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A new algorithm for concave quadratic programming

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Abstract
The main outcomes of the paper are divided into two parts. First, we present a new dual for quadratic programs, in which, the dual variables are affine functions, and we prove strong duality. Since the new dual is intractable, we consider a modified version by restricting the feasible set. This leads to a new bound for quadratic programs. We demonstrate that the dual of the bound is a semi-definite relaxation of quadratic programs. In addition, we probe the relationship between this bound and the well-known bounds in the literature. In the second part, thanks to the new bound, we propose a branch and cut algorithm for concave quadratic programs. We establish that the algorithm enjoys global convergence. The effectiveness of the method is illustrated for numerical problem instances.

Keywords Non-convex quadratic programming · Duality · Semi-definite relaxation · Bound · Branch and cut method · Concave quadratic programming

1 Introduction

We consider the following quadratic program (QP):

$$\begin{aligned} \min\ & x^T Qx + 2c^T x \\ s.t.\ & Ax \leq b, \end{aligned} \tag{QP}$$

where Q is a real symmetric $n \times n$ matrix, A is a real $m \times n$ matrix, $c \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$. Moreover, throughout the paper, it is assumed that the feasible set, $X = \{x \in \mathbf{R}^n : Ax \leq b\}$, is nonempty and bounded. It is well-known that (QP) is solvable in polynomial time when Q is positive semi-definite. Nevertheless, indefinite QPs, in which Q is an indefinite matrix, are NP-hard even when Q has just one negative eigenvalue [36,40]. In the paper, our focus is on non-convex QPs.

Duality plays a fundamental role in optimization, from both theoretical and numerical points of view [30]. It serves as a strong tool in stability and sensitivity analysis. For convex

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problems, duality is employed in some numerical methods for obtaining or verifying an optimal solution [4,7].

It is well-known that the (Lagrangian) dual of a convex QP is also a convex QP, and satisfies strong duality. However, for non-convex case the dual of QPs might be meaningless because the objective function of the dual problem might be $-\infty$ while the primal has a finite optimal value. In addition to convex QPs, the strong duality holds for some classes of non-convex problems. Optimizing a quadratic function on the sublevel set of a quadratic function is an archetype. S-lemma guarantees strong duality under Slater condition [37].

Global optimization methods for QPs are typically based on the convex relaxations and bounds. The most effective relaxations for QPs (with quadratic constraints) rest upon the semi-definite programming and the reformulation-linearization technique (RLT) [2,28,41]. The semi-definite relaxations were first applied to some combinatorial problems [29]. Due to their efficiency, these methods have been extended to QPs with quadratic constraints [34]. For more discussion on semi-definite relaxations and their comparisons, we refer the reader to the recent survey [2]. Moreover, recently it has been shown that the combination of the semi-definite relaxations and RLT leads to stronger relaxations [1,2].

In addition to the relaxation methods, scholars have proposed some bounds for classes of QPs [6]. Bounds similar to relaxation methods give a lower bound. The most effective bounds for QPs are based on semi-definite programming [6].

Another method which is able to give a bound for QPs is the so-called Lasserre hierarchy [25]. In fact, this method provides optimal value under some mild conditions [35]. Lasserre hierarchy is able to tackle polynomial optimization problems (optimizing a polynomial function on a given semi-algebraic set). It is well-known that polytopes are Archimedean. Hence, in view of the Putinar's Positivstellensatz theorem, the optimal value of (QP) is obtained by solving the following convex optimization problem:

$$\begin{aligned}
&\max\ \ell \\
&\ s.t. x^T Qx + 2c^T x - \ell = \sigma_0(x) - \sum_{i=1}^{m} \sigma_i(x)(A_i x - b_i), \\
&\qquad \sigma_i \in \Sigma[x],\ i = 0, 1, \ldots, m,
\end{aligned} \tag{1}$$

where $\Sigma[x]$ denotes the cone of polynomials which are sums of squares (SOS) [25]. By virtue of Lasserre hierarchy, generically, the optimal value of (QP) can be obtained by solving a finite number of semi-definite programs [35]. However, the dimension of semi-definite programs may increase dramatically [25,35].

Note that the Handelman's hierarchy can be also employed to produce a bound for QPs. Indeed, this method provides the optimal value under some mild conditions [27]. In this approach, each subproblem is a linear program, though the dimension of optimization problems may increase exponentially similar to the Lasserre hierarchy. For more details on the method, we refer the interested reader to [25,27].

Concave QPs are important both theoretically and practically. Concave QPs appear in many applications including fixed charge and risk management problems and quadratic assignment problems [10,14,24]. In addition, it has been shown that some class of QPs can be reformulated as concave QPs [14,24]. It is well-known that a concave QP realizes its minimum at some vertices [14]. So, the problem is equivalent to the combinatorial problem of optimizing a quadratic function on the vertices of a given polytope. This problem, as mentioned earlier, is NP-hard. Many avenues for tackling concave QPs have been pursued. Typical approaches are cutting plane methods, successive approximation methods and branch and bound approaches [19]. For more methods and details, see [14,19,46].

One of the effective approach for solving QPs is mixed integer programming reformulation [17,47]. As there exist current state-of-the-art mixed integer programming solvers including GUROBI and CPLEX, this method may be very efficient. It is worth mentioning that that some solvers including CPLEX exploits this idea to handle QPs [20]. Semi-definite relaxations have been also employed in branch and bound type methods for solving QPs [9,11].

In this paper, we introduce a new dual for QPs. Dual variables are affine functions and strong duality is established. As the dual problem is intractable, we take into account a subset of the feasible set, leading to a new bound for QPs. We establish that the bound is well-defined for QPs with bounded feasible set. Moreover, we show that the bound is invariant under invertible affine transformation, and is independent of the algebraic representation of X.

We probe the relationship between the new bound and the well-known bounds in the literature. We prove that the new bound is equivalent to the best known (quadratic) bound for standard QPs. Moreover, we show that for box constrained QPs the dual of the new bound is Shor relaxation with partial first-level RLT.

Thanks to the new bound, we develop a new branch and cut method for concave QPs. We investigate the global convergence of the proposed method. In addition, we illustrate the effectiveness of the method by presenting its numerical performance on some concave QPs.

The paper is organized as follows. After reviewing terminologies and notations, the new dual for QPs is introduced in Sect. 2. In Sect. 3, we investigate the relationship between the new bound and the conventional bounds. Section 4 is devoted to developing an algorithm for concave QPs. Numerical experiments are reported in Sect. 5.

1.1 Notation

The n-dimensional Euclidean space is denoted by $\mathbf{R}^n$. We denote the ith row of a given matrix A by A_i. Vectors are considered to be column vectors and the superscript T denotes the transpose operation. We use e and e_i to denote vector of ones and ith unit vector, respectively. The nonnegative orthant is denoted by $\mathbf{R}^n_+$. Notation $A \succeq 0$ means matrix A is positive semi-definite, and $A \geq 0$ means that A is entry-wise nonnegative. Furthermore, $A \bullet B$ denotes the inner product of A and B, i.e., $A \bullet B = trace(AB^T)$.

For a set $X \subseteq \mathbf{R}^n$, we use the notations $int(X)$ and $cone(X)$ for the interior and the convex conic hull of X, respectively. For a convex cone K, its dual cone is defined and denoted by $K^* = \{y : y^T x \geq 0,\ \forall x \in K\}$. Two notations $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$ stand for the gradient and Hessian of smooth function f at $\bar{x}$. For the affine function $\alpha : \mathbf{R}^n \to \mathbf{R}$ given by $\alpha(x) = a^T x + a_0$, its norm is defined and denoted by $\|\alpha\| = \max_{i=0,1,\ldots,n} |a_i|$.

2 A new dual for quadratic programs

In this section, we present a new dual for QPs. Throughout the section, it is assumed that X is a bounded polyhedral set. We propose the following convex optimization problem as a dual of (QP),

$$\max \ \ell$$
$$s.t. x^T Qx + 2c^T x - \ell + \sum_{i=1}^{m} \alpha_i(x)(A_i x - b_i) \in P[x],$$

$$\sum_{i=1}^{m} \alpha_i(x)(A_i x - b_i) \le 0, \quad x \in X, \tag{2}$$

where α_i, $i = 1, \ldots, m$, are affine functions and $P[x]$ denotes the set of nonnegative polynomials on $\mathbf{R}^n$. Note that ℓ and α_i ($i = 1, ..., m$) are decision variables in problem (2). It is readily seen that the above problem is a convex problem with infinite constraints. Remark that a quadratic function $q(x) = x^T Qx + 2c^T x + c_0$ is nonnegative on $\mathbf{R}^n$ if and only if $\begin{pmatrix} Q & c \\ c^T & c_0 \end{pmatrix} \succeq 0$. We prove that problem (2) is feasible and fulfills strong duality. Before we get to the proof, we need to present a lemma.

Lemma 1 *Let $X = \{x \in \mathbf{R}^n : Ax \le b\}$ be a full-dimensional polytope and let $q(x) = x^T Qx + 2c^T x + c_0$ be a quadratic function. Then there exist affine functions α_i for $i = 1, \ldots, m$ such that*

$$x^T Qx + 2c^T x + c_0 = \sum_{i=1}^{m} \alpha_i(x)(A_i x - b_i).$$

Proof Without loss of generality, we may assume that Q is symmetric. The existence of affine functions α_i, $i = 1, \ldots, m$, satisfying the desired equality is equivalent to the solvability of the following linear system

$$\frac{1}{2}\sum_{i=1}^{m} \begin{pmatrix} d_i \\ f_i \end{pmatrix} \begin{pmatrix} A_i & -b_i \end{pmatrix} + \frac{1}{2}\sum_{i=1}^{m} \begin{pmatrix} A_i^T \\ -b_i \end{pmatrix} \begin{pmatrix} d_i^T & f_i \end{pmatrix} = \begin{pmatrix} Q & c \\ c^T & c_0 \end{pmatrix}, \tag{3}$$

where $\begin{pmatrix} d_i^T & f_i \end{pmatrix}$, $i = 1, \ldots, m$, are variables and it has $\frac{(n+1)(n+2)}{2}$ constraints. Indeed, $\alpha_i(x) = \begin{pmatrix} d_i^T & f_i \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$ for $i = 1, \ldots, m$. To prove the solvability of (3), it is sufficient to show that the coefficient matrix of linear system (3) has full row rank. On the contrary, suppose that it does not have full row rank, that is, the rows of the coefficient matrix are linearly dependent. So, there exists a non-zero symmetric matrix D such that

$$\frac{1}{2}\sum_{i=1}^{m} D \bullet \left(\begin{pmatrix} d_i \\ f_i \end{pmatrix} \begin{pmatrix} A_i & -b_i \end{pmatrix} \right) + \frac{1}{2}\sum_{i=1}^{m} D \bullet \left(\begin{pmatrix} A_i^T \\ -b_i \end{pmatrix} \begin{pmatrix} d_i^T & f_i \end{pmatrix} \right) = 0.$$

Since $A \bullet (x^T y) = x^T Ay$, we have

$$\sum_{i=1}^{m} \begin{pmatrix} A_i & -b_i \end{pmatrix} D \begin{pmatrix} d_i \\ f_i \end{pmatrix} = 0.$$

The above equality implies that $\begin{pmatrix} A_i & -b_i \end{pmatrix} D = 0$ for $i = 1, \ldots, m$. As X is full-dimensional, the set $\left\{ \begin{pmatrix} A_1^T \\ -b_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_m^T \\ -b_m \end{pmatrix} \right\}$ generates $\mathbf{R}^{n+1}$. Hence, $z^T D = 0$ for each $z \in \mathbf{R}^{n+1}$, which implies that $D = 0$. This contradicts the assumption $D \neq 0$ and implies the solvability of (3). □

The following theorem shows that the proposed dual satisfies strong duality.

Theorem 1 *Let X be a polytope. The optimal values of problems (QP) and (2) are equal.*

Proof Without loss of generality, we may assume that X is full-dimensional. Otherwise it is enough to consider (QP) on the affine space generated by X. Suppose that $q^\star$ is the optimal value of (QP). By Lemma 1, there exists $\bar{\alpha}_i$, $i = 1, \ldots, m$, such that

$$x^T Qx + 2c^T x - q^\star = -\sum_{i=1}^m \bar{\alpha}_i(x)(A_i x - b_i).$$

So, $\bar{\alpha}_i$, $i = 1, \ldots, m$, and $\bar{\ell} = q^\star$ are feasible for (2) and the optimal value of problem (2) is greater than or equal to $q^\star$. The constraints of problem (2) imply that the optimal value of the dual problem may not be greater than $q^\star$. Hence, the aforementioned feasible point is optimal for the dual problem and the proof is complete. □

Although problem (1) is convex, it is not tractable. This difficulty is caused by the number of constraints. To take advantages of this formulation, we need to adopt some procedures. A natural approach is the maximization of the objective function on a subset of the feasible set, i.e. restricting the feasible set of the dual problem. In the rest of the paper, we restrict the variables, α_i ($i = 1, \ldots, m$), to nonnegative affine functions on X.

If the vertices of X are also available one can consider the following set which includes the aforementioned set. Let $v_1, \ldots, v_k$ denote the vertices of X. It is easily seen that the affine functions α_i, $i = 1, \ldots, m$, which fulfill the following inequalities are feasible for problem (2),

$$\sum_{i=1}^m (A_i v_j - b_i)\alpha_i(x) \le 0, \quad x \in X, \quad j = 1, \ldots, k. \tag{4}$$

Non-homogenous Farkas' Lemma, Theorem 8.4.2 in [32], provides an explicit form of the affine functions which satisfy (4). Thus, it can be formulated by a finite number of linear inequalities. If $cone(\{Ax - b : x \in X\}) = -\mathbf{R}^m_+$, then α_i, $i = 1, \ldots, m$, that fulfill (4) are nonnegative affine functions on X.

Let $\mathcal{A}_+(X)$ denote the set of nonnegative affine functions on non-empty polytope X. By non-homogenous Farkas' Lemma, $\alpha(x) = d^T x + f$ is nonnegative on nonempty polyhedron X if and only if there exist nonnegative scalars λ_i, $i = 0, \ldots, m$, with $\alpha(x) = \lambda_0 + \sum_{i=1}^m \lambda_i(b_i - A_i x)$; See Theorem 8.4.2 in [32]. It is easily seen that $\mathcal{A}_+(X)$ is a polyhedral cone with a nonempty interior [32]. To tackle problem (2), we consider the following restricted problem,

$$\begin{aligned}
&\max \ \ell \\
&s.t. x^T Qx + 2c^T x - \ell + \sum_{i=1}^m \alpha_i(x)(A_i x - b_i) \in P[x], \\
&\alpha_i \in \mathcal{A}_+(X), \quad i = 1, \ldots, m.
\end{aligned} \tag{5}$$

Considering affine functions instead of scalars for Lagrange multipliers can be found in the literature. Sturm et al. [45] applied an affine function as a Lagrange multiplier to extend S-lemma. To make the point clear, consider the optimization problem

$$\begin{aligned}
&\min x^T Q_1 x + 2c_1^T x \\
&s.t. x^T Q_2 x + 2c_2^T x + b_2 \le 0, \\
&c_3^T x + b_3 \le 0,
\end{aligned} \tag{6}$$

where $Q_2 \succeq 0$ and $\mathcal{X} = \{x : x^T Q_2 x + 2c_2^T x + b_2 \leq 0, c_3^T x + b_3 \leq 0\}$ has a nonempty interior. They showed that the optimal values of problem (6) and the following problem are the same,

$$\begin{aligned}
&\max \ \ell \\
&\quad s.t. x^T Q_1 x + 2c_1^T x - \ell + t(x^T Q_2 x + 2c_2^T x + b_2) + (d^T x + f)(c_3^T x + b_3) \in P[x], \\
&\qquad t \geq 0, \ \begin{pmatrix} d \\ f \end{pmatrix} \in \mathcal{H}^*,
\end{aligned}$$

where the cone

$$\mathcal{H} = \left\{ \begin{pmatrix} x \\ x_0 \end{pmatrix} : x^T Q_2 x + 2x_0 c_2^T x + b_2 x_0^2 \leq 0, 2c_2^T x + b_2 x_0 \leq 0, x_0 \geq 0 \right\}.$$

As seen, the Lagrange multiplier of the linear constraint is an affine function. Unlike problem (5), which the feasible affine functions are characterized by the feasible set, only the quadratic constraint determines feasible affine functions. It may be of interest to verify the validity of Sturm et al.'s result when the constraint $\begin{pmatrix} d \\ f \end{pmatrix} \in \mathcal{H}^*$ is replaced with $d^T x + f \in \mathcal{A}_+(\mathcal{X})$. The following proposition says that the result also holds under these conditions.

Proposition 1 *If $Q_2 \succeq 0$ and $int(\mathcal{X}) \neq \emptyset$, then the optimal value of the following problem is equal to that of problem* (6)*,*

$$\begin{aligned}
&\max \ \ell \\
&\quad s.t. x^T Q_1 x + 2c_1^T x - \ell + t(x^T Q_2 x + 2c_2^T x + b_2) + (d^T x + f)(c_3^T x + b_3) \in P[x], \\
&\qquad t \geq 0, \ d^T x + f \in \mathcal{A}_+(\mathcal{X}).
\end{aligned} \tag{7}$$

Proof It is easily seen that problem (7) is a bound for (6). Thus, its optimal value is less than or equal to that of problem (6). By Sturm et al.'s result, to show equality, it suffices to prove the inclusion $\left\{ d^T x + f : \begin{pmatrix} d \\ f \end{pmatrix} \in \mathcal{H}^* \right\} \subseteq \mathcal{A}_+(\mathcal{X})$. Assume on the contrary, there exists $\begin{pmatrix} \bar{d} \\ \bar{f} \end{pmatrix} \in \mathcal{H}^*$, but $\bar{d}^T x + \bar{f} \notin \mathcal{A}_+(\mathcal{X})$. So, there exists $\bar{x} \in \mathcal{X}$ with $\bar{d}^T \bar{x} + \bar{f} < 0$. The positive semi-definiteness of Q_2 and $\bar{x} \in \mathcal{X}$ imply that $2c_2^T \bar{x} + b_2 \leq -\bar{x}^T Q_2 \bar{x} \leq 0$. Therefore, $\begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \in \mathcal{H}$. Since $\begin{pmatrix} \bar{d} \\ \bar{f} \end{pmatrix} \in \mathcal{H}^*$, we must have $\bar{d}^T \bar{x} + \bar{f} \geq 0$, which contradicts the assumption that $\bar{d}^T \bar{x} + \bar{f} < 0$ and completes the proof. $\square$

Note that under the assumptions of Proposition 1 an affine function α is nonnegative on $\mathcal{X}$ if and only if there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ with

$$\alpha(x) + \lambda_1(x^T Q_2 x + 2c_2^T x + b_2) + \lambda_2(c_3^T x + b_3) \in P[x];$$

See [32] for proof. As the left-hand-side of the first constraint of (7) is a quadratic polynomial, this problem can be formulated as a semi-definite program.

Typically, SOS polynomials are used as Lagrange multipliers in polynomial optimization [12,22,26]. However, to the best knowledge of author, affine functions have not been applied in this manner as dual variables in finite optimization theory.

Let us return back to problem (5). When X is nonempty, (5) can be formulated as the semi-definite program:

$$\max \ \ell$$
$$s.t. \frac{1}{2}\sum_{i=1}^{n}\begin{pmatrix} d_i \\ f_i \end{pmatrix}\begin{pmatrix} A_i & -b_i \end{pmatrix} + \frac{1}{2}\sum_{i=1}^{n}\begin{pmatrix} A_i^T \\ -b_i \end{pmatrix}\begin{pmatrix} d_i^T & f_i \end{pmatrix} + \begin{pmatrix} Q & c \\ c^T & -\ell \end{pmatrix} \succeq 0,$$
$$\begin{pmatrix} d_i \\ f_i \end{pmatrix} \in cone\left\{ \begin{pmatrix} -A_j^T \\ b_j \end{pmatrix}, 1 \le j \le m, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad i = 1, \ldots, m. \tag{8}$$

Note that if X is empty, the second constraint of (8) is not necessarily equivalent to the constraint $\alpha_i \in \mathcal{A}_+(X)$, $i = 1, \ldots, m$. As the second constraint of (8) gives α_i explicitly, the above problem can be written as

$$\max \ \ell$$
$$s.t. \frac{1}{2}\begin{pmatrix} -A^T Y A - A^T Y^T A & A^T(Yb + Y^T b + y) \\ (Yb + Y^T b + y)^T A & -2y^T b - 2b^T Y b \end{pmatrix} + \begin{pmatrix} Q & c \\ c^T & -\ell \end{pmatrix} \succeq 0,$$
$$Y \ge 0, \ y \ge 0, \tag{9}$$

where $Y \in \mathbf{R}^{m\times m}$ and $y \in \mathbf{R}^m$. In the above semi-definite program, the matrix

$$\Gamma_{ij} = \frac{-1}{2}\begin{pmatrix} A_i^T A_j + A_j^T A_i & -b_i A_j^T - b_j A_i^T \\ -b_i A_j - b_j A_i & 2b_i b_j \end{pmatrix}$$

is the coefficient of y_{ij}. As the coefficients of y_{ij} and y_{ji} are the same, we can assume that Y is symmetric. So, problem (9) is reformulated as

$$\max \ \ell$$
$$s.t. \begin{pmatrix} -A^T Y A & A^T(Yb + 0.5y) \\ (Yb + 0.5y)^T A & -y^T b - b^T Y b \end{pmatrix} + \begin{pmatrix} Q & c \\ c^T & -\ell \end{pmatrix} \succeq 0,$$
$$Y \ge 0, \ y \ge 0, \ Y^T = Y. \tag{10}$$

The next lemma shows that, under the boundedness of X, problem (5) is feasible.

Proposition 2 *If X is a polytope, then problem (5) is feasible.*

Proof As X is bounded, there exist scalars f_i, $i = 1, \ldots, m$, such that $\alpha_i(x) = A_i x + f_i$ is positive on X and $\alpha_i \in \mathcal{A}_+(X)$. Due to the boundedness of X, the system $Ad \le 0$ does not have any non-zero solution. Thus, $H = \sum_{i=1}^m A_i^T A_i$ is positive-definite. By choosing γ sufficiently large and a suitable choice of ℓ, $\gamma\alpha_i$, $i = 1, \ldots, m$, accompanying ℓ satisfy all constraints of problem (5). □

Proposition 2 does not hold necessarily for QPs with unbounded feasible set. The following example illustrates the point.

Example 1 Consider the QP

$$\min x_1 x_2$$
$$s.t. x_1 = 0.$$

In this example, problem (5) is infeasible while the optimal value is zero.

Problem (5) can be also interpreted via S-lemma. Let $int(X) \ne \emptyset$. For given $\alpha_i \in \mathcal{A}_+(X)$, $i = 1, \ldots, m$, we have $X \subseteq \{x : \sum_{i=1}^m \alpha_i(x)(A_i x - b_i) \le 0\}$, which is an overestimation of

X with a sublevel set of a quadratic function. By S-lemma, the optimal value of x^TQx+2c^Tx on this set is obtained by solving the problem

$$\max\left\{\ell : x^TQx+2c^Tx-\ell+t\left(\sum_{i=1}^m\alpha_i(x)(A_ix-b_i)\right)\in P[x],\ t\geq 0\right\}.$$

As $\mathcal{A}_+(X)$ is a cone, the first constraint of problem (5) is merely S-lemma and problem (5) provides the greatest optimal value x^TQx+2c^Tx on the sublevel sets of quadratic functions which include X.

In the sequel, we denote the set of $\{\sum_{i=1}^m\alpha_i(x)(A_ix-b_i):\alpha_i(x)\in\mathcal{A}_+(X)\}$ by $\mathcal{N}(X)$. One can infer from the proof of Proposition 2 that $\mathcal{N}(X)$ is nonempty and involves a strictly convex function. It is easily seen that $\mathcal{N}(X)$ is a closed, convex cone with nonempty interior. Moreover, if the interior of X is nonempty, then $\mathcal{N}(X)$ is pointed, i.e. $\mathcal{N}(X)\cap-\mathcal{N}(X)=\{0\}$.

The cone $\mathcal{N}(X)$ can be applied for obtaining a convex underestimator of a quadratic function on a given polytope. For (QP), one can consider $F:X\to\mathbf{R}$, defined by

$$F(x)=\max\,\{x^TQx+2c^Tx+p(x):p\in\mathcal{N}(X),\nabla^2p+2Q\succeq 0\},$$

as a convex underestimator. Another problem in which $\mathcal{N}(X)$ might be useful is the approximation of Löwner–John ellipsoid for a given polytope. We refer the interested reader to [4,7] for more details about Löwner–John ellipsoid.

As we consider a subset of the feasible set of dual problem, the optimal value of (5) may be strictly less than that of (2) or equivalently (QP). As a result, we can regard problem (5) as a new bound for QPs. The following proposition states this fact.

Proposition 3 *Let $\bar{x}$ and $\bar{\alpha}_i$ $(i=1,\ldots,m)$, $\bar{\ell}$ be feasible for* (QP) *and problem* (5)*, respectively. Then $\bar{x}^TQ\bar{x}+2c^T\bar{x}\geq\bar{\ell}$.*

Proof The first constraint of problem (5) implies $\bar{x}^TQ\bar{x}+2c^T\bar{x}-\bar{\ell}+\sum_{i=1}^m\bar{\alpha}_i(\bar{x})(A_i\bar{x}-b_i)\geq 0$. Since $-\sum_{i=1}^m\bar{\alpha}_i(\bar{x})(A_i\bar{x}-b_i)\geq 0$, we get $\bar{x}^TQ\bar{x}+2c^T\bar{x}\geq\bar{\ell}$, which is the desired inequality. □

The next theorem provides sufficient conditions under which the bound obtained in (5) is exact. Let $I(x)$ denote the set of active constraints at x, i.e. $I(x)=\{i:A_ix=b_i\}$.

Theorem 2 *Let X be a polytope. The optimal values of* (QP) *and problem* (5) *are the same if there exist $\bar{x}\in\operatorname{argmin}_{x\in X}x^TQx+2c^Tx$ and $d_i^Tx+f_i\in\mathcal{A}_+(X)$ for $i=1,2,\ldots,m$ such that*

$$Q+\frac{1}{2}\sum_{i=1}^m d_iA_i+\frac{1}{2}\sum_{i=1}^m A_i^Td_i^T\succeq 0,$$
$$Q\bar{x}+c+\sum_{i\in I(\bar{x})}(d_i^T\bar{x}+f_i)A_i^T+\sum_{i\in\{1,2,\ldots,m\}\setminus I(\bar{x})}(A_i\bar{x}-b_i)d_i=0,$$

and

$$d_i^T\bar{x}+f_i=0,\quad\forall i\in\{1,2,\ldots,m\}\setminus I(\bar{x}).$$

Proof Consider the quadratic function $q(x)=x^TQx+2c^Tx+\sum_{i=1}^m(d_i^Tx+f_i)(A_ix-b_i)$. The first condition implies that q is convex. We can infer from the second and third conditions that $\nabla q(\bar{x})=0$. By the sufficient optimality conditions for convex quadratic functions, $\bar{x}$ is an optimal solution problem $\{\min q(x):x\in\mathbf{R}^n\}$ with the optimal value $\bar{x}^TQ\bar{x}+2c^T\bar{x}$.

Additionally, $\alpha_i(x) = d_i^T x + f_i$, $i = 1, 2, \ldots, m$, and $\ell = \bar{x}^T Q\bar{x} + 2c^T\bar{x}$ are feasible for problem (5). By virtue of Proposition 3, the optimal value of both problems are equal and the proof is complete. □

We notice that checking the sufficient conditions provided in Theorem 2 for a given point is not difficult. In fact, it can be done by testing the feasibility of a semi-definite program, for which there exist polynomial time algorithms [4]. Under convexity, the above theorem holds for any optimal solution, so the bound is exact in the convex case. However, in general, the assumptions in Theorem 2 may be not satisfied.

Another point concerning Theorem 2 is that it provides sufficient conditions for global optimality. Some sufficient global optimality conditions by virtue of semi-definite relaxation can be found in the literature, see e.g., [49] and the references therein. As we shall see later, the dual of problem (5) is a semi-definite relaxation of (QP). Therefore, most of the derived results with a little modification can be applied to problem (5). We refer the interested reader to [5,14] for some necessary and sufficient global optimality conditions for QPs.

In general, when the feasible set of a semi-definite program is unbounded, it is probable that it does not realize its optimal value [4]. In the following proposition, we prove that problem (5) achieves its optimal value while the feasible set of (5) may be unbounded.

Proposition 4 *If X is a polytope, then problem* (5) *achieves its optimal value.*

Proof Without loss of generality, we can assume $int(X) \neq \emptyset$. Otherwise, it is enough to consider the polytope on the affine space generated by itself.
By Proposition 3, the supremum of problem (5), denoted by $\bar{\ell}$, is finite. Hence, there exist sequences $\{\alpha_i^k\} \subseteq \mathcal{A}_+(X)$,$i = 1, \ldots, m$, and $\{\ell^k\}$ such that $\ell^k \to \bar{\ell}$ and $x^T Qx + 2c^T x - \ell^k + \sum_{i=1}^m \alpha_i^k(x)(A_i x - b_i) \in P[x]$. If the sequences $\{\alpha_i^k\}$, $i = 1, \ldots, m$, are bounded, then due to the closedness of $P[x]$ and $\mathcal{A}_+(X)$, the proof is complete. Otherwise, by setting $\mu_i^k = \frac{\alpha_i^k}{t_k}$ for $i = 1, \ldots, m$ with $t_k = \max_{1\leq i\leq m} \|\alpha_i^k\|$, and choosing appropriate subsequences if necessary, we can assume that $\mu_i^k \to \bar{\mu}_i$ for $i = 1, \ldots, m$ and there exists j such that $\bar{\mu}_j \neq 0$. Due to the closedness of the set of nonnegative polynomials, we have $q(x) = \sum_{i=1}^m \bar{\mu}_i(x)(A_i x - b_i) \in P[x]$. As $int(X) \neq \emptyset$, there exists some $\bar{x} \in X$ such that $\bar{\mu}_j(\bar{x}) > 0$ and $A_j\bar{x} < b_j$. Therefore, $q(\bar{x}) < 0$, which contradicts $q \in P[x]$. □

It follows from the proof of Proposition 4 that all optimal solutions of problem (5) are bounded when $int(X) \neq \emptyset$. One important question may arise about the bound obtained in (5) is that: Is the optimal value of problem (5) independent of the representation of X? The next theorem gives the affirmative answer to the question.

Theorem 3 *Let $\{x \in \mathbf{R}^n : \bar{A}x \leq \bar{b}\}$ and $\{x \in \mathbf{R}^n : \hat{A}x \leq \hat{b}\}$ be two different representations of the polytope X. Then the optimal values of problem* (5) *corresponding to these representations are equal.*

Proof Suppose that $\hat{\ell}$ and $\bar{\ell}$ are the optimal values of problem (5) corresponding to the representations $\{x \in \mathbf{R}^n : \hat{A}x \leq \hat{b}\}$ and $\{x \in \mathbf{R}^n : \bar{A}x \leq \bar{b}\}$, respectively, with $\hat{A} \in \mathbf{R}^{\hat{m}\times n}$ and $\bar{A} \in \mathbf{R}^{\bar{m}\times n}$. By Proposition 4, there exist $\hat{\alpha}_i$ $(i = 1, \ldots, \hat{m})$ which are optimal for

$$\max \ell$$

$$s.t. x^T Qx + 2c^T x - \ell + \sum_{i=1}^{\hat{m}} \alpha_i(x)(\hat{A}_i x - \hat{b}_i) \in P[x],$$

$$\alpha_i \in \mathcal{A}_+(X),\ i = 1, \ldots, \hat{m}. \tag{11}$$

As $\hat{b_i} - \hat{A}_i x \in \mathcal{A}_+(X)$, there exist nonnegative scalars Y_{ij}, $j = 0, 1, \ldots, \bar{m}$, such that

$$\hat{A}_i x - \hat{b_i} = \sum_{j=1}^{\bar{m}} Y_{ij}(\bar{A}_j x - \bar{b_j}) - Y_{i0}.$$

Similarly, according to $\hat{\alpha_i} \in \mathcal{A}_+(X)$, there exist nonnegative scalars W_{ij}, $j = 0, 1, \ldots, \bar{m}$, with $\hat{\alpha_i}(x) = -\sum_{j=1}^{\bar{m}} W_{ij}(\bar{A}_j x - \bar{b_j}) + W_{i0}$. By replacement, we have

$$\begin{aligned}
&x^T Qx + 2c^T x - \hat{\ell} + \sum_{i=1}^{\hat{m}} \hat{\alpha_i}(x)\Big(\sum_{j=1}^{\bar{m}} Y_{ij}(\bar{A}_j x - \bar{b_j}) - Y_{i0}\Big)\\
&= x^T Qx + 2c^T x - \hat{\ell} + \sum_{j=1}^{\bar{m}}(\bar{A}_j x - \bar{b_j})\Big(\sum_{i=1}^{\hat{m}} Y_{ij}\hat{\alpha_i}(x)\Big) - \sum_{i=1}^{\hat{m}} Y_{i0}\hat{\alpha_i}(x)\\
&= x^T Qx + 2c^T x - \left(\hat{\ell} + \sum_{i=1}^{\hat{m}} Y_{i0}W_{i0}\right) + \sum_{j=1}^{\bar{m}}(\bar{A}_j x - \bar{b_j})\Big(\sum_{i=1}^{\hat{m}}(Y_{ij}\hat{\alpha_i}(x) + W_{ij}Y_{i0})\Big)\\
&\in P[x].
\end{aligned}$$

As $\mathcal{A}_+(X)$ is a convex cone, $\tilde{\alpha_i}(x) = \sum_{i=1}^{\hat{m}}(Y_{ij}\hat{\alpha_i}(x) + W_{ij}Y_{i0}) \in \mathcal{A}_+(X)$ for $i = 1, \ldots, \bar{m}$. So, $\tilde{\ell} = \hat{\ell} + \sum_{i=1}^{\hat{m}} Y_{i0}W_{i0}$ and $\tilde{\alpha}_i$, $i = 1, \ldots, \bar{m}$, are feasible for problem (5) corresponding to the representation $X = \{x \in \mathbf{R}^n : \bar{A}x \leq \bar{b}\}$. This implies $\bar{\ell} \geq \hat{\ell}$, because $Y_{i0}W_{i0} \geq 0$. By a similar argument, one can establish $\ell \hat{\geq} \bar{\ell}$. Therefore, $\bar{\ell} = \hat{\ell}$ and the proof is complete. □

By arguments similar to the proof of Theorem 3, one can prove that if polytope X_1 is a subset of polytope X_2, then $opt(X_2, Q, c) \leq opt(X_1, Q, c)$. (Let $opt(X, Q, c)$ denote the optimal value of problem (5) corresponding to polytope X, matrix Q and vector c). We call this property as inclusion property. In the following proposition, we show that the optimal value of problem (5) is constant under invertible affine transformation.

Proposition 5 *The optimal value of problem* (5) *is invariant under affine invertible transformations.*

Proof Let T be an invertible affine transformation on $\mathbf{R}^n$. We set $Y = T^{-1}(X)$ and assume that $T(y) = Hy + h$ for some invertible matrix H and $h \in \mathbf{R}^n$. For $\alpha_i \in \mathcal{A}_+(X)$, $i = 1, \ldots, m$, and ℓ satisfying

$$x^T Qx + 2c^T x - \ell + \sum_{i=1}^{m} \alpha_i(x)(A_i x - b_i) \in P[x],$$

we have

$$y^T \bar{Q} y + 2\bar{c}^T y + c_0 - \ell + \sum_{i=1}^{m} \mu_i(y)(\bar{A}_i y - \bar{b}_i) \in P[y],$$

where $\mu_i = \alpha_i T \in \mathcal{A}_+(Y)$, $\bar{Q} = H^T QH$, $\bar{c} = Hc + H^T Qh$, $c_0 = h^T Qh + 2c^T h$, $\bar{A} = AH$ and $\bar{b} = b - Ah$. This statement implies $opt(X, Q, c) \leq opt(Y, \bar{Q}, \bar{c}) - c_0$. By a similar argument, one can derive that $opt(Y, \bar{Q}, \bar{c}) - c_0 \leq opt(X, Q, c)$. This completes the proof. □

The following proposition states if a polytope is singleton, then bound (5) is exact.

Proposition 6 *If $X = \{\bar{x}\}$, then the optimal values of* (QP) *and* (5) *are equal.*

Proof According to Proposition 5, the optimal value of problem (5) is independent of translation. Thus, without loss of generality, we may assume that $\bar{x} = 0$. In addition, by virtue of Theorem 3, we may assume $X = \{x : x = 0\}$. We define $\alpha_i, \mu_i \in \mathcal{A}_+(X)$ for $i = 1, \dots, n$, corresponding to (QP), as follows

$$\alpha_i(x) = \begin{cases} 0 & c_i \geq 0 \\ (-Qx)_i - 2c_i & c_i < 0 \end{cases} \quad \mu_i(x) = \begin{cases} (Qx)_i + 2c_i & c_i \geq 0 \\ 0 & c_i < 0 \end{cases}$$

Thus, $x^T Qx + 2c^T x + \sum_{i=1}^m \alpha_i(x)(x_i) + \sum_{i=1}^m \mu_i(x)(-x_i) = 0$. Due to Proposition 3, the optimal value of (5) is zero, which is the desired conclusion. $\square$

Since (5) is a convex optimization problem, it is natural to ask about its dual. To get the dual of problem (5), we consider problem (10). The dual of problem (10) is formulated as follows,

$$\begin{aligned}
&\min \begin{pmatrix} Q & c \\ c^T & 0 \end{pmatrix} \bullet \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} \\
&s.t. \begin{pmatrix} A_i^T A_j & -b_j A_i^T \\ -b_j A_i & b_i b_j \end{pmatrix} \bullet \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} \geq 0, \quad i \leq j = 1, \dots, n \\
&\quad \begin{pmatrix} 0 & -.5A_i^T \\ -.5A_i & b_i \end{pmatrix} \bullet \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} \geq 0, \quad i = 1, \dots, n \\
&\quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} = 1, \\
&\quad \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} \succeq 0.
\end{aligned} \tag{12}$$

By $(A_i^T A_j) \bullet X = A_i X A_j^T$ and Shor decomposition, the dual may be rewritten as

$$\begin{aligned}
&\min Q \bullet X + 2c^T x \\
&s.t. AXA^T - bx^T A^T - Axb^T \geq -bb^T, \\
&\quad Ax \leq b, \\
&\quad X \succeq xx^T.
\end{aligned} \tag{13}$$

Problem (13) is a semi-definite relaxation of (QP). Indeed, let $\bar{x}$ be a feasible point of (QP), it is easily seen that the matrix $\begin{pmatrix} \bar{x}\bar{x}^T & \bar{x} \\ \bar{x}^T & 1 \end{pmatrix}$ is feasible for (13) and $\bar{x}^T Q\bar{x} + 2c^T\bar{x} = Q \bullet (\bar{x}\bar{x}^T) + 2c^T\bar{x}$.

A semi-definite program is called strongly feasible if it is feasible and remains feasible for all sufficiently small perturbations of right side of linear constraints [38]. As problem (5) is feasible for any quadratic function, the semi-definite program (5) is strongly feasible. By Theorem 3.2.6 in [38], the strong duality holds, that is, the optimal values of both problems (5) and (13) are the same.

As problem (13) is the dual of problem (5), it follows from Proposition 3 that it is bounded from below, and there is no need to add more constraints. In general, semi-definite relaxations are not necessarily bounded from below [18]. Burer et al. [8] called this relaxation a strong relaxation. One may wonder to know about the relationship between this relaxation with other semi-definite relaxations existing in the literature. It is seen that problem (13) is Shor's relaxation of

$$\begin{aligned} &\min x^T Qx + 2c^T x \\ &s.t. - Ax \geq -b, \\ &\quad (A_i x - b_i)(A_j x - b_j) \geq 0, \; i \leq j = 1, \ldots, m \end{aligned}$$

which is exactly (QP) with $0.5m(m+1)$ redundant constraints [43]. Most relaxation methods, including RLT, use redundant constraints [42]. Gorge et al. [18] used convex combination of these redundant constraints as a cut for semi-definite relaxations. We refer the reader to [18] for more information on the applications of these redundant constraints for QPs. Note that the constraint $Ax \leq b$ is redundant for (13) and so it can be removed [42].

One important inquiry about bounds is how one can reduce the gap. In this context, one idea may be the replacement of nonnegative affine functions with nonnegative convex quadratic functions on the given polytope. Similar to the affine case, a new bound can be formulated as a semi-definite program. Furthermore, one can show that most presented results in Sect. 2 hold in this case. However, the number of variables is of the order of m^3, and makes this formulation less attractive. In addition, numerical implementations showed that the gap improvement was not considerable compared to the affine case. Pursuing this method by polynomials with degree greater than or equal to three is not practical since checking positivity of such a polynomial on a given polytope is not easy [25]. However, the procedure can be followed by restricting to some classes of polynomials [21]. Moreover, we refer the interested reader to [48] for the extension bound (5) to quadratically constrained quadratic programs.

As mentioned before Proposition 3, problem (5) can be written as

$$\max_{q \in \mathcal{N}(X)} \{\min x^T Qx + 2c^T x : q(x) \leq 0\},$$

and the bound (5) is exact provided $x^T Qx + 2c^T x - q^\star \in P[x] - \mathcal{N}(X)$ ($q^\star$ denotes the optimal value of (QP)). As a result, enlargement of $\mathcal{N}(X)$ may lead to the reduction of gap.

Let $\bar{x} \in int(X)$ and let $d \in \mathbf{R}^n$ be an arbitrary non-zero vector. The polytope X can be partitioned in two polytopes $X_1 = \{x \in X : d^T(x - \bar{x}) \leq 0\}$ and $X_2 = \{x \in X : d^T(x - \bar{x}) \geq 0\}$. It is readily seen that

$$\mathcal{N}(X) \subseteq \mathcal{N}(X_1) \cap \mathcal{N}(X_2).$$

It is likely that cone $\mathcal{N}(X)$ is strictly included in $\mathcal{N}(X_1) \cap \mathcal{N}(X_2)$ when bound (5) is not exact. To check $q \in \mathcal{N}(X_1) \cap \mathcal{N}(X_2)$ for two polytopes $X_1 = \{x \in \mathbf{R}^n : \hat{A}_i x \leq \hat{b}_i, \; 1 \leq i \leq \hat{m}\}$ and $X_2 = \{x \in \mathbf{R}^n : \bar{A}_i x \leq \bar{b}_i, \; 1 \leq i \leq \bar{m}\}$, one needs to solve the linear system

$$\begin{aligned} q(x) &= \sum_{i=1}^{\hat{m}} \hat{\alpha}_i(x)(\hat{A}_i x - \hat{b}_i) \\ q(x) &= \sum_{i=1}^{\bar{m}} \bar{\alpha}_i(x)(\bar{A}_i x - \bar{b}_i) \\ \hat{\alpha}_i &\in \mathcal{A}_+(X_1), \; i = 1, \ldots, \hat{m} \\ \bar{\alpha}_i &\in \mathcal{A}_+(X_2), \; i = 1, \ldots, \bar{m}. \end{aligned}$$

Thus, by the replacement of $\mathcal{N}(X)$ with $\mathcal{N}(X_1) \cap \mathcal{N}(X_2)$ in (5), the number of variables and constraints will be at least twice the former case. It is easily seen the optimal value of

(5) depends on the choice of $\bar{x}$ and d. Note that partitioning the feasible set is a wide-spread method for reducing the duality gap; See [13,44,46] and references therein. We will take advantage of this idea to develop a branch and cut algorithm for concave QPs.

In the same line, one can partition X to k polytopes and fattens $\mathcal{N}(X)$. In this case, the number of variables will be of $O(km^2)$. For instance, for $\bar{x} \in int(X)$, one could consider two different hyperplanes which pass through the given point. As a result, the polytope is divided into four polytopes.

Although problem (5) provides a lower bound for (QP), the number of variables is of $O(m^2)$, which makes this semi-definite program time-consuming in some cases. In the sequel, we propose the bounds whose decision variables are less than that of problem (5). However, this problem does not equip us with a better lower bound.

Consider problem (QP). Let L denote the subspace generated by the eigenvectors of Q corresponding to the negative eigenvalues. We propose the following bound for (QP),

$$\begin{aligned} &\max\ \ell \\ &s.t. x^T Qx + 2c^T x - \ell + \sum_{i=1}^{m} \alpha_i(x)(A_i x - b_i) \in P[x], \\ &\quad \alpha_i \in \mathcal{A}_+(X), \quad i = 1, \dots, m, \\ &\quad \nabla\alpha_i \in L, \quad i = 1, \dots, m. \end{aligned} \tag{14}$$

The above problem is reduced to problem (5) if Q is negative definite. Nevertheless, for the class of problems which Q has only one negative eigenvalue, problem (14) has $2m + 1$ variables. This follows from the point that the affine function α is feasible to problem (14) if and only if there exist scalars λ_0 and λ_1 such that

$$\alpha(x) = \lambda_1 o^T x + \lambda_0,\ \min\{\lambda_1 \bar{l} + \lambda_0, \lambda_1 \bar{u} + \lambda_0\} \geq 0,$$

where $o \neq 0$ is a constant vector in L and $\bar{l}$ and $\bar{u}$ denote the minimum and maximum of $\mu(x) = o^T x$ on X, respectively. In the viewpoint of computation time it would be more beneficial to tackle this problem instead of (5). We prove that the optimal value of problem (14) is finite in the next proposition.

Proposition 7 *Let X be a polytope. Then problem (14) has finite optimal value.*

Proof By Proposition 3, the optimal value of (14) is either finite or minus infinity. So, it suffices to prove the existence of a feasible point. Let $\{v_1, \dots, v_k\}$ be a basis for L. As X is bounded, by virtue of Farkas' Lemma, there exist nonnegative constants Y_{ij}, $j = 1, \dots, k$, such that $v_j = \sum_{i=1}^{m} Y_{ij} A_i^T$. Moreover, there are f_j, $j = 1, \dots, k$, with $v_j^T x + f_j \in \mathcal{A}_+(X)$. For γ sufficiently large, the matrix $Q + \gamma \sum_{j=1}^{k} v_j v_j^T$ is positive semi-definite. As X is bounded, for suitable choice of ℓ the affine functions $\alpha_i(x) = \gamma \sum_{j=1}^{k} Y_{ij}(v_j^T x + f_j)$, $i = 1, \dots, m$, fulfill all constraints of (14). □

The following example demonstrates that the optimal value of (14) may be strictly less than that of (5).

Example 2 Consider the QP,

$$\begin{aligned} &\min\ 2x_1 x_2 \\ &s.t.\ \ x_1, x_2 \leq 1 \\ &\qquad -x_1, -x_2 \leq 0. \end{aligned}$$

The optimal values of this QP and (5) is zero (Take into account $\alpha_1(x) = 0, \alpha_2(x) = 0, \alpha_3(x) = x_2, \alpha_4(x) = x_1, \ell = 0$). It is seen that $\alpha \in \mathcal{A}_+(X)$ and $\nabla\alpha \in L$ if and only if $\alpha \in cone(\{x_1 - x_2 + 1, -x_1 + x_2 + 1\})$. By solving problem (14), for any feasible point, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ell$, we have $\ell \leq \frac{-1}{8}$.

One can also formulate a semi-definite program for box constrained QPs with fewer variables in comparison with (5). In this case, one could consider the affine coefficient of constraint $x_k \leq u_k$ $(-x_k \leq -l_k)$, α_k, in the form

$$\alpha_k(x) = d_k x_k + f_i, \ \alpha_k \in \mathcal{A}_+(X),$$

where $d_k \in \mathbf{R}$. By arguments similar to Proposition 2, it is proved that the bound is finite in this case as well.

3 Comparison with existing bounds

One important question here is the relationship between bound (5) and the conventional bounds for QPs. This section sets out to answer this question. For comparison, we focus on two types of QPs, standard quadratic programs and box constrained quadratic programs. Let us first concentrate on standard quadratic programs. Consider the standard quadratic program,

$$\begin{aligned} &\min \ x^T Q x \\ &s.t. \ \textstyle\sum_{i=1}^n x_i = 1, \\ &\qquad x \geq 0. \end{aligned} \tag{StQP}$$

It is well-known that (StQP) is solvable in polynomial time if Q is either positive semi-definite or negative semi-definite on standard simplex (denoted by Δ hereafter). In general, (StQP) is NP-hard [6].

We denote the optimal value of (StQP) by ℓ_Q. Note that optimizing a quadratic function on standard simplex can be casted as (StQP). This follows form the point that for each $x \in \Delta$, we have $x^T Q x + 2c^T x = x^T(Q + ec^T + ce^T)x$. It is seen that $\ell_{Q+tee^T} = \ell_Q + t$ for each $t \in \mathbf{R}$. So, in our discussion for standard quadratic programs we make the assumption that Q is component-wise non negative.

Bomze et al. [6] proposed the best (quadratic) convex underestimation bound as follows

$$\ell_Q^{conv} = \sup\{\ell_S : S \succeq 0, \ Q - S \geq 0, diag(S) = diag(Q)\}, \tag{15}$$

where $diag(S)$ denotes the diagonal of S. They showed that the above problem gives better bound in comparison with other quadratic bounds. Problem (5) for (StQP) is formulated as follows,

$$\begin{aligned} &\max \ \ell \\ &s.t. \ x^T Q x - \ell + \sum_{i=1}^n \alpha_i(x)(-x_i) + \alpha_{n+1}(x)(e^T x - 1) \in P[x], \\ &\quad \alpha_i \in \mathcal{A}_+(\Delta), \ i = 1, \ldots, n. \end{aligned} \tag{16}$$

The next theorem shows that bounds (15) and (16) are equivalent.

Theorem 4 *Problems* (15) *and* (16) *give the same bound.*

Proof Let $\bar{\ell}$ denote the optimal value problem (16). First we show that $\ell_Q^{conv} \le \bar{\ell}$. Without loss of generality, we may assume that (15) attains an optimal solution S. As ith row of $Q - S$ is greater than or equal to zero, $\bar{\alpha}_i(x) = (Q_i - S_i)x \in \mathcal{A}_+(\Delta)$, $i = 1, \ldots, n$, and

$$x^T Qx - x^T Sx + \sum_{i=1}^{n} \bar{\alpha}_i(x)(-x_i) = 0.$$

By the optimality conditions for convex QPs, there are nonnegative scalars β_i, $i = 1, \ldots, n$ and $\beta_{n+1} \in \mathbf{R}$ such that

$$x^T Sx - \ell_Q^{conv} + \sum_{i=1}^{n} \beta_i(-x_i) + \beta_{n+1}(e^T x - 1) \in P[x].$$

By the above equalities, it is seen that $\alpha_i(x) = \bar{\alpha}_i(x) + \beta_i$, $i = 1, \ldots, n$, $\alpha_{n+1}(x) = \beta_{n+1}$ and $\ell = \ell_Q^{conv}$ are feasible for problem (16). Hence, $\ell_Q^{conv} \le \bar{\ell}$.
Now, let $\bar{\ell}$ and $\bar{\alpha}_i = a_i^T x + a_i^0$, $i = 1, \ldots, n+1$, be optimal for (16). We get

$$x^T Qx - \bar{\ell} - \sum_{i=1}^{n} x_i \bar{\alpha}_i(x) + (e^T x - 1)\bar{\alpha}_{n+1}(x) = (x - \bar{x})^T S(x - \bar{x}) + s, \tag{17}$$

where S and s are a positive semi-definite matrix and a nonnegative scalar, respectively, and $\bar{x} \in \mathbf{R}^n$. Let $e^T x \neq 0$. By replacing x with $(e^T x)^{-1} x$ and multiplying both sides of (17) by $(e^T x)^2$, we get

$$x^T(Q - \bar{\ell} ee^T)x - \sum_{i=1}^{n} x^T e_i (a_i + a_i^0 e)^T x = (x - (e^T x)\bar{x})^T S(x - (e^T x)\bar{x}) + s((e^T x))^2.$$

Since $\alpha_i \in \mathcal{A}_+(\Delta)$, $a_i^j + a_i^0 \ge 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. So, matrix $N = \sum_{i=1}^{n} e_i(a_i + a_i^0 e)^T \ge 0$. Since $S \succeq 0$ and $s \ge 0$, the homogenous quadratic function $(x - (e^T x)\bar{x})^T S(x - (e^T x)\bar{x}) + s((e^T x))^2$ is nonnegative on $\mathbf{R}^n$. Thus, $(x - (e^T x)\bar{x})^T S(x - (e^T x)\bar{x}) + s((e^T x))^2 = x^T \bar{S} x$ for some $\bar{S} \succeq 0$. Hence,

$$\max\,\{\ell : Q - \ell ee^T \succeq N,\ N \ge 0\} \ge \bar{\ell}.$$

Because $\ell_Q^{conv} = \max\,\{\ell : Q - \ell ee^T \succeq N,\ N \ge 0\}$ (see Section 6 in [6]), we know $\bar{\ell} \le \ell_Q^{conv}$ and the proof is complete. □

In the rest of the section, we continue our discussion for box constrained QPs. Consider the box constrained QP

$$\begin{aligned} &\min\ x^T Qx + 2c^T x \\ &s.t.\ \ a_i^T x = d_i, \quad i = 1, \ldots, m \\ &\qquad\ l \le x \le u, \end{aligned} \tag{18}$$

where $a_i \in \mathbf{R}^n$ and $d_i \in \mathbf{R}$ for $i = 1, \ldots, m$. For convenience, we may assume that $l = 0$ and $u = e$. By the combination of semi-definite programming relaxation (SDP) and RLT, Anstreicher [1] proposed some relaxations for problem (18). One of the most effective relaxations in this category is Shor relaxation with partial first-level RLT (SRLT).

This relaxation is formulated as

$$
\begin{array}{lll}
\min & Q \bullet X + 2c^T x & \\
s.t. & Xa_i = d_i x, & i = 1, \ldots, m \\
& a_i^T x = d_i, & i = 1, \ldots, m \\
& X \geq 0, & \\
& ex^T - X \geq 0, & \\
& X - ex^T - xe^T + ee^T \geq 0, & \\
& X - xx^T \succeq 0. &
\end{array}
\tag{19}
$$

Bao et al. [2] provided a full comparison of relaxation methods for box constrained QPs. They showed that SRLT dominates other relaxations. By applying formulation (13), the dual of bound (5) for problem (18) is formulated as follows,

$$
\begin{array}{lll}
\min & Q \bullet X + 2c^T x & \\
s.t. & a_i^T X a_j - d_i a_j x - d_j a_i x = -d_i d_j, & i \leq j = 1, \ldots, m \\
& Xa_i = d_i x, & i = 1, \ldots, m \\
& Xa_i - (a_i^T x - d_i)e = d_i x, & i = 1, \ldots, m \\
& X \geq 0, & \\
& ex^T - X \geq 0, & \\
& X - ex^T - xe^T + ee^T \geq 0, & \\
& X - xx^T \succeq 0, & \\
& a_i^T x = d_i, & i = 1, \ldots, m \\
& 0 \leq x \leq e, & \\
& X \succeq xx^T. &
\end{array}
\tag{20}
$$

By virtue of Remark 1 in [42], the constraints $a_i^T X a_j - d_i a_j x - d_j a_i x = -d_i d_j$ and $Xa_i - (a_i^T x - d_i)e = d_i x, i \leq j = 1, \ldots, m$, are redundant for problem (20). By removing these redundant constraints and the redundant constraints $0 \leq x \leq e$, we get SRLT. As strong duality holds for problem (5), SRLT and the bound obtained in (5) are equivalent for box constrained QPs.

One important issue with quadratic programs is how to convert a relaxation solution to an approximate solution [31]. As problem (5) not only provides a lower bound for quadratic programs, but also gives a convex underestimator, one may obtain an approximate solution by optimizing the given function on the feasible set. We use this strategy in the next section.

We conclude the section by addressing an interesting point about bound (5). This bound can be regarded as a special case of the following bound

$$
\begin{array}{ll}
\max & \ell \\
s.t. & x^T Q x + 2c^T x - \ell - \sum_{\tau \in \mathbb{N}_d^d} \lambda_\tau \prod_{i=1}^d (b_i - A_i x)^{\tau_i} = \sigma(x), \\
& \sigma \in \Sigma[x],\ \lambda \geq 0,
\end{array}
\tag{21}
$$

where $\mathbb{N}_d^d = \{\tau \in \mathbb{N}^d : \sum_{i=1}^d \tau_i \leq d\}$. One may regard the above-mentioned bound as a combination of Lasserre hierarchy and Handelman's hierarchy [26,27]. One can obtain bound (5) by setting $d = 2$ in (21). In fact, this follows from Non-homogenous Farkas' Lemma. Recently, Lasserre et al. [26] took advantage of this idea and proposed new bounds for general polynomial optimization problems.

4 A new algorithm for concave quadratic optimization

In this section, by virtue of the newly introduced bound, we propose a new algorithm for concave QPs. Throughout the section, it is assumed that X is a polytope with nonempty interior and Q is negative semi-definite. We introduce a branch and cut (B&C) algorithm. We use Konno's cut in the cutting step. Before we go into the details of the algorithm, let us remind a definition.

Definition 1 Let $\hat{x} \in X$ be a vertex. This vertex is called a local optimal if the value of the objective function at this point is less than or equal to that at the adjacent vertices.

As mentioned earlier, we are developing a B&C algorithm for concave QPs. The main steps of a B&C method are branching, bounding, fathoming and cutting. A typical branching approach for QPs is partitioning and for bounding is linear program relaxation based on RLT; See [19,41] for more details. Recently, Burer et al. and Chen et al. employed KKT optimality conditions and semi-definite relaxation for branching and bounding, respectively [9,11]. In the sequel, we present the details of the steps.

The proposed method regards (QP) as the root node of the B&C tree. Let the following quadratic program be the subproblem of a node,

$$\begin{aligned} &\min\ x^T Qx + 2c^T x \\ &s.t.\ \ \bar{A}x \leq \bar{b}, \end{aligned}$$

and let $\bar{X}$ denote the feasible set of the above problem. To get a lower bound for the node, we formulate the semi-definite program

$$\begin{aligned} &\max \ell \\ &s.t. \begin{pmatrix} -\bar{A}^T Y \bar{A} & \bar{A}^T (Y\bar{b} + 0.5y) \\ (Y\bar{b} + 0.5y)^T \bar{A} & -y^T \bar{b} - \bar{b}^T Y \bar{b} \end{pmatrix} + \begin{pmatrix} Q & c \\ c^T & -\ell \end{pmatrix} \succeq 0, \\ &\quad \ell \leq u \\ &\quad Y \geq 0,\ Y = Y^T,\ y \geq 0, \end{aligned} \tag{22}$$

where u is the best upper bound generated by the algorithm so far. Let $\bar{Y}$, $\bar{y}$ and $\bar{\ell}$ be optimal solutions to (22). Then, we formulate the convex QP,

$$\begin{aligned} &\min x^T (Q - \bar{A}^T \bar{Y} \bar{A})x + 2(c + \bar{A}^T \bar{Y} \bar{b} + 0.5\bar{A}^T \bar{y})^T x \\ &s.t. Ax \leq b. \end{aligned} \tag{23}$$

Let $\hat{x}$ be a solution of problem (23). Next, a local vertex optimal point $\bar{x} \in X$ is chosen such that $\bar{x}^T Q\bar{x} + 2c^T \bar{x} \leq \hat{x}^T Q\hat{x} + 2c^T \hat{x}$. This step is not time-consuming. Indeed, there exist some efficient approaches for computing $\bar{x}$ [46]. We use the value $\bar{x}^T Q\bar{x} + 2c^T \bar{x}$ to update the upper bound, u. We use $\bar{\ell}$ and other lower bounds obtained from fathomed nodes and child nodes to update the lower bound.

After the bounding step, if $\bar{\ell} - u \geq -\epsilon$ the node will be fathomed. (ϵ is the prescribed tolerance). Otherwise, the algorithm solves problem (23) corresponding to the feasible set of the subproblem, $\bar{X}$, and computes a local optimal point $\bar{x} \in \bar{X}$ for concave QP $\{\min x^T Qx + 2c^T x : x \in \bar{X}\}$.

In the next step, the algorithm produces a cut. Let us go into the details of the step. As mentioned above, we employ Konno's cut. For convenience, let $\bar{x} = 0$ be a non-degenerated local optimal vertex of $\{\min x^T Qx + 2c^T x : x \in \bar{X}\}$. Suppose that vectors e_i, $i = 1, \ldots, n$, denote extreme directions at $\bar{x} \in \bar{X}$. Let $q(x) = x^T Qx + 2c^T x$. To compute Konno's cut, first we need to obtain Tuy's cut given by

$$\sum_{i=1}^{n} \frac{x_i}{t_i} \geq 1,$$

where $t_i := \max\{\theta : q(\bar{x} + \theta e_i) \geq u - \epsilon\}$ and $\epsilon > 0$ is the prescribed tolerance. Then, the algorithm computes $y^i \in \operatorname{argmin}\{\bar{x}^T Qy + 2c^T y : y \in X, \sum_{i=1}^{n} \frac{y_i}{t_i} \geq 1\}$. If the following inequality holds it will continue the cutting step, for otherwise it goes to the branching step,

$$\min_{i=1,\ldots,n} \{q(y^i)\} \geq u - \epsilon.$$

We call $\bar{x}$ an eligible vertex if we have the above inequality. Let $\bar{x}$ be eligible. The method will add the valid cut $\sum_{i=1}^{n} \frac{x_i}{s_i} \geq 1$, where

$$s_i = \max\{\theta : -A^T\lambda + t^{-1}\mu - Q_i^T\theta = c, -b^T\lambda + \mu + c_i\theta \geq u - \epsilon, \lambda \geq 0, \mu \geq 0\},$$

for $i = 1, \ldots, n$ and $t^{-1} = \left(\frac{1}{t_1}, \cdots, \frac{1}{t_n}\right)^T$. We refer the interested reader to [23,46] for more information on Konno's cut.

After computing Konno's cut, the method updates the feasible set, that is, set $\bar{X} = \{x \in \bar{X} : \sum_{i=1}^{n} \frac{x_i}{s_i} \geq 1\}$. If $\bar{X}$ is empty, the node will be fathomed. Otherwise, the method goes to the branching step.

For branching, the algorithm divides the polytope $\bar{X}$ into two partitions X_1 and X_2 such that $X_1 = \{x : x \in \bar{X}, d^T x \leq d^T x_c\}$ and $X_2 = \{x : x \in \bar{X}, d^T x \geq d^T x_c\}$ where x_c is a Chebyshev center of $\bar{X}$ and $d \neq 0$ is a random vector in $\mathbf{R}^n$. It is worth noting that a Chebyshev center of a polytope is computed by solving a linear program [7].

Updating the upper bound is straightforward (it is enough to consider the minimum of the generated upper bounds). To update the lower bound, we must consider the lower bound of all fathomed and child nodes. Strictly speaking, it is seen that for father node n_p and its two child nodes, n_{p_1} and n_{p_2}, we have $l_p \leq \min\{l_{p_1}, l_{p_2}\}$, where l_p, l_{p_1} and l_{p_2} denote the generated lower bound of n_p, n_{p_1} and n_{p_2}, respectively. Therefore, if $\{l_i^f\}_{i=1}^k$ and $\{l_i^c\}_{i=1}^o$ are the lower bound fathomed and new nodes, respectively, then the new lower bound will be obtained by the following formula

$$l = \min\left\{\min_{i=1}^{k} l_i^f, \min_{i=1}^{o} l_i^c\right\}. \tag{24}$$

It is easily seen that the lower bound is increasing throughout the algorithm. All steps of the method are presented in Algorithm 1.

Algorithm 1 Branch and Cut Algorithm

Initialization:
  $k = 1, l = -\infty, u = \infty, s = 1, \epsilon > 0, L = \{\}, \Pi_1 = \{(P_1, X)\}$ and $\Pi_2 = \{\}$.
while s = 1 **do**
  while $\Pi_1 \neq \emptyset$ **do**
    $(P_k, X_k) \leftarrow select(\Pi_1)$ and $\Pi_1 \setminus (P_k, X_k)$.
    Solve semi-definite program (5) corresponding to (P_k, X_k) to get l_k.
    $L \leftarrow l_k$ and delete its father lower bound if exists in L.
    Solve convex QP (23) and obtain a local vertex optimal $\bar{x}$.
    $u \leftarrow \min(u, \bar{x}^T Q\bar{x} + 2c^T\bar{x})$.
    if $l_k + \epsilon < u$, **then**
      Solve convex QP (23) corresponding to X_k and obtain a local vertex optimal $\bar{x} \in \{\min x^T Qx + 2c^T x : x \in \bar{X}\}$.
      if $\bar{x}$ is eligible, **then**
        Compute Konno's cut $\sum_{i=1}^{n} \frac{x_i}{t_i} \geq 1$ at $\bar{x}$.
        $X_k \leftarrow \{X_k : \sum_{i=1}^{n} \frac{x_i}{t_i} \geq 1\}$
      end if
      if $X_k \neq \emptyset$, **then**
        Branch (P_k, X_k) to two nodes (P_{k+1}, X_{k+1}) and (P_{k+2}, X_{k+2}).
        $\Pi_2 \leftarrow \{(P_{k+1}, X_{k+1}), (P_{k+2}, X_{k+2})\}$ and $k \leftarrow k + 2$.
      end if
    end if
  end while
  update l by formula (24).
  if $\Pi_2 = \emptyset$ or stopping criteria are satisfied, **then**
    $s = 0$.
  else
    $\Pi_1 \leftarrow \Pi_2$ and $\Pi_2 \leftarrow \{\}$.
  end if
end while

Here, Π_1 and Π_2 denote the set of nodes and the set L contains lower bounds of fathomed and child nodes. Additionally, P_k and X_k denote the kth node and its feasible set, respectively. Stopping criteria which one may use in Algorithm 1 can be absolute gap tolerance, a limit on the maximum running time, etc.

In the rest of the section, we investigate the finite convergent of the algorithm with the stopping accuracy $\epsilon > 0$. To this end, we consider function $\Phi : \mathbf{R}^n \times \mathbf{R}^n_+ \to \mathbf{R}$ where $\Phi(y, d)$ is defined as the optimal value

$$\begin{aligned} &\max \ell \\ &s.t. x^T Qx + 2c^T x - \ell + \sum_{i=1}^{n} [\alpha_i(x)(x - y_i - d_i) + \mu_i(x)(-x + y_i - d_i)] \in P[x] \\ &\quad \alpha_i, \mu_i \in \mathcal{A}_+(X(y, d)), \end{aligned} \tag{25}$$

where $X(y, d) = \{x : y - d \leq x \leq y + d\}$. The well-definedness of Φ on its domain follows from Proposition 2. The next lemma lists some properties of Φ.

Lemma 2 *The function $\Phi : \mathbf{R}^n \times \mathbf{R}^n_+ \to \mathbf{R}$ has the following properties.*

(i) $\Phi(y, 0) = y^T Qy + 2c^T y, \quad \forall y \in \mathbf{R}^n$;
(ii) Φ is continuous on $X \times \mathbf{R}^n_+$.

Proof The first part follows immediately from Proposition 6.
For the second part, first we prove the lower semi-continuity of Φ. Let $(\bar{y}, \bar{d}) \in X \times \mathbf{R}^n_+$ and

$\bar{\alpha}_i, \bar{\mu}_i \in \mathcal{A}_+(X(\bar{y}, \bar{d}))$, $i = 1, \ldots, n$, and $\bar{\ell}$ be optimal for (25). For every $\epsilon > 0$, there exist $\hat{\alpha}_i, \hat{\mu}_i \in \mathcal{A}_+(X(\bar{y}, \bar{d}))$, $i = 1, \ldots, n$, such that the quadratic function

$$x^T Qx + 2c^T x - (\bar{\ell} - \epsilon)$$
$$+ \sum_{i=1}^{n} \Big((\bar{\alpha}_i(x) + \hat{\alpha}_i(x))(x - \bar{y}_i - \bar{d}_i) + (\bar{\mu}_i(x) + \hat{\mu}_i(x))(-x + \bar{y}_i - \bar{d}_i) \Big),$$

is strictly convex and positive on $\mathbf{R}^n$. Indeed, it belongs to the interior of $P[x]$. As a result, for small perturbations of $\bar{\alpha}_i$, $\bar{\mu}_i$, $\bar{y}$ and $\bar{d}$, the above-mentioned quadratic function belongs to $P[x]$, which implies

$$\liminf_{(y,d)\to(\bar{y},\bar{d})} \Phi(y, d) \geq \bar{\ell} - \epsilon.$$

As the above inequality holds for every $\epsilon > 0$, Φ is lower semi-continuous at $(\bar{y}, \bar{d})$. Now, we prove the upper semi-continuity of Φ. First, we consider the case $\bar{d} \in int(\mathbf{R}^n_+)$. Let the sequence $\{(y_k, d_k)\} \subseteq X \times \mathbf{R}^n_+$ converges to $(\bar{y}, \bar{d})$. Suppose that α_i^k, μ_i^k, $i = 1, \ldots, n$, and ℓ^k are optimal for problem (25) corresponding to (y_k, d_k). If the sequences $\{\alpha_i^k\}$, $\{\mu_i^k\}$, $i = 1, \ldots, n$, are bounded, then without loss of generality we may assume that $\alpha_i^k \to \bar{\alpha}_i$, $\mu_i^k \to \bar{\mu}_i$ and $\ell_k \to \bar{\ell}$. In addition, due to the lower semi-continuity of the set-valued mapping $X(.,.)$, we have $\bar{\alpha}_i, \bar{\mu}_i \in \mathcal{A}_+(X(\bar{y}, \bar{d}))$ and

$$x^T Qx + 2c^T x - \bar{\ell} + \sum_{i=1}^{n} \big(\bar{\alpha}_i(x)(x - \bar{y}_i - \bar{d}_i) + \bar{\mu}_i(x)(-x + \bar{y}_i - \bar{d}_i)\big) \in P[x],$$

which implies upper semi-continuity. In the case of the existence of at least one unbounded sequence, without loss of generality we may assume that $t_k^{-1}\alpha_i^k \to \bar{\alpha}_i$ and $t_k^{-1}\mu_i^k \to \bar{\mu}_i$, where $t_k = \max_{1\leq i\leq m}\{\|\alpha_i^k\|, \|\mu_i^k\|\}$. Moreover, there exists $\bar{\alpha}_i \neq 0$ and

$$q(x) = \sum_{i=1}^{n} \big(\bar{\alpha}_i(x)(x - \bar{y}_i - \bar{d}_i) + \bar{\mu}_i(x)(-x + \bar{y}_i - \bar{d}_i)\big) \in P[x].$$

By arguments similar to Proposition 4, since $int(X(\bar{y}, \bar{d})) \neq \emptyset$, there exists $\bar{x}$ such that $q(\bar{x}) < 0$, which contradicts the nonnegativity of q. So, the sequences cannot be unbounded, and in this case the upper semi-continuity of Φ is derived. Likewise, one can show for the case that some components of $\bar{d}$ are zero, Φ is upper semi-continuous at $(\bar{y}, \bar{d})$ on $X \times \mathbf{R}^n_{\bar{d}}$, where $\mathbf{R}^n_{\bar{d}} = \{x \in \mathbf{R}^n_+ : x_i = 0 \; if \; \bar{d}_i = 0\}$. Suppose that the sequence $\{(y_k, d_k)\} \subseteq X \times \mathbf{R}^n_+$ converges to $(\bar{y}, \bar{d})$. We decompose the sequence $\{d_k\}$ as $d_k = d_k^1 + d_k^2$ where d_k^1 is the projection of d_k on $\mathbf{R}^n_{\bar{d}}$. It is seen that $d_k^1 \to \bar{d}$. Hence, we have

$$\limsup_{k\to\infty} \Phi(y_k, d_k) \leq \limsup_{k\to\infty} \Phi(y_k, d_k^1) \leq \Phi(\bar{y}, \bar{d}).$$

The first inequality results from the inclusion property. Therefore, Φ is continuous on $X \times \mathbf{R}^n_+$ and the proof is complete. □

The diameter of polytope $\Delta \subseteq R^n$ is defined and denoted by

$$\delta(\Delta) = \max_{x,y\in\Delta} \|x - y\|.$$

A branching procedure is called exhaustive if for each infinite sequence of nested polytopes $X_1 \supset X_2 \supset \ldots \supset X_k \supset \ldots$ such that (P_{k+1}, X_{k+1}) is a child node of (P_k, X_k), we have $\lim_{k\to\infty} \delta(X_k) = 0$ [19,46].

Theorem 5 *Let the branching procedure of Algorithm 4 be exhaustive. Then Algorithm 4 is finitely convergent with the stopping accuracy* $\epsilon > 0$.

Proof Consider the function Φ on the compact set $X \times (\mathbf{B} \cap \mathbf{R}^n_+)$, where $\mathbf{B}$ stands for the closed unit ball. Owing to Lemma 2, we can infer uniform continuity of Φ on the given domain, that is, we have the following property

$$\forall \epsilon > 0,\ \exists \delta > 0, \forall x \in X, \forall d \in \mathbf{R}^n_+;\ \|d\|_\infty < 2\delta\ \Rightarrow |\Phi(x,d) - x^T Qx - 2c^T x| < \epsilon.$$

Since X is compact, the objective function is Lipschitz continuous on it. Without loss of generality, suppose that Lipschitz modulus is one. Let $\Delta \subseteq X$ be a polytope with diameter less than $0.5\min\{\epsilon, \delta\}$. As a result, there are $\bar{y} \in X$ and $\bar{d} \in \mathbf{R}^n_+$ such that $\Delta \subseteq X(\bar{y}, \bar{d})$ and $\|\bar{d}\|_\infty < 2\delta$. Due to the inclusion property and the provided results, we have

$$\begin{aligned}
&|opt(\Delta, Q, c) - \min_{x\in\Delta}(x^T Qx + 2c^T x)| \le |\Phi(\bar{y}, \bar{d}) - \min_{x\in\Delta}(x^T Qx + 2c^T x)| \\
&\le |\Phi(\bar{y}, \bar{d}) - \bar{y}^T Q\bar{y} - 2c^T \bar{y}| + |\bar{y}^T Q\bar{y} + 2c^T \bar{y} - \min_{x\in\Delta}(x^T Qx + 2c^T x)| \\
&\le \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

By exhaustive property after a finite number of branching, the feasible set of subproblems, Δ, is included in $X(y, d)$ for some $y \in X$ and $d \in \mathbf{R}^n_+$ ($\|d\|_\infty < \min\{\epsilon, \delta\}$), so all nodes will be fathomed and algorithm will stop after finite steps. □

It is worth mentioning that for having the finite convergent, one should adopt an exhaustive branching procedure. However, if the method selects nonzero vector d randomly, with probability of one, we have exhaustive property.

5 Computational results

In this section, we illustrate numerical performance of Algorithm 1 on four groups of test problems. The code and the test problems are publicly available at https://github.com/molsemzamani/quadproga.

We implemented the algorithm using MATLAB 2018b. The computations were run on a Windows PC with Intel Core i7 CPU, 3.4 GHz, and 16 GB of RAM. To solve semi-definite program (10), we employed MOSEK [33]. We used CPLEX to solve convex QPs and linear programs.

To evaluate the performance of Algorithm 1, we compared the numerical results with three non-convex quadratic optimization solvers: BARON 18.11.12 [39], Couenne 1.0 [3] and CPLEX 12.8 [20]. All solvers were run on MATLAB 2018b and we applied AMPL [15] to pass the problems to BARON and Couenne.

In our numerical experiments, we used two stopping criteria to terminate the solvers: absolute gap tolerance and running time limit. The absolute gap is defined as a difference between the computed lower and upper bounds.

For the first group, we selected seventeen concave instances from Globallib folder in [11]. This folder contains all non-convex instances of Globallib test problems [16]. The dimension of problems range from five to fifty.

We set the absolute gap tolerance and the maximum running time to 10^{-4} and 100 s, respectively. Since all methods could give us global optimum with the prescribed gap tolerance, we just report the running time. The performance of all solvers are summarized in

Table 1 Globallib instances

Instance	n	BARON	Couenne	CPLEX	Algorithm 1
st-qpc-m1	5	0.08	0.12	0.04	0.11
st-bsj4	6	0.23	0.09	0.05	0.31
ex2-1-6	10	0.29	0.19	0.05	0.56
st-fp5	10	0.14	0.11	0.07	0.15
st-qpk3	11	0.39	1.19	0.07	0.14
qudlin	12	0.13	0.1	0.01	0.12
ex2-1-7	20	0.76	10.06	0.09	1.38
st-fp7a	20	0.55	0.58	0.08	0.52
st-fp7b	20	0.45	0.89	0.07	0.48
st-fp7d	20	0.31	0.46	0.06	0.18
st-fp7e	20	0.81	10.13	0.13	1.81
st-m1	20	0.26	0.18	0.08	0.24
ex2-1-8	24	0.19	0.08	0.01	0.19
st-m2	30	0.41	0.35	0.18	0.65
st-rv7	30	0.49	0.87	0.12	0.41
st-rv8	40	0.58	0.77	0.13	1.05
st-rv9	50	2.1	3.69	0.43	1.8

Table 1, in which n denotes the dimension of instances and the rest columns denote the execution time for solvers.

The second group of examples involves twenty concave QPs with dense data. The test problems were generated as follows. The feasible set, X, was given by the following linear system

$$Ax \leq 10b, \sum_{i=1}^{n} x_i \leq 100, x \geq 0,$$

where the square matrix A and the vector b were generated by MATLAB's function randn and rand, respectively. The randn function generates a sample of a Gaussian random variable, with mean 0 and standard deviation 1, while rand generates a uniformly distributed random number between 0 and 1. We also generated the vector c via randn function. We generated the square matrix Q with the formula $Q = -U^T DU$, where U is an orthogonal matrix obtained from the singular value decomposition of some random matrix and D is a diagonal matrix whose components are chosen by rand function. We generated twenty concave QPs in $\mathbf{R}^{40}$ and $\mathbf{R}^{45}$.

We set the absolute gap tolerance and the maximum running time to 10^{-3} and 1000 s, respectively. We report the generated lower bound and running time. The running time of less than 1000 s denotes that the solver succeeded in solving the problem with the prescribed absolute gap. Table 2 reports computational performances of methods. In this table, $q^{\star}$ denotes the optimal value and columns lb and $time$ show the generated lower bound and spent CPU time, respectively. To evaluate the quality of the generated upper bound for the cases that the running time exceeded the time limit, we measured the difference between the generated upper bound and the optimal value. Table 3 reports the maximum of the differences for twenty dense examples.

Table 2 Dense instances

Instance	$q^\star$	BARON		Couenne		CPLEX		Algorithm 1	
		lb	*time*	*lb*	*time*	*lb*	*time*	*lb*	*time*
Ex1-40	−2286.1	−7421.5	1000	−2842.8	1000	−2286.1	36	−2286.1	7
Ex2-40	−3821.4	−13627	1000	−6390.5	1000	−3821.4	323	−3821.4	204
Ex3-40	−2756.6	−10617	1000	−4522.3	1000	−2756.6	100	−2756.6	11
Ex4-40	−2341.6	−5714	1000	−4287	1000	−2341.6	39	−2341.6	9
Ex5-40	−2808.2	−5660	1000	−4195.5	1000	−2808.2	124	−2808.2	11
Ex6-40	−4341.8	−25538	1000	−5319.4	1000	−4341.8	30	−4341.8	4
Ex7-40	−2465.4	−5916.9	1000	−3326.3	1000	−2465.4	91	−2465.4	3
Ex8-40	−2554.6	−6570.7	1000	−5246.1	1000	−2554.6	564	−2554.6	424
Ex9-40	−4599.6	−16653	1000	−5381.7	1000	−4599.6	26	−4599.6	3
Ex10-40	−3446.6	−8798.8	1000	−4835.9	1000	−3446.6	40	−3446.6	4
Ex1-45	−4493.2	−6463.3	1000	−2842.8	1000	−4493.2	59	−4493.2	13
Ex2-45	−2705.9	−13214	1000	−6039.8	1000	−2722.9	1000	−2705.9	94
Ex3-45	−3057.8	−19086	1000	−6271.7	1000	−3057.8	461	−3057.8	196
Ex4-45	−2714.1	−8607.3	1000	−6092.4	1000	−2714.1	689	−2714.1	698
Ex5-45	−3028.2	−13511	1000	−6822.5	1000	−3075.1	1000	−3028.2	888
Ex6-45	−2354.4	−10756	1000	−6215.4	1000	−2549.2	1000	−2354.4	657
Ex7-45	−3391.4	−15958	1000	−6561	1000	−3391.4	197	−3391.4	96
Ex8-45	−1948.2	−6923.2	1000	−3675.8	1000	−1948.2	341	−1948.2	9
Ex9-45	−2710.2	−8781.2	1000	−5014	1000	−2710.2	172	−2710.2	35
Ex10-45	−3099	−8431.3	1000	−6239.7	1000	−3099	193	−3099	91

Table 3 Dense instances

BARON	Couenne	CPLEX	Algorithm 1
2.1	180	50	0

The results in Table 2 show that Algorithm 1 outperforms other methods. Indeed, it manages to solve all examples with the prescribed absolute tolerance gap in one thousand seconds while BARON and Couenne could not be successful for any instance. CPLEX succeeds in seventeen instances. In addition, BARON against other solvers generated the loosest lower bound.

As seen in Table 3, while BARON does not meet the absolute gap tolerance, it can generate better upper bounds than Couenne and CPLEX. Indeed, it could give the optimal value for most instances. As Algorithm 1 met absolute tolerance gap for all examples, the column corresponding to this method is zero.

For the third group of the instances, we regarded concave QPs with sparsity. We selected twenty examples from RandQP folder in [11]. In most of the instances, Q were indefinite. We shifted eigenvalues such that Q was transformed to a negative semi-definite matrix. Moreover, we considered the instances without equality constraints. As there were box constraints in all instances, the feasible set was bounded. Table 4 summarizes the computational performances. For this group of test problems, all solvers managed to generate the optimal value.

From Table 4, it is clearly seen that CPLEX outperforms other methods. As the dimension of examples in the second and third group are roughly in the same range, comparison of

Table 4 Sparse instances

Instance	$q^\star$	BARON		Couenne		CPLEX		Algorithm 1	
		lb	$time$	lb	$time$	lb	$time$	lb	$time$
qp40-20-2-1	−286.31	−286.31	3	−286.31	10	−286.31	1	−286.31	14
qp40-20-2-2	−169.572	−169.572	21	−169.572	23	−169.572	1	−169.572	556
qp40-20-2-3	−152.31	−152.31	91	−152.31	51	−152.31	4	−152.31	500
qp40-20-3-1	−219.664	−219.664	35	−219.665	1000	−219.664	1	−219.664	28
qp40-20-3-2	−171.255	−171.255	35	−171.255	43	−171.255	2	−171.255	501
qp40-20-3-3	−101.248	−101.248	25	−101.248	39	−101.248	2	−101.248	255
qp40-20-3-4	−118.119	−118.119	136	−118.12	1000	−118.119	5	−118.119	53
qp40-20-4-1	−240.464	−240.464	188	−240.464	772	−240.464	6	−240.464	104
qp40-20-4-2	−168.813	−168.813	117	−168.813	90	−168.813	5	−168.813	28
qp40-20-4-3	−93.511	−93.511	662	−93.511	337	−93.511	38	−93.511	561
qp50-25-1-1	−430.892	−430.892	49	−430.892	518	−430.892	3	−430.892	557
qp50-25-1-2	−131.88	−131.88	71	−131.88	189	−131.88	8	−131.88	619
qp50-25-1-3	−137.567	−137.567	167	−137.569	1000	−137.567	7	−137.567	799
qp50-25-1-4	−133.52	−134.151	1000	−133.521	1000	−133.52	10	−133.52	716
qp50-25-2-1	−269.924	−269.924	84	−269.924	136	−269.924	7	−269.924	648
qp50-25-2-2	−204.733	−204.733	654	−204.733	369	−204.733	28	−204.733	612
qp50-25-2-3	−167.34	−167.341	1000	−167.34	934	−167.34	8	−167.34	8
qp50-25-2-4	−129.209	−129.21	1000	−129.209	107	−129.209	7	−129.209	615
qp50-25-3-1	−393.76	−393.76	40	−393.761	1000	−393.76	2	−393.76	40
qp50-25-3-2	−224.266	−224.268	1000	−224.267	1000	−224.266	14	−224.266	23

Table 5 Max norm instances

Instance	$q^\star$	BARON		Couenne		CPLEX		Algorithm 1	
		lb	*time*	*lb*	*time*	*lb*	*time*	*lb*	*time*
Ex1-40	−1087.4	−1087.4	360	−1087.4	8	−1087.4	45	−1087.4	4
Ex2-40	−1425	−1425	221	−1448.2	1000	−1425	34	−1425	100
Ex3-40	−1514.5	−1514.5	205	−1514.5	618	−1514.5	7	−1514.5	15
Ex4-40	−1324.1	−1324.1	18	−1324.1	126	−1324.1	3	−1324.1	4
Ex5-40	−1206.1	−1206.1	214	1206.1	1000	−1206.1	20	-1206.1	24
Ex6-40	−2104.8	−2104.8	4	−2104.8	288	−2104.8	3	−2104.8	3
Ex7-40	−1150.4	−1150.8	1000	−1204.1	1000	−1150.4	37	−1150.4	36
Ex8-40	−1268.6	−1268.6	305	−1268.6	702	−1268.6	24	−1268.6	112
Ex9-40	−2090.7	−2090.7	10	−2090.7	80	−2090.7	2	−2090.7	12
Ex10-40	−1503.3	−1503.3	62	−1503.3	527	−1503.3	3	−1503.3	3
Ex1-45	−2097.9	−2097.9	12	−2097.9	85	−2097.9	3	−2097.9	40
Ex2-45	−1190.4	−1190.4	578	−1301.6	1000	−1190.4	51	−1190.4	59
Ex3-45	−1636.8	−1636.8	472	−1785.1	1000	−1636.8	19	−1636.8	178
Ex4-45	−1527.3	−1527.3	64	−1527.3	537	−1527.3	7	−1527.3	81
Ex5-45	−1484.8	−1484.8	803	−1557.3	1000	−3.18896	64	−1484.8	531
Ex6-45	−1106.1	−1117.8	1000	−1191.2	1000	−1106.1	65	−1106.1	502
Ex7-45	−1489.6	−1489.6	244	−1519.4	1000	−1489.6	10	−1489.6	37
Ex8-45	−939.6	−1021.7	1000	−985	1000	−939.6	33	−939.6	6
Ex9-45	−1235.2	−1235.2	243	−1277.7	1000	−1235.2	17	−1235.2	47
Ex10-45	−1557.8	−1557.8	102	−1557.8	870	−1557.8	10	−1557.8	38

Table 6 Max norm instances

BARON	Couenne	CPLEX	Algorithm 1
0	0.9	0	0

Table 2 and Table 4 reveals that the performance of BARON, Couenne and CPLEX improve considerably. On the contrary, the performance of Algorithm 1 on both groups are similar. The reason for this behavior may be that Algorithm 1 does not exploit sparsity.

We considered the norm maximization problem for the last group. This problem can be formulated as concave QP

$$\begin{aligned} &\min\ -x^T x \\ &s.t.\ Ax \leq b, \end{aligned}$$

where $X = \{x : Ax \leq b\}$ is a polytope. Unlike norm minimization problem, the above problem is NP-hard. To assess the performance of the solvers, we considered the polytopes which were generated for the second group. Tables 5 and 6 give computational performances.

As can be seen in Table 5, CPLEX outperforms other methods. The instances in the second and last group are the same with different objective function. In fact, the objective function of examples in the last group are sparser than that in the second group. From Tables 2 and 5, one can infer that the performance of BARON, Couenne and CPLEX against Algorithm 1 improve as Q becomes sparser. Nevertheless, similar to the former case, the performance of Algorithm 1 is not sensitive to sparsity. Table 6 shows that for examples where BARON and

Couenne do not satisfy the absolute gap tolerance BARON gives an optimal solution while Couenne is not successful to generate an exact optimal solution.

On average, CPLEX outperforms other solvers in most instances. After CPLEX, Algorithm 1 has better performance than BARON and Couenne in most instances. Especially, it has the best performance on the second group of examples, which suggests its efficiency for dense problems. Nevertheless, its performances on the third group of instances is not satisfactory. Algorithm 1 needs to be supplied with some techniques to handle sparse problems. Taking advantage of sparsity may improve its performance. As the most time-consuming step of Algorithm 1 is solving semi-definite program (22), handling efficiently this problem, for instance by using the information of parent node, will be another approach to improve the performance. We leave trying such techniques as a future work.

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