that $n = 0$ for $\mathbb{Z}_n = \mathbb{Z}$). Then G is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.*

Although Burns and Ellis [5] gave an example of a capable group which is not 2-capable, they could show that for finitely generated abelian groups, c-capability and capability coincide. More precisely, they proved the following theorem.

Theorem 18. *Fix $c \geq 1$. A finitely generated abelian group G is c-capable if and only if it is capable.*

For an arbitrary group G, let $G \cong F/R$ be a free presentation for G and consider the canonical surjections

$$\pi : F/[R,_c F] \twoheadrightarrow G; \qquad c \geq 0,$$

where $[R, {}_{c}F]$ is defined inductively to be $[[R, {}_{c-1}F], F]$ and $[R, {}_{1}F] = [R, F]$. The c-th epicentral subgroup $Z_c^*(G)$ of G is the image in G of the c-th term of the upper central series of $F/[R, {}_c F]$, that is, $Z_c^*(G) = \pi(Z_c(F/[R, {}_c F]))$. We recall that $Z_1^*(G)$ is the epicenter of G and it seems to have been first studied in [4], and some authors refer to it as the precise center of G.

The next proposition provides some basic properties of epicentral subgroups of a group (see [5] for details.)

PROPOSITION 19

Let G be a group and $c \geq 1$.

   (i) *$Z_c^*(G)$ is the smallest normal subgroup of G contained in $Z_c(G)$, such that the quotient $G/Z_c^*(G)$ is c-capable.*
  (ii) *$Z_{c+1}^*(G)$ contains $Z_c^*(G)$.*
 (iii) *$Z_c^*(G) = 1$ if and only if G is c-capable.*

Now according to Proposition 19, we yield the upper epicentral series for a group G which is a family of characteristic subgroups

$$1 = Z_0^*(G) \subseteq Z_1^*(G) \subseteq Z_2^*(G) \subseteq \cdots .$$

It is obvious that the class of all c-capable groups is neither subgroup closed nor under homomorphic images. But this class is closed under direct product.

PROPOSITION 20 [14, Corollary 2.7]

Let $\{G_i\}$ for $1 \leq i \leq n$ be a family of groups. Then $Z_c^*(\prod_{i=1}^{n} G_i) \subseteq \prod_{i=1}^{n} Z_c^*(G_i)$.

According to Theorem 14, a finite group G in χ can be written as a direct product of its center and a suitable subgroup H. In this section, we intend to show that the c-capability of G depends only on the structure of $Z(G)$. Then some necessary and sufficient conditions for the c-capability of G are given.

In the sequel, we recall the definition of c-nilpotent multiplier which is a generalization of the Schur multiplier, and then state some related fundamental statements which will be needed here.

The notion of c-nilpotent multiplier for a group G with the free presentation F/R is denoted by $\mathcal{M}^{(c)}(G)$ and defined to be

$$\frac{\gamma_{c+1}(F) \cap R}{[F, {}_c R]}.$$

This notion is actually the Baer invariant of the group G with respect to the variety of nilpotent groups of class at most c, and it has been introduced in [2]. $\mathcal{M}^{(1)}(G)$ is the Schur multiplier of G which is simply denoted by $\mathcal{M}(G)$. One can see that the group $\mathcal{M}^{(c)}(G)$ is always abelian and independent of the choice of the free presentation for G (see [13]). There are many papers involving this concept and it can be found, for instance, in [12].

In 1998, Ellis [6] presented a formula for the c-nilpotent multiplier of the direct product of two groups as follows.

Theorem 21 [6, Theorem 5]. *Let S and T be groups whose abelianisations are finitely generated. Then there is an isomorphism*

$$\mathcal{M}^{(c)}(S \times T) \cong \mathcal{M}^{(c)}(S) \oplus \mathcal{M}^{(c)}(T) \oplus \Gamma_{c+1}(S^{ab}, T^{ab}).$$

The following two theorems which are proved in [14] will be also needed for our main theorem.

Theorem 22. *Let G be a group and $N \subseteq Z_c(G)$. Then there is a natural exact sequence*

$$\mathcal{M}^{(c)}(G) \xrightarrow{\theta} \mathcal{M}^{(c)}(G/N) \to \gamma_{c+1}(G) \cap N \to 1.$$

Theorem 23. *Let G be a group and $N \subseteq Z_c(G)$. Then $N \subseteq Z_c^*(G)$ if and only if the natural map $\mathcal{M}^{(c)}(G) \to \mathcal{M}^{(c)}(G/N)$ is monomorphism.*

The proof of the following lemma is straightforward.

Lemma 24. *Let G be a finite abelian group. The following statements are equivalent*:

(1) *G is an elementary group.*
(2) *G is a direct product of some its elementary abelian Sylow subgroups.*
(3) *The exponent of G is square free.*

Now, let G be a finite group such that $\Phi(G) \cap Z(G) = 1$. Then by Theorem 14, $G = H \times Z(G)$, in which $Z(H) = 1$ and $Z(G)$ is an elementary abelian group. Obviously H is c-capable and by Lemma 24, each Sylow p-subgroup of $Z(G)$ is an elementary abelian p-group. Now, if the rank of each Sylow p-subgroup of $Z(G)$ is greater than one, then $Z(G)$ is capable and therefore we can conclude the c-capability of G by the following theorem.

Theorem 25. *Let G be a finite group such that $\Phi(G) \cap Z(G) = 1$. If each Sylow subgroup of $Z(G)$ is non cyclic, then G is c-capable.*

Proof. Clearly $Z(G)$ is a finite capable group. Therefore $Z(G)$ is c-capable, by Theorem 18. Now, the result follows by Theorem 14 and Proposition 20.   □

Now, the case that $Z(G)$ has a cyclic Sylow p-subgroup will be verfied in the next two theorems.

Theorem 26. *Let G be a finite group such that $\Phi(G) \cap Z(G) = 1$ and $Z(G)$ has a cyclic Sylow p-subgroup. Then G is c-capable, if for each cyclic Sylow p-subgroup of $Z(G)$ we have $p \mid |G/G'Z(G)|$.*

Proof. Since $Z(G)$ is nilpotent, we can write $Z(G)$ as the direct product of its Sylow subgroups. Now, let $Z(G) = H_0 \times K$ be such that K is the direct product of cyclic Sylow subgroups of $Z(G)$, and H_0 is the direct product of the others. By Proposition 20, $Z_c^*(G) \subseteq Z_c^*(K)$, since $Z_c^*(H_0)$ is trivial. Therefore $Z_c^*(G)$ is a subgroup of cyclic group

K. Hence $Z_c^*(G)$ contains at least one cyclic Sylow subgroup of $Z(G)$, whenever $Z_c^*(G)$ is non trivial.

Now, suppose by the way of contradiction, G is non c-capable. Let $S_{p_1}, S_{p_2}, \ldots, S_{p_t}$ be the Sylow subgroups of $Z(G)$, in which $S_{p_1} \cong \mathbb{Z}_{p_1}$ be the cyclic Sylow p_1-subgroup of $Z(G)$ contained in $Z_c^*(G)$. By Theorem 14, $G \cong H \times Z(G)$ and therefore $G \cong H_1 \times \mathbb{Z}_{p_1}$, in which $H_1 = H \times S_{p_2} \times \cdots \times S_{p_r}$. Since G is finite, so $H_1/H_1' = H_1^{ab}$ is a finitely generated abelian group. Let $H_1^{ab} \cong \oplus_{i=1}^s \mathbb{Z}_{m_i}$. Since $G' \cap Z(G) = 1$, we have $\gamma_{c+1}(G) \cap \mathbb{Z}_{p_1} = 1$ and hence one can conclude that $\mathcal{M}^{(c)}(G) \cong \mathcal{M}^{(c)}(H_1)$, by using Theorems 22 and 23. But $\mathcal{M}^{(c)}(\mathbb{Z}_{p_1}) = 1$, so $\mathcal{M}^{(c)}(G) \cong \mathcal{M}^{(c)}(H_1) \oplus \Gamma_{c+1}(H_1^{ab}, \mathbb{Z}_{p_1})$, by Theorem 21. Notice that each direct summand of $\Gamma_{c+1}(H_1^{ab}, \mathbb{Z}_{p_1})$ is a tensor product of $(c+1)$ cyclic groups which involves at least one cyclic summand of H_1^{ab} and at least one $\mathbb{Z}_{p_1}$. If $L = (\otimes_{i=1}^c \mathbb{Z}_{p_1}) \otimes H_1^{ab}$, then it is easy to see that L is a subgroup of $\Gamma_{c+1}(H_1^{ab}, \mathbb{Z}_{p_1})$ and also $L \cong \oplus_{i=1}^s \mathbb{Z}_{(m_i, p_1)}$. But $\Gamma_{c+1}(H_1^{ab}, \mathbb{Z}_{p_1}) = 0$, because of the isomorphism $\mathcal{M}^{(c)}(G) \cong \mathcal{M}^{(c)}(H_1)$. Thus L is trivial and therefore $(m_i, p_1) = 1$, for each $1 \le i \le s$. It follows that $(p_1, |H_1^{ab}|) = 1$. This contradiction completes the proof. $\qquad\square$

Theorem 27. *Let G be a finite group such that $\Phi(G) \cap Z(G) = 1$ and $Z(G)$ has a cyclic Sylow p-subgroup. If G is c-capable, then for each cyclic Sylow p-subgroup of $Z(G)$ we have $p \mid |G/G'Z(G)|$.*

Proof. Using the notation used in the proof of Theorem 26, G can be written as the product $H \times H_o \times K$, in which K is the direct product of cyclic Sylow subgroups of $Z(G)$. Assume by the way of contradiction, there exists a cyclic Sylow subgroup of $Z(G)$, $S_{p_1} \cong \mathbb{Z}_{p_1}$ say, such that $p_1 \nmid |G/G'Z(G)|$. Assuming $N = S_{p_1}$, it is concluded that $G = H_1 \times N$. It is easy to see that $N \subseteq Z_c(G)$. Moreover $|G/G'Z(G)| = |H/H'|$, thus $(p_1, |H_1^{ab}|) = 1$. Using Theorem 21, we have $\mathcal{M}^{(c)}(G) \cong \mathcal{M}^{(c)}(H_1) \oplus \Gamma_{c+1}(H_1^{ab}, N)$. One can easily check that each tensor product of $(c+1)$ cyclic groups which involves at least one cyclic summand of H_1^{ab} and at least one $\mathbb{Z}_{p_1}$ is trivial. Therefore, $\Gamma_{c+1}(H_1^{ab}, N)$ is trivial and so $|\mathcal{M}^{(c)}(G)| = |\mathcal{M}^{(c)}(H_1)|$. On the other hand, $\gamma_{c+1}(G) \cap N = 1$. Hence $\mathcal{M}^{(c)}(G) \cong \mathcal{M}^{(c)}(H_1)$ and $N \subseteq Z_c^*(G)$, by Theorem 23. This gives the required result. $\qquad\square$

Invoking Theorems 25, 26 and 27, we have the following criterion for c-capability of finite groups in χ.

COROLLARY 28

Let G be a finite group such that $\Phi(G) \cap Z(G) = 1$ and $c \ge 1$. Then G is c-capable if and only if either every Sylow subgroup of $Z(G)$ is non cyclic or for every cyclic Sylow p-subgroup of $Z(G)$, we have $p \mid |G/G'Z(G)|$.

As an example, for Corollary 28, one can consider the group G to be $S_n \times \mathbb{Z}_m$, in which S_n is the symmetric group of degree $n > 2$ and $\mathbb{Z}_m$ is the cyclic group of order m with m a square free number. Then it is easy to see that G is c-capable if and only if $m = 1$ or 2.

Let G be a group such that $\Phi(G) \cap Z(G) = 1$, and c be a positive integer number. According to Lemma 8, any n-stem group which is n-isoclinic to G is c-capable. Therefore, we have the following result.

COROLLARY 29

Let n be a non negative integer and c ≥ 1. Then any n-isoclinic family of groups in χ has at least one c-capable group.

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COMMUNICATING EDITOR: B Sury