Simplification through regression analysis on the dynamic response of plates with arbitrary boundary conditions excited by moving inertia load

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Abstract

Dynamic response of a thin rectangular plate traversed by a moving inertia load with arbitrary boundary condition is investigated through this paper. The inertia effect of mass is considered and relevant formulation is established based on the full-term of acceleration, employing the method of Boundary Characteristic Orthogonal Polynomials, BCOP. To acquire the complete solution of partial differential equations governing on the plate, the Galerkin method is used to separate the temporal function from the spatial one. The problem is formulated in the state space and applying the numerical method of Matrix Exponential the complete solution would be achieved. In the numerical studies, a comprehensive parametric study is performed for both cases of loading when inertia effect is included or neglected. Several mass and aspect ratios for the plate with major types of boundary conditions CCCC, SSSS, CFCF and SFSF are accounted for presenting the results. Dynamic amplification factor against velocity parameter is scrutinized within many graphs alongside with a time history analysis of dynamic deflection for the plate’s mid-span. Investigating on the dynamic response concludes to the critical boundary condition upon moving mass. By introducing a conversion factor, the margin of inertia and the critical velocity where happened would be achieved, then through a regression analysis a curve fitting model of polynomials is proposed. Corresponding coefficients testify the goodness of fit for such regression which are reported within tables. Referring to this simplified model of conversion factor pertaining to the specific boundary condition, it would be possible to handle the problem in moving load case without undertaking the complexities arisen from inertia contribution into the formulation. Having derived the factor from simplified model which has been calculated for a specific mass and velocity ratio, then multiplying into the moving load response, the complete solution for moving mass would be achieved.

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1. Introduction

Dynamic analysis of structures under moving load excitation is one of the most interesting field in structural dynamics. Because of the vast application of these loadings in engineering area, many researchers have paid their attention to this issue. Rail roads, runways, bridges, cranes and deck of ships under action of planes in takeoff or landing are just a few

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examples of the moving load applications in the real world. Beams and plates are two main choices for dynamic behavior modeling of real structures, have been utilized in the most studies.

Ouyang [1] has summarized a variety of engineering problems which are pertaining to the structures under moving load excitations. Fryba [2], established a perfect reference for simple situations with several analytical solution methods, provided the first and most comprehensive studies on the moving load problem. In a general classification one can study a moving load problem with and without inertia effects of moving load, in which moving mass phrase is usually assigned to the problems including the inertia effects of load. In this way, Akin and Mofid [3] studied the Euler-Bernoulli beam with various boundary conditions under moving mass excitation using discrete element technique. Their results highlighted the importance of moving mass inertia, specifically in high velocities. However their studies were only concerned with the vertical component of acceleration of moving mass, thus disregarding the other convective acceleration terms. Esmailzadeh and Ghsaishi [4] investigated the dynamic behavior of simply supported Euler-Bernoulli beam acted upon by the uniform partially distributed moving masses or loads. They reported the significance of moving mass inertia as well as the length of the distributed moving load that has a substantial effect on the dynamic response of the system. Lee [5] studied the dynamic behavior of simply supported Timoshenko beam excited by a concentrated moving mass. He also examined the possibility of separation of moving mass from the supporting beam during the course of motion by monitoring the variation of the contact force between them. In similar investigations, the effect of mass weight and velocity of the moving mass on the maximum dynamic response of the Timoshenko beam were reported as crucial [6]. Nikkhoo et al. [7] studied the effect of convective acceleration components of moving mass on dynamic behavior of a simply supported Euler-Bernoulli beam by utilizing the Eigenfunction Expansion Method (EEM), to solve the constitutive equation of motion. In their results, a so called critical velocity is introduced as a function of beam’s span and fundamental period as well as the moving mass weight. They concluded that the effect of convective acceleration terms could no longer be disregarded for masses moving with velocities larger than this critical velocity. Their findings were also cited by other researchers [8,9]. Rao [10] explored the behavior of simply supported beam under moving load excitation including its inertial effects, employing the mode superposition and the Multiple Scale Method (MSM). He showed that the effect of moving mass on the transient response of beam is notable. In this study the vibration of plate under moving mass excitation is in consideration which could reflect a better model of “bridge-like” structure’s response under moving load excitations. A vast studies have been performed for free vibration analysis on the various types of plates, but as brief review of literatures in plate assessment traversed by moving loads in a general term, Cifuentes and Lalapet [11] used a finite element discretization of the plate consistent with the trajectory of the moving mass in order to study the dynamic behavior of rectangular Kirchhoff plates excited by an orbiting mass. They investigated the deflection time history of the plate at its center point under moving mass and load excitation. Although only the vertical component of acceleration orbiting mass was considered in their study, the results showed the importance of orbiting mass inertia. Gbadeyan and Oni [12] explored the dynamic behavior of Kirchhoff rectangular plates as well as Euler-Bernoulli and Rayleigh beams with various boundary conditions under moving masses. They employed modified generalized finite integral transforms and modified Struble’s method to solve the resulting differential equations. In their works the complete acceleration terms of moving mass acted on rectangular plate are considered, with the motion path being parallel to any of the plate’s boundaries. Moreover their results were limited to some time-history responses of the plate’s center point deflection for moving force and mass excitation highlighted the significance of moving mass inertia. Shadnam et al. [13] investigated the dynamic response of a simply supported Kirchhoff rectangular plate excited by a moving mass traveling on an arbitrary trajectory. They employed the Eigenfunction Expansion Method to solve the differential equation of motion. Their results again pointed out the importance of the moving mass inertia as well as the effect of higher vibrational modes in capturing the exact dynamic response of the system. However, only the vertical acceleration component of the moving mass was considered in their studies. The full term formulation of this problem has been provided by Fryba [2] in which he concluded that the solution to such problems is very difficult to manage. Nikkhoo et al. [14] investigated the parametric study of a rectangular Kirchhoff plate traversed by a moving mass. The mass was traveled on an arbitrary trajectory on the plate surface. They evaluated the approximate and exact expression of inertial effects on the dynamic response of the system. Eigenfunction Expansion Method was employed to transform the partial differential equations of motion into a number of coupled ordinary differential equations. A parametric study was performed on the effect of different mass weights and velocities of moving mass as well as the aspect ratio of the plate on the maximum dynamic response of its center point. The results indicate that for any specific mass weight and plate’s aspect ratio, the output of the approximate formulation will no longer acceptable for mass velocities above certain limits. Therefore, for large moving mass velocities, the exact formulation problem should be considered instead. Takabatake [15] considered a discontinuous vibration of bending stiffness and mass of the plate with variation of the thickness using a characteristic function and presented an analytical method for rectangular plates under moving loads. Huang et al. [16] developed a procedure incorporating the finite strip method to treat the response of rectangular plate on elastic foundation subjected to a moving point loads. Ghazvini et al. [17] introduce a robust computational approach to perform the transverse vibration of a thin rectangular plate of varying thickness under a traveling mass using Eigenfunction Expansion Method. Nikkhoo et al. [18] proposed a semi-analytical model to study the response of a thin rectangular plate subjected to series of moving inertial loads by using Eigenfunction Expansion Method. Gbadeyan and Dada [19] investigated the elasto-dynamic response of a rectangular Mindlin plate subjected to a distributed moving mass by using finite difference algorithm to transform the differential equation into a set of linear algebraic equations. Amiri et al. [20] based on first-order shear deformation plate theory, studied the response of a rectangular Mindlin plate under a moving mass by using direct separation of variables and Eigenfunction Method. Eftekhari and Jafari [21] presented
a combined application of Ritz method, Differential Quadrature Method (DQM) and Integral Quadrature Method (IQM) to conduct the vibration response of rectangular plate subjected to accelerated traveling mass. In that paper the Ritz method with beam eigenfunctions were used to discretize the spatial partial derivatives, and the Differential Quadrature Method and Integral Quadrature Method were employed to analogize the resultant system of partial differential equations, then the Newmark time integration scheme was used to solve the ordinary differential equations. Wu [22] investigated the vibration of rectangular plate under moving force along a circular path using Finite Element Method (FEM) associated with equivalent forces and moments. Wu [23,24] presented a moving distributed element to perform the dynamic analysis of an inclined plate under moving (distributed) loads using Finite Element Method. An equivalent finite element to analyze the transverse vibration of the plate under a moving point mass excitation was studied by Esen [25]. Recently a comprehensive investigation of a thin plate that vibrated by a moving load (force and mass) with arbitrary boundary conditions was performed by Qinghua et al. [26]. In that study the governing differential equations was derived using the Lagrange equation and updated Rayleigh-Ritz method associated with courant's penalty was employed to deal with the spatial partial derivatives. The admissible functions just satisfy a totally unconstrained condition. Then the Differential Quadrature Method was used for discretization of temporal derivatives.

State-space formulation as a powerful method in the vibration analysis of plates is vastly applied within papers such as [27–30], which is adoptive method in current paper. Now, through this study the dynamic analysis of thin rectangular plate with arbitrary boundary condition excited by moving load and mass is assessed so that the results reflect the importance of mass weight and velocity of load, traveling on the length of plate. Four major types of CCCC, SSSS, CCF and SFSF of boundary conditions would be upon the evaluation and the dynamic response of plate’s center point in a dimensionless form of deflection against the velocity parameter of moving load is depicted through several graphs for both types of loadings (i.e. included or excluded of inertia effect). Partial differential equation of motion governing on the plate is derived [2], and using Galerkin method the complete response is proposed as a series of two separated spatial and temporal functions multiplied to each other. In the problem formulation incorporates the full terms of acceleration when the mass with all inertia effects moves on the plate with constant velocity. Having applied the Boundary Characteristic Orthogonal Polynomials (BCOP) method [31] the mode shapes are generated and a complete free vibration analysis would be carried out employing 62 computational modes, against the most previous studies where the mode numbers usually don’t exceed 25. This could testify the accuracy of results and the convergence of the method. Unlike the common proposed Eigenfunction Expansion Method (EEM), this solution is very handy and straightforward independent of limitations imposed by boundary conditions which confined the EEM to use in the problems with at least two simply supported edges against each other. Proceeding the work by utilizing the mode shapes into the form of Galerkin method leads to separate of the proposed response function into spatial and temporal ones. Substituting this series into the original Partial Differential Equation yields to an Ordinary Differential Equation in time domain. To capture the results in time domain, the problem would be modeled in the state space formulation, then using the powerful method of Matrix Exponential, the complete solution of dynamic response for the plate at plate’s center point would be achieved. Through the study, three types of results would be presented within several graphs which covered four above mentioned major types of boundary conditions and the problem has been solved for moving load and moving mass conditions, simultaneously. This provide an insight for readers to see and compare the influence of inertia within all case studies. The results would be reported as three types of time-history of dynamic deflection, Dynamic Amplification Factor (DAF) and Corrective factor ($\beta$) spectrums, which the first one is dimension-based and presented through graphs against time, and both last ones are dimensionless which listed within graphs against non-dimensional velocity, $V$. The reference point for calculations is considered at the plate’s center point. Velocity parameter, $V$, is defined as dividing the velocity of load moving on the rectilinear path parallel to the length of plate, by the critical velocity of moving load [7]. By performing a comprehensive parametric study in this paper, the problem would be explored for five mass ratios and varieties of velocity parameter, along with the three values for plate’s aspect ratio. One of the novel aspects of this study is to investigate on the aspect ratio effect on the dynamic behavior of plates with arbitrary boundary conditions, where such sensitivity analysis on the aspect ratio would bring out important insights into bridge-like structures design for engineers. As mentioned earlier, in the calculation of dynamic response of plate, including the inertia effect of moving load where all terms of acceleration is encountered, the complexity of problem is risen up remarkably. Due to this complexity, most engineers prefer to avoid engagement with such sophistications in computations. In this work, as an important simplification, a conversion factor $\beta$, is introduced which is defined as a ratio of dynamic response of the plate for two situations (i.e. when inertia effects is considered or neglected). This factor is evaluated for several mass ratios and through a regression analysis simplified to a polynomial function of mass and velocity ratios. The results are presented in form of three-dimensional surfaces, derived from a regression analysis. Based on the robust dynamic analysis of plates with arbitrary boundary condition through the paper, this simplified model for the plate’s vibration is presented for the first time. Having chosen the mass and velocity, one can easily get the corresponding $\beta$ factor. The main idea behind this simplification is to run the problem without inertia incorporation, when the factor is near unity, and necessity of incorporating the inertia terms into the problem treatment when the factor takes values far from unity. The conversion coefficients in regression model are listed into tables and by using them, it can remarkably reduce the volume of computations. By evaluating on the regression model for various boundary condition of plates, the most critical boundary condition which needs to consider the inertia effects within formulation would be recognized, a thing which has not investigated before.
2. Governing equations

A thin rectangular plate with arbitrary boundary condition is under consideration in a way that always satisfies the Kirchhoff Theory assumptions. Let consider a mass moving on the plate surface in arbitrary trajectory with constant velocity with time dependent coordinates of \(x_0(t)\) and \(y_0(t)\), denote the path in Fig. 1.

Deflection of the middle plane points of plate are specified by \(w(x, y, t)\) with coordinates of \(x\) and \(y\) in any time of \(t\). On the other hand, the initial conditions for plate represented by continuous functions, \(g_1(xy)\) and \(g_2(xy)\) that \(w(x, y, 0) = g_1(xy)\) and \(\frac{\partial w(x, y, 0)}{\partial t} = g_2(x, y)\). Considering the inertial effects of mass in moving on the plate and assuming the infinitesimal strains into Hamilton principle lead to the below governing partial differential equation:

\[
D \nabla^4 w + \mu \frac{\partial^2 w(x, y, t)}{\partial t} = m \left(-g - \frac{d^2 w_0(t)}{dt^2}\right) \delta(x - x_0(t)) \delta(y - y_0(t))
\]  

(1)

In the above equation, \(\mu\) is the mass per unit area of the plate, \(D\) is the bending stiffness of the plate and is defined as \(\frac{Eb^3}{12(1-\nu^2)}\), in which \(E\), \(b\) and \(\nu\) are module of elasticity, plate thickness and Poisson Ratio, respectively, and assumed to be constant. The parameter \(\delta\) is Delta-Dirac function, \(m\) and \(g\) denote the magnitude of moving load and gravity acceleration respectively, \(w_0(t)\), measures the vertical displacement of mass \(m\) and is defined as \(w_0(t) = w(x(t), y(t), t)\) with the assumptions that the full contact condition between moving mass and plate’s surface is met. Expanding the time derivate of \(w_0(t)\) leads to:

\[
\frac{d^2 w_0(t)}{dt^2} = \begin{bmatrix}
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \left(\frac{dx}{dt}\right)^2 + \frac{\partial^2 w}{\partial y^2} \left(\frac{dy}{dt}\right)^2 + 2 \frac{\partial^2 w}{\partial x \partial y} \left(\frac{dx}{dt}\right) \left(\frac{dy}{dt}\right) + 2 \frac{\partial^2 w}{\partial x \partial t} \frac{dx}{dt} + 2 \frac{\partial^2 w}{\partial y \partial t} \frac{dy}{dt}
\end{bmatrix}
\]

\[
\begin{aligned}
\frac{\partial w}{\partial x} \left(\frac{dx}{dt}\right)^2 + \frac{\partial w}{\partial y} \left(\frac{dy}{dt}\right)^2 + \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}\end{aligned}
\]

\[
\begin{cases}
x = x_0(t) \quad y = y_0(t)
\end{cases}
\]

(2)

Some researchers and engineers truncate the convective terms of above equation for simplification in calculations that yields to an approximate formulation. Nikkhou and Rofoei [14] applied the full term formulations and performed a comparative study showing the importance of the plate’s response, affected by a moving mass. Here, the full term formulation of Eq. (2) is considered and all convective terms would have contribution into computations. In order to solve Eq. (1), the Galerkin method in a general application is applied, which separate the proposed solution function into spatial and temporal ones as Eq. (3), in which \(w(x, y, t)\) denotes the plate deflection function.

\[
w(x, y, t) = \sum_{j=1}^{N} \varphi_j(x, y) Q_j(t)
\]

(3)

Where \(\varphi_j(x, y)\) denotes the mode shape of vibrated plate under moving load that would be generated through Boundary Characteristic Orthogonal Polynomials, BCOP, procedure here. This method is a powerful and reliable tool for analysis of the vibrated plates, specifically for varieties of boundary conditions, unlike the analytical and most common used methods that are just applicable for simply supported edges. In this method, mode shapes are generated by continues functions that must satisfy all of the boundary conditions and the orthogonality properties between each two functions are complied. The time dependent modal amplitude of plate describing the time variation of any point in middle plane of plate is denoted by \(Q_j(t)\) where \(N\), shows the number of computational modes.
3. Gram–Schmidt orthogonalization procedure

As mentioned in the previous section, in this study, the mode shapes are generated by polynomials that are orthogonal to each other. The well-known Gram–Schmidt procedure constitutes the basis of BCOP method which insure this property. A series of original functions like \( f_i(x) \) is proposed, then by following steps, one can easily generate a corresponding orthogonal set of functions from the original ones, that is called as \( \varphi_i(x) \) here:

\[
\varphi_1 = f_1
\]

\[
\varphi_i = f_i - \sum_{j=1}^{N} \alpha_{ij} \varphi_j
\] (4)

In which the coefficients of \( \alpha_{ij} \) are defined as below:

\[
\alpha_{ij} = \frac{\langle f_i, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} = \frac{\int_a^b w(x)f_i(x)\varphi_j(x)dx}{\int_a^b w(x)\varphi_j(x)\varphi_j(x)dx}
\] (5)

\( \langle f_i, f_j \rangle \), generally denotes the inner product of two functions as defined in integral form regarding an appropriate weight function of \( w(x) \)[31]. Throughout all of this study, the variables which imply as a function, the matrices and also all of variables with vector entity have been denoted by bold letters. Usually for common shapes of isotropic plates with constant thickness, the weight function is taken as unity. Applying the orthogonal functions, specifically polynomials in the above mentioned method, the procedure will significantly be simplified and results in a straightforward solution. Variable, \( \mathbf{x} \), in the original functions of \( f_i(x) \) is regarded as a vector, so plates (as a special case of two-dimensional domains) are modeled by two independent variables function of \( x \) and \( y \). Bhat (1978) utilized the two variables orthogonal polynomials to analyze the dynamic behavior of plates through Ritz method but in its procedure the boundary conditions were satisfied just for the first mode. So, by increasing the computational modes in higher natural frequencies, Bhat’s method was deviated from accuracy. Chakraverty [31] overcame this problem by applying a new approach of utilizing the orthogonal polynomials where boundary conditions must be complied in all mode shapes using a geometrical function pre-multiplied to each orthogonal function, ensuring that the boundary conditions are held in all vibrational modes. This method is usually addressed by Boundary Characteristic Orthogonal Polynomials (BCOP) in literatures. The work is initiated by a set of well-known triangle of \( 1, x, y, x^2, xy, y^2, \ldots \). Utilizing an easy transformation, one can derive a non-dimensional form of mode functions in \( \xi - \eta \) coordinates as Eq. (6) that resulted in significant simplification of the procedure.

\[
x = a\xi + (b \cos \alpha)\eta
\]

\[
y = (b \sin \alpha)\eta
\] (6)

In which \( a, b \) are length and width of the plate respectively, and \( \alpha \) shows the acute angle of parallelogram (that in special case of rectangles is taken as 90°). So, the mode shapes are generated in new coordinates, over a unit square. By an inverse form of Eq. (6), the original mode shapes would be accessible.

Now, the procedure could be initiated by below set of polynomials of \( \xi \) and \( \eta \):

\[
g(\xi, \eta)\{1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \ldots\}
\] (7)

Where \( g(\xi - \eta) \) is the function that ensures all geometric boundary conditions are satisfied for all modes. This property provides the most significant effect on the convergence of numerical computations while the computational modes are increased.

\[
g(\xi, \eta) = \xi^p(1-\xi)^q\eta^r(1-\eta)^s
\] (8)

Eq. (8) defines a general form of \( g \) function. Through assignment of appropriate values of 0, 1 and 2 to any power parameters of \( p, q, r \) and \( s \), all types of boundary conditions could be generated easily. In a fashion that, 0 denotes the free edge, 1 for simply supported constraints and 2 is utilized to showing the clamped supports. So, the Gram-Schmidt procedure over new coordinates are formulated as below [31]:

\[
\varphi_i = F_i \quad \varphi_i = F_i - \sum_{j=1}^{i-1} \alpha_{ij} \varphi_j \\
\alpha_{ij} = \frac{\langle F_i, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle}, \quad j = 1, 2, \ldots, (j-1)
\] (9)

In which

\[
F_i = g(\xi, \eta)f_i(\xi, \eta)
\] (10)
Utilizing these BCOPs in the Rayleigh-Ritz method, one can easily derive the frequency as below:

\[ w(x, y) = \sum_{j=1}^{N} C_j \varphi_j \]  
\[ \omega^2 = \frac{\int_{R} \left( \nabla^2 w \right)^2 + 2(1 - \nu) \left\{ w_{xy}^2 - w_{xx} w_{yy} \right\} \, dx \, dy}{\rho h \int_{R} w^2 \, dx \, dy} \]

In the above expressions, \( w(x, y) \) in Eq. (11) denotes the proposed shape functions in Rayleigh–Ritz method. By replacing the mode shape functions in terms of \( \xi - \eta \) into Eq. (12), Rayleigh quotient, the standard eigenvalue equation is derived:

\[ \sum_{j=1}^{N} (a_{ij} - \lambda^2 b_{ij}) C_j = 0 \]

\( a_{ij}, \, b_{ij} \) and \( \lambda \) are defined as below:

\[ a_{ij} = \int_{R} \left\{ \varphi_i^{\xi} \varphi_j^{\xi} + B_1 \left( \varphi_i^{\xi \eta} \varphi_j^{\eta} + \varphi_i^{\xi \eta} \varphi_j^{\xi} \right) + B_2 \left( \varphi_i^{\xi \xi} \varphi_j^{\eta} + \varphi_i^{\xi \eta} \varphi_j^{\eta} + \varphi_i^{\xi \eta} \varphi_j^{\xi} \right) + B_3 \varphi_i^{\xi \xi} \varphi_j^{\xi \eta} \right\} \, d\xi \, d\eta \]

\[ b_{ij} = \int_{R} \varphi_i \varphi_j \, d\xi \, d\eta \]

\[ \lambda^2 = \frac{a^4 \omega^2 \rho h}{D} \]

In the above equations upper scrips show the derivatives respect to \( \xi \) and \( \eta \) parameters where \( \lambda \) represents the frequency parameter. Also the coefficients of \( B_1, B_2, ..., B_5 \) are listed below:

\[ B_1 = -2r \cos \alpha \]
\[ B_2 = r^2 \left( \nu \sin^2 \alpha + \cos^2 \alpha \right) \]
\[ B_3 = 2r^2 \left( 1 - \nu \sin^2 \alpha + \cos^2 \alpha \right) \]
\[ B_4 = -r^3 \cos \alpha \]
\[ B_5 = r^4 \]
\[ r = \frac{a}{b} \]

In the Eqs. (17–22), \( r \) and \( \nu \) represent the aspect ratio of plate and Poisson ratio respectively.

4. Matrix Exponential Method (MEM)

Solution in time domain and computing the function \( Q(t) \) is derived by a powerful iterative numerical method based on state-space formulation named Matrix Exponential Method [32]. By substituting the Eq. (3) into Eq. (1) and employing the orthogonality of modes, one can rearrange the differential equation of motion of plate in the matrix form as below:

\[ \mathbf{M}(t) \ddot{\mathbf{Q}}(t) + \mathbf{C}(t) \dot{\mathbf{Q}}(t) + \mathbf{K}(t) \mathbf{Q}(t) = \mathbf{E}(t) \]

\[ \mathbf{Q}(t_0) = \mathbf{Q}_0 \]

\[ \dot{\mathbf{Q}}(t_0) = \dot{\mathbf{Q}}_0 \]

In which, Eqs. (24)–(25) represent the initial conditions of vibrating plate as initial modal amplitude and initial modal velocity, respectively. In this study, it is proposed that the plate is in rest situation initially. \( \mathbf{M}(t), \, \mathbf{C}(t) \) and \( \mathbf{K}(t) \) in the
Eq. (23) represent the mass, damping and stiffness matrices respectively, which have been constituted by below components:

\[ M_{ij} = m_{ij} + m \phi_i(x_0(t), y_0(t))\left[ \varphi_j(x_0(t), y_0(t)) \right] \]  \hspace{1cm} (26)

\[ C_{ij} = 2m \varphi_i(x_0(t), y_0(t))\left[ \dot{x}_0(t) \varphi_{j,y}(x_0(t), y_0(t)) + \dot{y}_0(t) \varphi_{j,y}(x_0(t), y_0(t)) \right] \]  \hspace{1cm} (27)

\[ K_{ij} = k_{ij} + m \varphi_i(x_0(t), y_0(t))\left[ \ddot{x}_0(t) \varphi_{j,xx}(x_0(t), y_0(t)) + \ddot{y}_0(t) \varphi_{j,yy}(x_0(t), y_0(t)) + \dot{x}_0(t) \varphi_{j,x}(x_0(t), y_0(t)) + \dot{y}_0(t) \varphi_{j,y}(x_0(t), y_0(t)) + 2\dot{x}_0(t)\dot{y}_0(t) \varphi_{j,xy}(x_0(t), \dot{y}_0(t)) \right] \]  \hspace{1cm} (28)

Where,

\[ m_{ij} = \int_{R} \rho h \varphi_i(x, y) \varphi_j(x, y) dx dy \]  \hspace{1cm} (29)

\[ k_{ij} = \int_{R} \left[ D \frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial^2 \varphi_j}{\partial x^2} + vD \left( \frac{\partial^2 \varphi_i}{\partial y^2} + \frac{\partial^2 \varphi_i}{\partial x^2} \right) + D \frac{\partial^2 \varphi_i}{\partial y^2} + 4 \left( \frac{Gh^3}{12} \right) \left( \frac{\partial^2 \varphi_i}{\partial x \partial y} \frac{\partial^2 \varphi_j}{\partial x \partial y} \right) \right] dx dy \]  \hspace{1cm} (30)

\[ D = \frac{Eh^3}{12(1 - \nu^2)} \]  \hspace{1cm} (31)

\[ E_j = -mg \varphi_j(x_0(t), y_0(t)) \]  \hspace{1cm} (32)

The state-space representation of equation is:

\[ \dot{X}(t) = A(t)X(t) + F(t) \]  \hspace{1cm} (33)

\[ A(t) = \begin{bmatrix} 0 & M^{-1}K & I \\ -M^{-1}E & 0 & 0 \end{bmatrix}_{2N \times 2N} \]  \hspace{1cm} (34)

\[ F(t) = \begin{bmatrix} 0 \\ -M^{-1}F \end{bmatrix}_{2N \times 1} \]  \hspace{1cm} (35)

\[ Q(t) = \begin{bmatrix} Q_1(t) \\ \vdots \\ Q_N(t) \end{bmatrix}_{2N \times 1} \]  \hspace{1cm} (36)

\[ X(t) = \begin{bmatrix} Q(t) \\ \dot{Q}(t) \end{bmatrix}_{2N \times 1} \]  \hspace{1cm} (37)

In the above formulations, \( U(t) \) is called fundamental solution matrix defined as below:

\[ X(t) = U(t)U^{-1}(t_0)X(t_0) + \int_{t_0}^{t} \left[ U(t)U^{-1}(t)[F(\tau)] \right] d\tau \]  \hspace{1cm} (38)

\[ \dot{U}(t) = A(t)U(t)X(t_0), \quad U(t_0) = I \]  \hspace{1cm} (39)

\[ X(t) = U(t)X(t_0) \]  \hspace{1cm} (40)

Moreover, a transfer matrix is utilized to obtain \( U(t) \) such as:

\[ \phi(t, \tau) = U(t)U^{-1}(\tau) \]  \hspace{1cm} (41)

\[ X(t) = \phi(t, \tau)X(\tau) \]  \hspace{1cm} (42)
An approximate solution can be used to obtain \( \Phi \):

\[
\Phi(t_{k+1}, t_k) = e^{A(t_k)\Delta t_k},
\]

in which \( \Delta t_k = t_{k+1} - t_k \) is a specific time interval. Assuming the existence of \( A^{-1}(t_k) \), the Eq. (33) would easily be solved and finally yields to:

\[
X(t_{k+1}) = A_1(t_k)X(t_k) + F_1(t_k)
\]

\[
A_1(t_k) \equiv e^{A(t_k)\Delta t_k}
\]

\[
F_1(t_k) \equiv [A_1(t_k) - I]A_1^{-1}(t_k)F(t_k)
\]

5. Numerical studies

A typical Aluminum plate with various boundary conditions under moving mass excitation is considered here. The modulus of elasticity of plate is \( E = 73.1 \) Gpa, the mass density is \( \rho = 2700 \) kgm\(^{-3} \), plate thickness, \( h = 17 \) mm and Poisson ratio is \( \nu = 0.33 \) are assumed. The length of plate, \( a = 2 \) m and width is \( b = 2 \) m that is the plate’s aspect ratio parameter which defined as \( r = \frac{a}{b} \) would be equal to 1 (for a square plate). In this study the mass is traveling along the length of plate, on a rectilinear path with constant velocity. It is assumed that the plate originally is at rest and the pure rolling and full contact condition between traveling mass and plate is met. Natural frequencies and mode shapes are derived through a free vibration analysis by means of aforementioned method in the previous sections. The magnitude of mass weight is considered by non-dimensional quantity as a ratio of the moving mass magnitude divided by total plate’s mass, within values 0.05, 0.1, 0.15, 0.2 and 0.25, denoted by \( M \).

For instance, a mass ratio of 0.2 is interpreted as 20 percent of plate’s mass, travelling on the plate and showed by \( M = \frac{m_{p}}{\rho hab} = 0.2 \), where \( h \) is the plate’s thickness. Accordingly, non-dimensional velocity of moving mass is calculated by a velocity parameter defined as \( v' = \frac{v}{\sqrt{f_{p}}} \) where, \( a \) is the length of plate (proposed here to be 2 m) and \( f_{p} \) is the first fundamental period of plate derived from free vibration analysis for each boundary condition through Rayleigh–Ritz method. So, non-dimensional velocity could easily be defined as \( v' = \frac{v}{\sqrt{f'}} \), which could range within 0 to 1 interval. A time history analysis for each case of boundary condition is presented at the first of each case studies. The selected numerical value for the thickness has been assigned to comply the Kirchhoff plate hypothesis. As well as, the assigned values for mass magnitude in time history analysis has been performed by just mathematical justification to show better the dynamic response of plates. Actually, the strength mechanics of plate’s material has not been involved within the numerical analysis, when the time histories were being produced. To investigate the dynamic response of plate’s center point, the parameter of Dynamic Amplification Factor (DAF) is calculated for each case and depicted in graphs showing the trend of this parameter against dimensionless velocity, increasing up to unity. One can derived the DAF through, \( \text{DAF} = \frac{W_{\text{max, dynamic}}}{W_{\text{static}}} \), in which \( W_{\text{static}} \) denotes the plate deflection at center point upon static concentrated load and \( W_{\text{max, dynamic}} \) shows the maximum deflection of reference point of the plate where is located at mid-span point upon a concentrated mass passes the plate’s surface. In each case, the problem would be solved for both cases of moving load without inertia contribution and moving mass for inertia inclusion. This means that the inertia effects are neglected in the first case and considered for the later. The results are presented through figures, in a way that each figure include the DAF of plate acted upon loadings with and without inertia effects simultaneously. By tracing each DAF curve carefully, varying the velocity of load moving on the plate it would be deduced that the DAF difference between figures are increased therefore the inertia effects play an important role on the dynamic response of plate and should not be neglected in computations. As mentioned above, the plate is proposed to be on rest condition before arriving the moving mass, hence we can write \( Q_{0} = Q = 0 \). Four types of most applicable boundary conditions are investigated separately, summarized as \( CCCC, SSSS, CFCF \) and finally \( SFSS \), in which, \( C, S \) and \( F \) letters denote the clamped, simple and free condition of supports, respectively. These models reflect the dynamic behavior of plate from most stiff condition toward most soft. Surveying on the literatures related to numerical methods using in the dynamic analysis of integrated systems, the number of computational modes usually limited to 25. It also has to be mentioned that increasing the number of modes contributed into calculations, would lead to mitigate the errors as deviation from exact solution, but on the other hand, the risk of numerical instability and the run time of procedure could be risen up seriously. Here, as another important result of this study, we can see the robustness of BCOP method for using in the moving load problem while the computational mode numbers are increased. This would be testified by assessing the frequency parameter while the number of computational modes are increased as reported within Tables 2,4,6 and 8 upon follow subsections. This fact attests the other important advantage of this method beyond its simplicity and straightforwardness. Regarding to aforementioned comments, in this study, all modals calculations are performed for 40 computational modes that could greatly affect the accuracy of the results. Before arrive into the work, a comparative study would be performed that shows an interesting consistency for output curves, derived from BCOP method and Eigenfunction Expansion solution, depicted in a same graph through the next section.

5.1. Verification

Many problems in structural dynamics analysis could simply be run through analytical procedures, but boundary conditions are the main restriction, there. This has pushed researchers to use such numerical methods instead of analytical
ones. In spite of the fact that approximate methods are efficient and the only way to solve complicated problems, their results must be assessed and verified in the cases that analytical solutions are available, or an experimental work must be performed to prove their efficiency.

For this reason and to show the precision of BCOP method in dynamic analysis of plates, a problem is investigated here. A four edges, simply supported plate, SSSS, under moving mass excitation was analyzed by Nikkho et al. [14], in which, the analytical procedure of Eigenfunction Expansion Method was employed and several results were presented there. As a verification, a typical case of that study is selected and investigated here through BCOP method. The response curves of plate’s center point are presented within a same graph, Fig. 2.

All the assumptions in that paper are held here and the procedure for 62 computational modes is run again, such that Fig. 3 shows a very good agreement (especially for high velocities), reflecting a complete adaptation in the results. Many literatures have been produced in the moving mass problem, emphasizing on the fact that as the velocity of load moving on the structure increases, inertia effect of load, plays an important role in the stability and dynamic response of the structure and hence cannot be disregarded in formulations.

As mentioned above, Fig. 3 could testify the accuracy of BCOP method. Based on the remarkable and satisfactory results obtained (especially for the velocity parameters within 0.8 to 1), the next sections will be established and presented.

5.2. Fully-Clamped plate, CCCC

Fig. 4 shows a square plate with clamped boundary condition on all edges, typically undergoes a mass moving on a rectilinear path. This case could be considered as an extreme model which denotes the most-stiffest condition of such structures, geometrically. Some metallic plates in the bridge structure which are welded in all edges to their foundations could be regarded as an application of this case of boundary conditions, when oscillate due to moving masses. For this case, an analytical solution could not be found easily. Applying the BCOP method to take the parameters p, q, r and s to 2 as mentioned in Section 3, the mode shapes are generated accordingly:

\[ F = \xi^2 (1 - \xi)^2 \eta^2 (1 - \eta)^2 \left\{ 1, \xi, \eta, \xi^2, \eta^2, \xi \eta, \xi^3, \eta^3, \ldots, \xi^3 \eta^3 \right\} \]
Fig. 4. A square plate with all edges clamped boundary condition, CCCC, under moving mass traveling on a rectilinear path with constant velocity.

Table 1
The fundamental period of plate with boundary condition CCCC for three values of aspect ratio.

<table>
<thead>
<tr>
<th>Aspect ratio (r)</th>
<th>Boundary condition: CCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental period (T)</td>
<td>1 (a = -2 m, b = -2 m)</td>
</tr>
<tr>
<td>0.0258 (s)</td>
<td>0.0378 (s)</td>
</tr>
</tbody>
</table>

Table 2
Convergence evaluation of frequency parameter derived by BCOP method for CCCC plates.

<table>
<thead>
<tr>
<th>Number of modes, N</th>
<th>25</th>
<th>35</th>
<th>40</th>
<th>50</th>
<th>62</th>
</tr>
</thead>
<tbody>
<tr>
<td>r = 1, λ (rad/s)</td>
<td>35.98751</td>
<td>35.98625</td>
<td>35.98622</td>
<td>35.98545</td>
<td>35.98526</td>
</tr>
<tr>
<td>r = 2, λ (rad/s)</td>
<td>98.31293</td>
<td>98.31290</td>
<td>98.311715</td>
<td>98.31143</td>
<td>98.31102</td>
</tr>
<tr>
<td>r = 3, λ (rad/s)</td>
<td>208.77008</td>
<td>208.77007</td>
<td>208.76793</td>
<td>208.76790</td>
<td>208.76715</td>
</tr>
</tbody>
</table>

Substituting Eq. (46) into Eq. (9) leads to generation of BCOP mode shapes, then applying them into Eq. (3), the proposed solution function will be presented as:

\[ w(x, y, t) = \sum_{j=1}^{40} \varphi_j(x, y)Q_j(t) = \varphi_1(x, y)Q_1(t) + \varphi_2(x, y)Q_2(t) + \ldots + \varphi_{40}(x, y)Q_{40}(t) = [\varphi_1 \ldots \varphi_{40}] \left[ \begin{array}{c} Q_1 \\ Q_2 \\ \vdots \\ Q_{40} \end{array} \right] \]  

(47)

Having modeled the problem in state-space formulation, state variable could be defined as a vector matrix with order of 80 x 1 as below:

\[ X(t) = \left[ \begin{array}{c} Q_1 \\ Q_2 \\ \vdots \\ Q_{40} \\ Q_1 \\ \vdots \\ Q_{40} \end{array} \right]_{80 \times 1} \]  

(48)

Before paying the attention to the plate under moving load or mass, a complete free vibration analysis is performed which leads to derive the characteristics of structure such as fundamental period of the plate, as reported in the Table 1. Also Table 2 assesses the convergence of frequency parameter for increasing the computational modes in the BCOP method.

Considering the proposed assumptions for the material of Aluminum plate, the solution is performed and the results are presented in Fig. 6, representing the DAF curves against non-dimensional velocity for the mass weight ratios of moving load within 0.05, 0.1, 0.15, 0.2 and 0.25. This figure actually derived from a dynamic analysis for many values of velocities of mass moving on the plate then capturing the maximum values of dynamic deflection of reference point, defined at the plate’s mid-span. Fig. 5, shows a time history for a fully-clamped plate acted upon a moving mass for both cases of inertia contributing or neglecting. Changing the length of plate with taking the width as constant value of 2m, provides a sensitivity analysis based on the dimensionless parameter of aspect ratio, r. The curves of Fig. 5 are derived for r = 3 means a = 6m and b = 2m. This figure also includes three separated curves based on three values of the mass magnitude corresponding
**Fig. 5.** Time history of dynamic deflection for a plate with boundary condition CCCC and aspect ratio, 3, under moving mass/load with the constant velocity, 10 m/s, presented for three mass magnitudes, \( m = 50 \text{ kg}, 150 \text{ kg} \) and 300 kg.

**Fig. 6.** The effect of velocity on the DAF for a square plate with CCCC boundary condition upon moving mass and load with the ratios of 0.05 (circle marker), 0.1 (square marker), 0.15 (triangle marker), 0.2 (diamond marker) and 0.25 (star marker).

to 50 kg, 150 kg and 300 kg, while it is travelling with constant velocity of 10 m/s, on the plate. As a general pattern which is followed in this study, dashed lines over the curves pertaining to the numerical case studies imply on the *moving load* case, when the inertia effects are ignored, and the solid lines denote moving inertia load case, called *moving mass*. Utilizing the makers on each curve help to indicate any parameter which is investigated at the specific figure. The time history of dynamic deflection calculated at the mid-span point has two different region where has been denoted by forced and free vibration phases of response, represents the situation of plate undergoes a moving mass for the first part, and free from any loading just the mass left the plate, for the second. Because of the absence of any damping element on the calculation, the altitude of free vibration would be unchanged. For the various mass ratios, figures include both results for *contributing* or *neglecting* the inertia effects. This provides the possibility of comparison between two cases of loading while the velocity of mass movement is increased. Furthermore, it is pointed out that the separation between DAF values are amplified as the velocity and mass weight are increased, as clearly as interpreted from Fig. 6. It could also be explained, as an important result, that in high velocities and mass weights, the influence of inertia on the plate’s dynamic behavior, play a crucial role. Hence, it can be concluded that the corresponding terms should be considered in calculations.

*Fig. 6* generally shows that as the velocity parameter goes up, it will cause to grow the DAF values for each specific mass ratio and on the other hand by tracing the effect of mass weight, at each specific velocity parameter, it is pointed out that curves relevant to the higher masses, shift in upper positions. Again it emphasizes on the importance of inertia terms contributed into calculations.
Another fact that could be derived from the above figure is that at around velocity parameter 0.55, the maximum response of plate is observed. One can conclude that in design of plates, with CCCC boundary condition under moving mass excitation, this speed must be addressed as critical where the largest displacement of plate has occurred.

5.2.1. Sensitivity analysis on the aspect ratio

As a further evaluation of plate’s response under moving loads, a plate with four edges clamped supports is investigated for two other length of 4 m and 6 m making the aspect ratio \( r \), equal to 2 and 3, respectively, and the solution is presented accordingly.

Fig. 7, consists of two separated figures in which (a) is a plate with four edges clamped boundary condition with \( r = 2 \) and (b) is derived for the plate with same boundary condition when, \( r = 3 \).

With enlargement of plate’s length, the aspect ratio will increase and this leads to decrease of geometric stiffness while the boundary conditions are kept unchanged. The results reported in Table 1 could testify this claim, where by increasing the aspect ratio the fundamental period of structure would be grown, naturally the frequency parameter adopts a descending fashion. Graphs in Fig. 7 show an interesting trend of changes in dynamic behavior of plate through changing both velocity and mass weight parameters of moving loads. When the aspect ratio of plate is increased, the stiffness is decreased subsequently and under moving load excitation, the peak value of DAF is increased at lower critical speed of moving load. In fact, the curves in Fig. 7(b) have more contraction than Fig. 7(a) and move toward left such that the picks are occurred at the velocity of 0.4 instead of 0.6. The other point which could be derived from trends of figures, is the rate of decrement on curves, especially when the velocity parameter approaches unity.
the table, free plate simply supported load supported 5.3.

Fig. 8. A square plate with all edges simply supported boundary condition, SSSS, under moving mass traveling on a rectilinear path with constant velocity.

![A square plate with all edges simply supported boundary condition, SSSS, under moving mass traveling on a rectilinear path with constant velocity.](image)

The plate with boundary condition SSSS under moving mass, r = 3 (a = 6m, b = 2m) and T = 0.08 sec

![The plate with boundary condition SSSS under moving mass, r = 3 (a = 6m, b = 2m) and T = 0.08 sec.](image)

Fig. 9. Time history of dynamic deflection for a plate with boundary condition SSSS and aspect ratio, 3, under moving mass/load with the constant velocity, 10 m/s, presented for three mass magnitudes, m = 50 kg, 150 kg and 300 kg.

Table 3

<table>
<thead>
<tr>
<th>Boundary condition: SSSS</th>
<th>Fundamental period (T)</th>
<th>Aspect ratio (r)</th>
<th>1 (a = 2 m, b = 2 m)</th>
<th>2 (a = 4 m, b = 2 m)</th>
<th>3 (a = 6 m, b = 2 m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.0470 (s)</td>
<td>0.0753 (s)</td>
<td>0.0847 (s)</td>
<td></td>
</tr>
</tbody>
</table>

5.3. Simply-Supported plate, SSSS

As the second case and one of the most applicable boundary conditions of plates, a square plate with four edges simply supported boundary condition, SSSS, is investigated here. Fig. 8 shows a typical model of such plate which is acted upon load moving on a rectilinear path with constant velocity. Most of bridge decks in the real world are usually simulated by the simply-supported plates within research works. On the other hand because of the fact that this type of boundary conditions have the analytical solutions, are strictly brought into the numerical studies from validation point of view, as performed in Section 5.1.

All procedures are proceeded here similar to previous section except for mode shape functions which must meet the simply supported condition and could be regarded as below:

\[ F = \xi (1 - \xi)\eta(1 - \eta)\{1, \xi, \eta, \xi^2, \xi\eta, \xi^3, \eta^2, \eta^3, \xi^3\eta^3\} \quad (49) \]

Substituting Eq. (49) into Eq. (9) and following the steps of mode shape generation, the problem for simply supported plate would be solved. To prevent repetition, we just report the results here as Fig. 9. Again, having carried out a complete free vibration analysis yields to the characteristics of plate like fundamental period of vibration listed into the Table 3. In the table, a plate with three values of aspect ratio parameter has been analyzed and increasing manner of T is derived, while the length of plate is enlarged. As well as, Table 4 reports the results derived from free vibration analysis for frequency
parameter. In this table the convergence of procedure has been testified in a way that by increasing the computational modes there is not seen any divergence or instability within numerical method.

Reserving the comments and formulation in the Section 5.2, the work is launched with a dynamic analysis of plate after a free vibration procedure. Surveying the Fig. 9, pertaining to a time history analysis of dynamic deflection of plate, it is observable that the amplitude of vibrations in the free vibration phase is more than the case of CCCC. It may be derived from changing in the plate stiffness. The other point which could deduce from the figure is the shape of curves in the forced vibration margin and their adoption for moving load and mass cases. One can easily see that deviation of these curves for solid and dashed lines in this region is more appreciable, which could rise from changes in the stiffness. The importance of inertia effects of moving loads in dynamic response of plate (especially for high weights of higher magnitude) is clearly seen through Fig. 10. For mass ratio of 0.25, the maximum response of plate if inertia effect is considered has a value as great as over 1.8 times respect to the loading condition when inertia effect is neglected. On the other hand, increasing in the velocity would lead to rise in each graph for specific mass weight ratio, which emphasizes on the importance of velocity parameter, in response of the plate under moving loads. The other point which could be gotten from Fig. 10, is related to velocity parameter of 0.2. As an engineering simplification and according to results showed at Fig. 10 the values of DAF around unity denote a condition that dynamic effects of loading in the plate’s response are not too remarkable and one can analyze the problem in statics condition. This situation is happen for the velocities below 0.2 for loading without inertia effects and for all mass weights. Against, regarding the inertia effects of loading would lead to the fact that such assumption for static analysis just is acceptable for velocities below 0.2 such a way that with increase of mass weight, this velocity move left more and more, that is the importance of inertia contribution into calculation is proved specifically when the mass weight has a high magnitude.

5.3.1 Sensitivity analysis on the aspect ratio

Here, again the plate with aforementioned assumptions and boundary condition put into consideration with aspect ratios of 2 and 3. Fig. 11 reflects the effect of aspect ratio on the dynamic response of plate when velocity of load traveling on the plate is constant. Preserving the comments mentioned in Section 5.2.1 about the stiffness of plate when aspect ratio parameter is changed, dynamic behavior of plate is denoted as below figures.

As a similar fashion with treatment for the plates, CCCC, when aspect ratio was increased there, the plates with SSSS edges also show a behavior in the dynamic response in a way that increasing the aspect ratio would cause to more contraction in curves for all mass weights and move the graphs toward left. Also the pick of each curves which denotes the maximum dynamic effect due to moving load excitation is risen up at lower speed. In other word along with mitigation in geometrical stiffness the pick of DAFs happen at velocities around 0.6 for aspect ratio, 3, while this maximum value is evaluated at velocities around 0.8 for aspect ratio, 2.

<table>
<thead>
<tr>
<th>Number of modes, N</th>
<th>25</th>
<th>35</th>
<th>40</th>
<th>50</th>
<th>62</th>
</tr>
</thead>
<tbody>
<tr>
<td>r = 1, λ (rad/s)</td>
<td>19.73921</td>
<td>19.73920</td>
<td>19.73920</td>
<td>19.73920</td>
<td>19.73920</td>
</tr>
<tr>
<td>r = 2, λ (rad/s)</td>
<td>49.34804</td>
<td>49.34802</td>
<td>49.34802</td>
<td>49.34802</td>
<td>49.34802</td>
</tr>
<tr>
<td>r = 3, λ (rad/s)</td>
<td>98.69609</td>
<td>98.69604</td>
<td>98.69604</td>
<td>98.69604</td>
<td>98.69604</td>
</tr>
</tbody>
</table>

Fig. 10. The effect of velocity on the DAF for a square plate with SSSS boundary condition upon moving mass and load with ratios of 0.05 (circle marker), 0.1(square marker), 0.15(triangle marker), 0.2(diamond marker) and 0.25(star marker).
The plate under moving mass with boundary condition, SSSS, and \( r = 2 \) (\( a = 4 \) m, \( b = 2 \) m)

(a)

The plate under moving mass with boundary condition, SSSS, and \( r = 3 \) (\( a = 6 \) m, \( b = 2 \) m)

(b)

Fig. 11. The effect of velocity on the DAF for a plate with SSSS boundary condition upon moving mass and load with the ratios of 0.05 (circle marker), 0.1(square marker), 0.15(triangle marker), 0.2(diamond marker) and 0.25(star marker). (a) aspect ratio, \( r = 2 \) and (b) aspect ratio, \( r = 3 \).

Table 5

The fundamental period of plate with boundary condition CFCF for three values of aspect ratio.

<table>
<thead>
<tr>
<th>Aspect ratio ( (r) )</th>
<th>Boundary condition: CFCF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( (a = 2 ) m, ( b = 2 ) m)</td>
<td>2 ( (a = 4 ) m, ( b = 2 ) m)</td>
</tr>
<tr>
<td>Fundamental period ( (T) )</td>
<td>0.0420 (s)</td>
</tr>
</tbody>
</table>

5.4. Clamped-Free plate, CFCF

According to the previous section all formulations would be derived for plate that has two opposite free and clamped boundary conditions typically is shown at Fig. 12. Probably, the best mechanical model of plates or bridge-decks with larger spans when both ends are prevented from any displacement could be analyzed by CFCF boundary condition. As well as, to complete the trend of evaluation on the various boundary conditions from the most stiffest to the softest cases, this model is scrutinized here.

By substituting the appropriate mode shapes which satisfy the required conditions, the results easily could be achieved and presented here. Eq. (50) shows the polynomial function using in the BCOP method corresponding to the CFCF plates.

\[
F = \xi^2 (1 - \xi)^2 \{1, \xi, \eta, \xi^2, \xi \eta, \eta^2, \xi^3, \ldots, \xi^3 \eta^5\} \tag{50}
\]

Similar to the path which was followed in the previous sections, a free vibration analysis is carried out at first, yields to fundamental period of structure for various aspect ratios, reported in the Table 5. We are to show the effect of computational
mode numbers on the frequency parameter, so the free vibration analysis has been performed for several mode numbers and the results have been listed within Table 6, accordingly. Proceeding the work with dynamic analysis leads to Fig. 13, where an interesting behavior is achieved from CCF square plate under moving mass. That is, the dynamic amplitude in the free vibration phase is more appreciable than the previous types of boundary conditions. Also, the difference between two cases of loading, special for the higher mass magnitude is more considerable. For example, when the plate undergoes a load of 300kg of mass magnitude, one can easily observe that in the curves corresponding to the forced vibration, denoted by triangle marker, have a less adoption than the other mass magnitude. It could be interpreted as the importance of inertia terms in the dynamic behavior of plate and the requirement of dynamic analysis while inertia terms are encountered.

Accordingly, if an analysis on the mass weight ratio traveling on the plate is carried out and the results for DAF against velocity depicted in a graph, the Fig. 14 would be generated which including both curves for inertia effects when they are included and neglected.

Similar to the fashion which mentioned at before, the response of a plate when is acted upon moving mass could be evaluated, in a way that by increasing the velocity and mass weight of moving load, difference between responses for the case with and without inertia effects are increased accordingly. This issue emphasizes again on the fact that inertia effect in high velocities and mass weights must be imported into the formulation.
Fig. 14. The effect of velocity on the DAF for a square plate with CFCF boundary condition upon moving mass and load with ratios of 0.05 (circle marker), 0.1 (square marker), 0.15 (triangle marker), 0.2 (diamond marker) and 0.25 (star marker).

Fig. 15. The effect of velocity on the DAF for a plate with CFCF boundary condition upon moving mass and load with the ratios of 0.05 (circle marker), 0.1 (square marker), 0.15 (triangle marker), 0.2 (diamond marker) and 0.25 (star marker). (a) aspect ratio, $r=2$ and (b) aspect ratio, $r=3$. 
Fig. 16. Plate with two opposite free and simply supported edges undergoes moving mass excitation.

Fig. 17. Time history of dynamic deflection for a plate with boundary condition SFSF and aspect ratio, 3, under moving mass/load with the constant velocity, 10 m/s, presented for three mass magnitudes, \( m = 50 \) kg, 150 kg and 300 kg.

Fig. 18. The effect of velocity on the DAF for a square plate with SFSF boundary condition upon moving mass and load with ratios of 0.05 (circle marker), 0.1 (square marker), 0.15 (triangle marker), 0.2 (diamond marker) and 0.25 (star marker).
5.4.1. Sensitivity analysis on aspect ratio

Problem is investigated for two other lengths of plate, 4m and 6m which correspond to $r = 2$ and $r = 3$, respectively. The results are presented through below figures including several mass weights traveling on the plate with constant velocity, like previous sections, Fig. 15.

As a general conclusion this point could be interpreted that by increasing the aspect ratio, distance between response of plate under moving load and mass are decreased. So this issue reflect the fact that analysis of plates in lower aspect ratios deserve more importance of attention keeping the inertia terms in calculations. So, performing the simplification through considering the problem just for moving load without any effect of inertia would lead to deviate from the accuracy.

On the other hand, similar to the other cases when aspect ratio is increased the critical velocity move left and also decreased, which maximum non-dimensional response of the plate occurred there.

5.5. Simple-Free plate, SFSF

Plate with SFSF boundary condition is the last case which is assessed here. Two opposite free edges and simply supported constraints are showed in Fig. 16, undergoes a mass moving along the length of plate with constant velocity. This model could be accounted for the most common case in the real structures after SSSS ones, like happened in some modeling of cable-bridges. On the hand, because of free edges on longitudinal directions and pin type on the width, this model is appropriately investigated as the basic for the multi-span plates. Similar to the past, having performed a time history analysis as Fig. 17, the work would be proceede by importing the appropriate mode function into BCOP method the procedure would
Fig. 20. (a) Time history of dynamic deflection for the plate with aspect ratio, 2, and various boundary condition CCC, SSSS, CFCF and SFSF. (b) The same graph in zoomed view for better showing the behavior of dynamic deflection when the boundary condition is changed, special for the forced vibration interval.

Table 7
The fundamental period of plate with boundary condition SFSF for three values of aspect ratio.

<table>
<thead>
<tr>
<th>Aspect ratio (r)</th>
<th>Fundamental period (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (a = 2 m, b = 2 m)</td>
<td>0.0970 (s)</td>
</tr>
<tr>
<td>2 (a = 4 m, b = 2 m)</td>
<td>0.3950 (s)</td>
</tr>
<tr>
<td>3 (a = 6 m, b = 2 m)</td>
<td>0.8910 (s)</td>
</tr>
</tbody>
</table>

be conducted and the results presented in Fig. 18.

\[ F = \xi (1 - \xi) \{1, \xi, \eta, \xi^2, \xi \eta, \eta^2, \xi^3, \xi^* \eta^3\} \]  

Table 7, shows the fundamental period of the plate with SFSF boundary condition, varying the aspect ratio parameter. A descending trend of stiffness for the plate by increasing the aspect ratio would be derived from the Table 7, clearly. On the other hand, an interesting result is acquired from Table 8 based on the fact that by increasing the aspect ratio the convergence behavior of frequency parameter is enhanced. In the other word, the modal convergence would be happened for the lower computational modes while the aspect ratio is increased. Free edges is categorized under the Natural type of boundary conditions which usually has no analytical solutions when is incorporated into the problem defini-
Table 8
Convergence evaluation of frequency parameter derived by BCOP method for CFCF plates.

<table>
<thead>
<tr>
<th>Number of modes, N</th>
<th>25</th>
<th>35</th>
<th>40</th>
<th>50</th>
<th>62</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = 1 ), ( \lambda ) (rad/s)</td>
<td>9.57479</td>
<td>9.57439</td>
<td>9.57439</td>
<td>9.57439</td>
<td>9.57439</td>
</tr>
<tr>
<td>( r = 2 ), ( \lambda ) (rad/s)</td>
<td>9.43170</td>
<td>9.43168</td>
<td>9.43168</td>
<td>9.43168</td>
<td>9.43168</td>
</tr>
<tr>
<td>( r = 3 ), ( \lambda ) (rad/s)</td>
<td>9.37669</td>
<td>9.37669</td>
<td>9.37669</td>
<td>9.37669</td>
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</tr>
</tbody>
</table>

tion. As it has been represented through above sections, the BCOP method could handle the problems with such complicated boundary conditions, easily. Where the analytical methods encounter with considerable problems to solve the dynamic equations with such boundary conditions, the numerical method of BCOP which generates the mode shapes by using the polynomials attributes could play an important role in the problem solution through straightforward and accurate procedure.

Clearly, following the trend of curves in the Fig. 18, shows again that plate’s behavior analysis upon high mass weights and velocities needs to consider the inertia effects in formulations. Refer to Fig. 18, it is pointed out the fact that for masses traveling on the plate with speeds below 0.2 the effect of inertia could be neglected and by moving far away from this speed the deviation of dashed lines from continues ones is grown up, due to the inertia terms contribution into formulation.

5.5.1. Sensitivity analysis on aspect ratio

Through a similar fashion, the plate with SFSF condition would be under consideration here, regarding two aspect ratios 2 and 3. Fig. 19 consists of two separated graphs which both of them include curves showing dynamic response of plate

Fig. 21. Conversion factor for a square plate under several magnitudes of moving loads indicated by circle, square, triangle, diamond and star markers corresponding to \( M = 0.05, 0.1, 0.15, 0.2 \) and 0.25, respectively. Each graph is assigned to four types of boundary conditions as (a) CCCC, (b) SSSS, (c) CFCF and (d) SFSF.
Fig. 21. Continued

at reference point under moving load with and without inertia effects, indicating by dashed and solid lines and has been individualized aided by several markers.

Fig. 19 shows that the aspect ratio parameter has a little effect on the dynamic behavior of plate. Both Fig. 19-(a) and Fig. 19-(b) in a general investigation represent a similar trend of changes, where if we survey each curves of two Figs for specific mass weight ratio, this fact could be derived that all changes and differences are located at a very close interval, when the aspect ratio is increased. This interesting observation reflect us the point that increasing in the aspect ratio has no remarkable effect on the plate’s response, when two sides of plate have no any constraint.

A comparison study has been carried out for the dynamic behavior of the plate with various boundary condition, as a final result presented at this section. Fig. 20 is dedicated to the time history of a plate with aspect ratio 2 which supported by all cases of boundary conditions that were scrutinized though the paper. The second part (b), of the figure is a zoomed view of the first part with the aim of showing the more detailed of curves special for the forced vibration phase. Dashed lines denotes the loading without inertia and the solid ones are for inertia contribution. A very interesting achievement was acquired here, where, a most large dynamic deflection is seen for the SFSF boundary condition while a mass with magnitude of 100 kg is moving with constant velocity of 15 m/s. This issue was predictable because of the fact that SFSF imposes the lowest geometrical stiffness on the plate, so most large deflection would be achieved. But the matter with a higher degree of importance is the effect of inertia terms contributed into the plate behavior under moving mass, where could deduced by assessing the difference between the dashed and solid lines within the figure. It could be observed that for the CFCF boundary condition the shifting of the curves is more appreciable which testifies the considerable effect of inertia terms for the plate acted upon a moving mass.
A) Curve fitting surface for conversion factor against non-dimensional mass and velocity, $V = [0.14, 0.34]$ proposed for the plate with boundary condition CCCC and $r = 1$

B) Curve fitting surface for conversion factor against non-dimensional mass and velocity, $V = [0.34, 0.7]$ proposed for the plate with boundary condition CCCC and $r = 1$

C) Curve fitting surface for conversion factor against non-dimensional mass and velocity, $V = [0.7, 1]$ proposed for the plate with boundary condition CCCC and $r = 1$

Fig. 22. Best fit model of surface for a square plate with CCCC BCs acted upon inertia moving loads. (A) $V = 0.14$ to $0.34$, (B) $V = 0.34$ to $0.7$, (C) $V = 0.7$ to $1$. 
Fig. 23. Best fit model of surface for a square plate with SSSS BCs acted upon inertia moving loads. (A) $V = 0.18$ to $0.44$, (B) $V = 0.44$ to $0.6$, (C) $V = 0.6$ to $1$. 
Fig. 24. Best fit model of surface for a square plate with CFCF BCs acted upon inertia moving loads. (A) $V = 0.16$ to 0.4, (B) $V = 0.4$ to 0.68, (C) $V = 0.68$ to 1.
6. Simplification through regression analysis

As showed before the inertia effect in dynamic response of plate has emerged specifically in high speed of moving loads, then must be included in formulations. On the other hand the complete solution of equation would be available when full term of acceleration contributes in the formulation which yields to add a remarkable complexity into problem. Because of this, many researchers and engineers have been forced to implement the numerical methods. Regarding to the fact that the problem has been solved for both model of load and mass moving on the plate, a corrective coefficient could be defined as below:

\[ \beta = \frac{\text{DAF}_{\text{mass}}}{\text{DAF}_{\text{load}}} \]  \hspace{1cm} (52)

\( \beta \) defines a conversion factor which is derived by dividing two cases of DAFs when inertia effects are included or ignored. By this way one can easily run the problem for moving load in the simplest form of formulation where has no any contribution of inertia into the problem, then having referred to relevant coefficient of the factor, could obtain the corresponding value of \( \beta \) and finally by multiplying it into the moving load solution, the complete result would be achieved. As well as, by investigation on this factor, the pick of most difference between two cases of loading is introduced as the critical velocity of moving load and guide designers to consider the required preparations. During following sections some investigations on conversion factor is carried out, for a square plate acted upon a moving load with a parametric study on the mass magnitude.

6.1. Conversion factor

Fig. 21 consists of four separated figures in which any of them is assigned for a specific boundary condition, in a fashion that corresponding coefficient, \( \beta \), for CCCC, SSSS, CFCF and SFSF conditions could be found through Fig. 21(a), Fig. 21(b), Fig. 21(c) and Fig. 21(d), respectively.

For instance, it is pointed out that the maximum difference between responses of plate upon two types of loading occurs in the velocity around 0.55 for CCCC condition while for CFCF conditions this velocity is risen up to around 0.65, so allows the load moves on the plate with a higher speed.

6.2. Regression analysis

Now we can go to run a regression analysis of the results presented in Fig. 21 for a square plate. As the figure shows, in low values of normalized mass velocity, \( V \), because of the fluctuations of spectra where is occurred at values near to unity evidenced by \( 0.9 < \beta < 1.1 \), the effects of inertia are no further studied. So, one can easily consider and run the problem in moving load case. For the velocities corresponding to the values of conversion factor far away from this interval a regression analysis is held here through best fit model based on polynomial function of order four in \( V \) and \( M \), as primary variables. According to Fig. 21, the upper limiting speed values of above mentioned regime are \( V_{\text{CCCC}} = 0.14 \), \( V_{\text{SSSS}} = 0.18 \), \( V_{\text{CFCF}} = 0.16 \) and \( V_{\text{SFSF}} = 0.56 \), respectively, for corresponding BCs. These values are dependent on the physical properties of plate at assigned BCs, in accordance with normalized parameter defined by aforementioned expressions.
Table 9
Regression coefficients and goodness of fit statistics assessment for a square plate with CCCC boundary condition.

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>CCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>V interval</td>
<td>(0.14,0.34)</td>
</tr>
<tr>
<td>Regression coefficients</td>
<td></td>
</tr>
<tr>
<td>$p_{00}$</td>
<td>$-2.416$</td>
</tr>
<tr>
<td>$p_{10}$</td>
<td>$61.54$</td>
</tr>
<tr>
<td>$p_{01}$</td>
<td>$-18.66$</td>
</tr>
<tr>
<td>$p_{20}$</td>
<td>$-401.2$</td>
</tr>
<tr>
<td>$p_{11}$</td>
<td>$218$</td>
</tr>
<tr>
<td>$p_{02}$</td>
<td>$26.49$</td>
</tr>
<tr>
<td>$p_{12}$</td>
<td>$18.64$</td>
</tr>
<tr>
<td>$p_{21}$</td>
<td>$-7.79$</td>
</tr>
<tr>
<td>$p_{13}$</td>
<td>$-182.4$</td>
</tr>
<tr>
<td>$p_{03}$</td>
<td>$-21.48$</td>
</tr>
<tr>
<td>$p_{22}$</td>
<td>$-114.8$</td>
</tr>
<tr>
<td>$p_{31}$</td>
<td>$910.5$</td>
</tr>
<tr>
<td>$p_{23}$</td>
<td>$258.8$</td>
</tr>
<tr>
<td>$p_{14}$</td>
<td>$93.87$</td>
</tr>
</tbody>
</table>

Goodness of fit

<table>
<thead>
<tr>
<th>SSE</th>
<th>0.01347</th>
<th>0.02298</th>
<th>0.0009331</th>
</tr>
</thead>
<tbody>
<tr>
<td>R-square</td>
<td>0.94999</td>
<td>0.98989</td>
<td>0.9986</td>
</tr>
<tr>
<td>Adjusted R-square</td>
<td>0.9341</td>
<td>0.9883</td>
<td>0.9984</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.01813</td>
<td>0.01684</td>
<td>0.00376</td>
</tr>
</tbody>
</table>

As reported in the plots of Fig. 21, the conversion factor, $\beta$, do not vary monotonically. Hence to have a better approximation, the normalized mass velocity intervals beyond the above mentioned limiting interval is subdivided into three subintervals for all cases of CCCC, SSSS and CFCF except for SFSF which an acceptable goodness of fit parameter would be achieved for all speeds, from limiting value up to end instantly. The adopted general form of fitting function based on $V$ and $M$ could be proposed as below:

$$
\beta = \frac{DAF_{mass}}{DAF_{load}} \equiv P_{00} + P_{10} \ast V + P_{01} \ast M + P_{20} \ast V^2 + P_{11} \ast V \ast M + P_{02} \ast M^2 + P_{30} \ast V^3 + P_{21} \ast V^2 \ast M + P_{12} \ast V \ast M^2 + P_{03} \ast M^3 + P_{40} \ast V^4 + P_{31} \ast V^3 \ast M + P_{22} \ast V^2 \ast M^2 + P_{13} \ast V \ast M^3
$$

(53)

Where constants $P_{ij}$ (linked though indexes $i$ and $j$ to the power of terms in $V$ and $M$, respectively) are the sought and looked for regression coefficients.

Through performing a least-square procedure, the results of surface fitting is gotten and as shown in Fig. 22. As commented above for have a more accurate fitting of function the velocities beyond limiting values subdivided into three length which are distinct in the figure by titles of A, B and C, for all boundary conditions except for SFSF.

The regression coefficients $P_{ij}$ for all boundary conditions and goodness of fit parameters are presented through Tables 9–12, which are assigned for each constraint condition. In a way that, Table 9 lists the results for CCCC boundary conditions, Table 10 for SSSS, Table 11 for CFCF and finally Table 12 for SFSF. SSE called for Summed Square of Residuals, $R^2$ shows the R square, $R^2$ is adjusted $R^2$, and Root Mean Squared Error is denoted by RMSE in the table. Tables 5–8 and corresponding Figs. 22–25 testify that the proposed regression model provides a very good agreement between the results in a way that for most cases, SSE and RMSE have values below 0.01 and also $R^2$ and $R^2$ are very close to unity.

As an instance, by conducting a simple and rapid dynamic analysis of plate excited by moving loads without any inertia effects, at first, and then employing the conversion factors the inertia effects of moving mass could be appropriately accounted for.

7. Discussion and conclusion

A thin rectangular single span plate with arbitrary boundary condition under moving load excitation traveling on an arbitrary trajectory was considered in this paper. The problem was solved for two cases of loading when inertia effects of moving force are included or ignored. In terms of loading with inertia effects on the plate’s surface, the problem was called as moving mass. In moving mass section, the full term formulation of acceleration calculated at reference point located at the mid-span due to the inertia effects of loading was established, which leads to achieve the results more accurate, although the complexity of equations would be increased. Four common major types of boundary conditions was investigated as CCCC, SSSS, CFCF and finally SFSF, denoting four edges clamped, four edges simple, clamped and free edges alternatively and simple and free conditions on the opposite edges, respectively. For mode shape generation the method of BCOP was employed in a way that vibrational modes were created by polynomials, satisfying all of geometrical constraints on the plate over all computational modes. The Galerkin method in a general form of formulation was held and the BCOPs was considered as mode shapes multiplying into a temporal function as time dependent modal amplitude of vibrated plate, where make
the complete solution of plate's response under moving mass excitation. The powerful method of Matrix Exponential was used to acquire the solution in time domain. Having performed a comprehensive numerical study on an Aluminium plate, the results presented through several graphs for each boundary condition, comprising the normalized form of dynamic deflection of reference point located at mid-plane surface of plate in vertical axis against the normalized velocity of moving mass sited at horizontal coordinate. The analysis was run for five mass weight ratios, 0.05, 0.1, 0.15, 0.2 and 0.25 for both cases of loading i.e. with and without inertia effects resulting the fact that by increasing mass ratios as well as velocity of mass, the differences between results are considerable and the inertia effects must be accounted for. Several important results are pointed out from this study which could be listed as follow. As a general conclusion, it is seen that the DAF parameter is increased when the mass magnitude and the velocity of movement are increased, for all boundary conditions. But as specific conclusions achieved for each boundary conditions, it could be mentioned that for CCCC boundary condition as the most stiffest case, the DAF parameter is increased by enlarging the aspect ratio parameter. But the relevant curves are approaching together that implies on the fact that the effect of inertia is mitigated by increasing the aspect ratio. On the other hand, increasing the aspect ratio would yields to lessen the critical speed in a way that \( V_{critical} \) is mitigated from

<table>
<thead>
<tr>
<th>Table 10</th>
<th>Regression coefficients and goodness of fit statistics assessment for a square plate with SSSS boundary condition.</th>
</tr>
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<tbody>
<tr>
<td>Boundary Condition</td>
<td>SSSS</td>
</tr>
<tr>
<td>V interval</td>
<td>(0.18,0.44)</td>
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<tr>
<td>Regression coefficients</td>
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</tr>
<tr>
<td>( P_{00} )</td>
<td>-4.834</td>
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<tr>
<td>( P_{10} )</td>
<td>80.54</td>
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<td>( P_{11} )</td>
<td>-400.2</td>
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<tr>
<td>( P_{02} )</td>
<td>89.14</td>
</tr>
<tr>
<td>( P_{21} )</td>
<td>32.11</td>
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<tr>
<td>( P_{30} )</td>
<td>853.7</td>
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<td>( P_{21} )</td>
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</tr>
<tr>
<td>( P_{32} )</td>
<td>-151.7</td>
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<tr>
<td>( P_{30} )</td>
<td>-34.72</td>
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<td>( P_{13} )</td>
<td>103</td>
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<td>Goodness of fit</td>
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<tr>
<td>SSE</td>
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<tr>
<td>R-square</td>
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<tr>
<td>Adjusted R-square</td>
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<td>RMSE</td>
<td>0.02036</td>
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<tr>
<th>Table 11</th>
<th>Regression coefficients and goodness of fit statistics assessment for a square plate with CFCF boundary condition.</th>
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</thead>
<tbody>
<tr>
<td>Boundary Condition</td>
<td>CFCF</td>
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<tr>
<td>V interval</td>
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<td>Regression coefficients</td>
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<tr>
<td>( P_{00} )</td>
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<td>( P_{10} )</td>
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<td>( P_{20} )</td>
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<tr>
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<td>( P_{11} )</td>
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<td>( P_{02} )</td>
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<td>( P_{10} )</td>
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<tr>
<td>( P_{21} )</td>
<td>131.2</td>
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<tr>
<td>( P_{31} )</td>
<td>36.4</td>
</tr>
<tr>
<td>( P_{22} )</td>
<td>21.12</td>
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<tr>
<td>Goodness of fit</td>
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<tr>
<td>SSE</td>
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<tr>
<td>R-square</td>
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<tr>
<td>Adjusted R-square</td>
<td>0.9776</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.00629</td>
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Table 12
Regression coefficients and goodness of fit statistics assessment for a square plate with SFSF boundary condition.

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>V interval</th>
<th>SFSF</th>
<th>(0.56,1)</th>
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<table>
<thead>
<tr>
<th>Regression coefficients</th>
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<tr>
<td>$P_{00}$</td>
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<td>$P_{10}$</td>
<td>−58.44</td>
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<tr>
<td>$P_{01}$</td>
<td>−12.24</td>
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</tr>
<tr>
<td>$P_{20}$</td>
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</tr>
<tr>
<td>$P_{11}$</td>
<td>40.81</td>
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<td></td>
</tr>
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<td>$P_{02}$</td>
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<td>$P_{30}$</td>
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<td>$P_{21}$</td>
<td>−43.69</td>
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<tr>
<td>$P_{12}$</td>
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<tr>
<td>$P_{03}$</td>
<td>3.091</td>
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<tr>
<td>$P_{40}$</td>
<td>27.03</td>
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</tr>
<tr>
<td>$P_{31}$</td>
<td>16.71</td>
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</tr>
<tr>
<td>$P_{22}$</td>
<td>1.5</td>
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</tr>
<tr>
<td>$P_{13}$</td>
<td>−5.998</td>
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</table>

Goodness of fit

<table>
<thead>
<tr>
<th>SSE</th>
<th>R-square</th>
<th>Adjusted R-square</th>
<th>RMSE</th>
</tr>
</thead>
</table>

0.6 to 0.4 while parameter $r$ is changed within 1 to 3, respectively. For the case SSSS, it is clearly seen that amplitude of free vibration analysis is increased respect to the CCCC case, which could be justified due to lessen in stiffness. In similar fashion with the CCCC plates, the effect of aspect ratio enhancement would cause to decrease the critical velocity from 0.95 down to 0.6 coresponding to change in aspect ratio from 1 up to 3. Along with decreasing the stiffness of structure, CFCF plate denotes a larger amplitude in free vibration, but a notable phase difference between moving mass and moving load excitation is clearly seen. This interesting result emphasizes on the necessity of running the problem upon inertia inclusion for this boundary condition. On the other hand, an important achievement could be extracted here, such that changing in aspect ratio has no considerable effect on the DAF magnitude and it remain almost unchanged while the aspect ratio is increased. But the critical velocity is decreased similar to the previous cases of boundary conditions while $r$ is boosted from 1 to 3. The last boundary condition SFSF as the softest case investigated here, shows the largest amplitude in free vibration phase in a way that this phase could be regarded as designative measurement in dynamic design of structure. An important achievement could be reported here for convergence analysis of BCOP method and the effect of aspect ratio parameter on it. It has been shown that by increasing the aspect ratio, the convergence for frequency parameter would be achieved at the lower number of computational modes. Once again, it was observed that by descending the stiffness of plate the sensitivity to aspect ratio changing was mitigated and the magnitude of DAF almost stays unchanged. Having presented a comparative diagram for the various boundary conditions based on time history analysis of dynamic deflection of mid-span plate, it was deduced that the most sensitive condition to inertia effect of moving mass is derived for CFCF, then concluded that simplification for ignoring the inertia effects has no longer be established when dynamic analysis of plates is investigated. The work was prolonged by proposing a conversion factor to derive DAFs for the case of inertia mass moving on the plate, so that one can easily assess the problem upon load acted on the plate without endurance of any complexity of formulation due to inertia contribution, and then by referring to the relevant figures assigned for each boundary conditions and mass ratios could estimate the appropriate conversion factor and considered solution accordingly. Finally, a regression analysis was performed to find and proposed a best fit function to this conversion factor. Through employing a least squared procedure, the results of surface fitting was plotted into figures special for each boundary conditions and for have a more accurate adopting model, the considered velocity interval was subdivided into three sub intervals except for SFSF constraint, where appropriate goodness of fit values were achieved, needless to subdividing. The relevant coefficients for polynomial fitting model function were reported through tables testified an acceptable accuracy of fitting. As well as, the numerical results reported within tables testified the graphical ones for importance of analysis in full complete form of inertia contribution.

This estimating model could play an important role in simplifying the whole procedure of plate’s dynamic investigation, in a way that by choosing a specified velocity and mass ratio as two variables of fitting function, the estimated conversion factor would be easily achieved and dynamic response of the plate in a simple fashion would be available, yields to save the time through a straightforward solution.

References