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On the q-capability of groups

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Given a positive integer q, in this paper, we investigate some more properties of the q-capability of groups. For instance, the relationship between q-capability and the varietal capability is determined. Moreover, we introduce the notion of q-epicenter for a group and then we obtain some criteria for the q-capability of groups. Finally, as an application, we characterize all q-capable extra-special p-groups when q is a power of p.

Keywords: q-capable groups; q-exterior product of groups; capable groups.

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1. Introduction

First, Baer in [2] has presented conditions for a group G to be the inner automorphism group of another group H ($G \cong H/Z(H)$). In the same paper, all capable groups which are the direct sums of cyclic groups are classified. Later, Hall and Senior in [11] used the word "capable" to call such a group G. Ellis in [9] found a new characterization of capable groups and then defined the notion of q-capable groups for $q \geq 0$. Recall that the q-center of a group G is the subgroup $Z_q(G)$ of the center $Z(G)$ consisting of those elements of order dividing q. A group G is said to

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be q-capable if there exists a group E such that $G \cong E/Z_q(E)$ and $Z(E) = Z_q(E)$. Note that q-capability implies capability (that is, 0-capability). Moghaddam and Kayvanfar in [13] have generalized the concept of capability to any variety ν of groups which is called ν-capability. For the variety $\mathcal{N}_c$ of nilpotent groups of class c or less, $\mathcal{N}_c$-capability is called c-capability for $c \geq 1$. The notion of c-capability of groups is studied in [7, 13]. Following the terminology of [7, 13] a group G is called c-capable if $G \cong H/Z_c(H)$ for some group H, where $Z_c(H)$ is the cth term of the upper central series of H. Therefore, there are differences between the c-capability and the q-capability of groups. Ellis in [9] have defined the q-exterior center of G, $Z_q^\wedge(G)$, to be the set of all elements g of G for which $g \wedge h = 1_{G\wedge^q G}$ for all $h \in G$ and $\wedge^q$ denotes the operator of the non-abelian q-exterior square. (See [8] for more details.) The 0-exterior center of G, $Z_0^\wedge(G)$, is called the exterior center of G, $Z^\wedge(G)$. Ellis has also proved that a group G is q-capable if and only if its q-exterior center is trivial. Then, Beyl *et al.* in [3] have defined the epicenter $Z^*(G)$ for a group G. It gives a necessary and sufficient condition for the capability of groups. In fact, a group G is capable if and only if $Z^*(G) = 1$. It is shown that $Z^\wedge(G) = Z^*(G)$.

The non-abelian q-tensor product plays a fundamental role to determine the q-capability of groups and so we need to know the structure of it. Several authors have introduced a construction related to q-tensor product in [6, 14, 16]. They have used a slightly different approach which is particularly useful for computer computations of the non-abelian tensor square of polycyclic groups. (See for instance references [6, 16].) We will determine the q-capability of extra-special p-groups by Rocco's results. The purpose of this paper is further investigations into the q-capability of groups. We will also be able to find a relation between q-capability and varietal capability via the variety ν and then state some more results of q-capability. In this paper, we have found criteria for recognizing q-capable group by introducing the notion of q-epicenter $\mathfrak{Z}_q^*(G)$ for a group G. Finally, we classify the q-capable extra-special p-groups.

2. Preliminaries

In this section, we introduce some notations and present some known results without proofs which are useful in the next sections.

Recall that, given a group G, a G-crossed module is a group homomorphism $\mu : M \to G$ with an operation of G on M such that, for $g \in G$ and for $m, m' \in M$ one has

$$\text{(i)} \quad \mu({}^g m) = g\mu(m)g^{-1},$$
$$\text{(ii)} \quad {}^{\mu(m)}m' = mm'm^{-1}.$$

Let $\mu : M \to G$ and $\nu : N \to G$ be two G-crossed modules and consider the pullback

$$\begin{array}{ccc} M \times_G N & \xrightarrow{\pi_2} & N \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\mu} & G \end{array}$$

Let $K = M \times_G N = \{(m, n) \,|\, m \in M, n \in N, \mu(m) = \nu(n)\}$. In this diagram, each group acts on any other group via its image in the group G. Now we continue with the following definition.

Definition 2.1 ([8]). Let $q \geq 0$. The tensor product module q, $M \otimes^q N$, of the G-crossed modules μ and ν is a group generated by the symbols $m \otimes n$ and $\{k\}$ such that $m \in M$, $n \in N$, $k \in K$. In this group, the following relations hold. For $m, m' \in M, n, n' \in N$ and $k, k' \in K$,

$$m \otimes (nn') = (m \otimes n)({}^n m \otimes {}^n n'), \tag{2.1}$$

$$mm' \otimes (n) = ({}^m m' \otimes {}^m n)(m \otimes n), \tag{2.2}$$

$$\{k\}(m \otimes n)\{k\}^{-1} = ({}^{k^q} m \otimes {}^{k^q} n), \tag{2.3}$$

$$[\{k\}, \{k'\}] = \pi_1 k^q \otimes \pi_2 k'^q, \tag{2.4}$$

$$\{(m {}^n m^{-1}, {}^n m m^{-1})\} = (m \otimes n)^q, \tag{2.5}$$

and

$$\{kk'\} = \{k\} \left(\prod_{i=1}^{q-1} (\pi_1 k^{-1} \otimes ({}^{k^{1-q+i}} \pi_2 k')^i) \right) \{k'\}. \tag{2.6}$$

The group $M \wedge^q N$ is the quotient of the q-tensor product $M \otimes^q N$ by the relation

$$\pi_1 k \otimes \pi_2 k = 1 \quad \text{for all } k \in K. \tag{2.7}$$

In other words, the group $M \wedge^q N$ is the quotient of the group $M \otimes^q N$ by the subgroup generated by the set $\{m \otimes n \,|\, (m, n) \in K\}$. Consider $m \wedge n$ as the image of $m \otimes n$ in $M \wedge^q N$.

For $q = 0$, the tensor product $M \otimes^0 N$ is the group generated by the symbols $m \otimes n$ for $m \in M$, $n \in N$ subject to the relations (2.1) and (2.2). In fact the tensor product $M \otimes^0 N$ is the non-abelian tensor product $M \otimes N$ defined in [5]. The following corollary is useful in the next results.

Corollary 2.1 ([8, Proposition 1.6]). *Let $[M, N]$ be a normal subgroup of $K = M \times_G N$ generated by the elements $(m^n m^{-1}, {}^m n n^{-1}), m \in M, n \in N$. Then there exists a commutative diagram with exact rows.*

$$\begin{array}{ccccccc} M \otimes N & \xrightarrow{\varphi} & M \otimes^q N & \longrightarrow & K/[M,N] & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow id & & \\ M \wedge N & \xrightarrow{\psi} & M \wedge^q N & \longrightarrow & K/[M,N] & \longrightarrow & 1 \end{array}$$

We assume that the reader is familiar with the notion of the verbal subgroup $V(G)$ and the marginal subgroup $V^*(G)$, associated with a variety of group ν and a given group G. See [15] for more information on variety of groups.

Definition 2.2 ([13, Definition 2.1]). Let ν be a variety of groups by the set of laws V. A group G is said to be ν-capable if there exists a group E such that $G \cong E/V^*(E)$.

Recall that the subgroup of G is defined in [13, Definition 2.1] as follows:

$$(V^*)^*(G) = \bigcap \{\psi(V^*(E)) \,|\, \psi : E \to G \text{ is a surjective homomorphism with } \ker \psi \subseteq V^*(E)\}.$$

The following theorem gives a necessary and sufficient condition for the ν-capability of groups.

Theorem 2.1 ([13, Corollary 2.4])). *A group G is ν-capable if and only if $(V^*)^*(G) = 1$.*

For a given group G, the q-exterior center of G for $q \geq 0$, which is useful in deciding whether a group G is q-capable (see [9]), is defined by Ellis as follows:

$$Z_q^{\wedge}(G) = \{g \in G \,|\, g \wedge x = 1_{G \wedge^q G} \text{ for all } x \in G\}.$$

3. The Concept of q-Capability

Throughout this section, we find some structural properties of q-capability that relates to varietal capability.

For a given group G, G^q and G' denote the subgroup $\langle g^q | g \in G\rangle$ and the derived subgroup of G, respectively. Now, suppose that $q \geq 0$ and $q \neq 1$. Let $V_q = \{x_1^q, [x, y]\}$ be the set of laws. Clearly, the verbal and marginal subgroups of a group G are $V_q(G) = G'G^q$ and $V_q^*(G) = Z_q(G)$, respectively. Recall that φ is a ν_q-extension of the group G if $\varphi : E \to G$ is a surjective homomorphism such that

$\ker\varphi \subseteq V_q^*(E) = Z_q(E)$. The intersection of all subgroups of the form $\varphi(V_q^*(E))$ is denoted by $(V_q^*)^*(G)$ ([3]). A ν_q-capable group is the varietal capable group via the variety ν_q of groups defined by the set of laws V_q.

Lemma 3.1. *Let G be a q-capable group for $q \geq 0$ and $q \neq 1$. Then G is ν_q-capable.*

Proof. It is straightforward. □

Now, we provide an example of a q-capable group. For this, let C_n denote the cyclic group of order n.

Example 3.1. Consider G to be the dihedral group of order 8. Clearly, $Z_q(G) = Z(G)$ if $q = 2^t$ and $t \geq 1$. Since $G/Z_q(G) \cong C_2 \times C_2$, the group Klein four-group, V_4, is q-capable.

Note that we can similarly state all concepts and conclusions in [13] for the q-capability of groups.

Recall that from [4] a q-central extension of a group G is an exact sequence $1 \to \ker \xrightarrow{\varphi} E \to G \to 1$ such that $\ker\varphi \subseteq Z_q(E)$ and then we define the following central subgroup as follows:

$$(Z_q^*)^*(G) = \bigcap\{\ker\varphi(Z_q(E)) | \varphi : E \to G \text{ is a } q\text{-central extension}\}.$$

Corollary 3.1. *Let G be a group. Then $(Z_q^*)^*(G) \subseteq Z_q^\wedge(G)$ for all $q \geq 0$ and $q \neq 1$.*

Proof. It is easily obtained from [13, Corollary 2.4; 9, Proposition 16(vii)]. □

Therefore, we can find a criterion for detecting ν_q-capable groups by the q-epicenter subgroup of group G, which is defined in next chapter. We now present some properties of ν_q-capability and q-capability.

Lemma 3.2. *Let G be a group. Then for any integer $n, q \geq 1$ holds*

$$G \supseteq Z_q^\wedge(G) \supseteq Z_{q^2}^\wedge(G) \supseteq \cdots \supseteq Z_{q^{n-1}}^\wedge(G) \supseteq Z_{q^n}^\wedge(G) \supseteq \cdots.$$

Proof. By [8, Theorem 1.22(2)], we have the following homomorphism

$$\phi : G \wedge^{q^n} G \to G \wedge^{q^{n-1}} G$$

$$g \wedge g_1 \mapsto g \wedge g_1$$

$$\{k\} \mapsto \{k^q\}$$

for all $g_1, g, k \in G$. Therefore, $Z_{q^{i+1}}^\wedge(G) \subseteq Z_{q^i}^\wedge(G)$ for all $i \geq 1$. The result holds. □

Obviously, the q^n-capability of groups implies the (q^{n+1})-capability for all n, $q \geq 1$. The following result will help us to determine the q-capability of some interesting groups. Notice that finite p-groups are some interesting examples of finite nilpotent groups. What can we say about q-capable p-groups? The following proposition provides a condition under which we can identify a q-capable p-group, where p is prime and $q \geq 0$.

Proposition 3.1. *Let G be a non-trivial q-capable finite p-group such that p is prime and $q \geq 0$. Then p divides q.*

Proof. According to the definition of q-capability, there exists a group E such that $G \cong E/Z(E)$ and $Z(E) = Z_q(E)$. By [12, Lemma 2.1], we can choose E as a finite nilpotent group. Thus $E \cong S_{p_1} \times \cdots \times S_{p_r}$ where S_{p_i} is a p_i-Sylow subgroup of G for all $1 \leq i \leq r$. Set $p_1 = p$. Then,

$$E/Z(E) \cong (S_{p_1}/Z(S_{p_1})) \times \cdots \times (S_{p_r}/Z(S_{p_r})).$$

Since $E/Z(E)$ is a p_1-group and $Z_q(E) = Z(E)$, we have $Z_q(E) = Z(E) = Z(S_{p_1}) \times S_{p_2} \times \cdots \times S_{p_r}$. As a result, the order of x divides q for all $x \in Z(S_{p_1})$, as required. □

Remark 3.1. Let G be a q-capable finite nilpotent group of order $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ such that p_i is prime and $p_i \neq p_j$ for $i \neq j$. Then p_i divides q for all $1 \leq i \leq s$.

Now, we discuss the structure of all ν_q-capable groups. Here, we use $o(x)$ and $\exp(G)$ to denote the order of x in a group G and the exponent of G, respectively.

Lemma 3.3. *Let $q \geq 0$ and $q \neq 1$. If G is a ν_q-capable group and (E, φ) is a q-central extension of G with $\ker\varphi = V_q^*(E) = Z_q(E)$, then we can define a bilinear homomorphism (bihomomorphism) $\alpha : Z_q(G) \times G \to Z_q(E)$ such that the left kernel of α is trivial and right kernel of α contains $G'G^q$.*

Proof. Let

$$A = \{t_x Z_q(E) \in E/Z_q(E) \mid \varphi(t_x) = x \text{ for all } x \in G\}.$$

One can easily check that $\varphi([t_x, t_y]) = 1$ for every $x \in Z_q(G)$ and $y \in G$, so $[t_x, t_y] \in Z_q(E)$ for every $x \in Z_q(G)$ and $y \in G$. We can also conclude that for every $x_1, x_2 \in G$

$$t_{x_1 x_2} = t_{x_1} t_{x_2} z \tag{3.1}$$

for some $z \in Z_q(E)$, since φ is a homomorphism. Now we can define the map

$$\alpha : Z_q(G) \times G \to Z_q(E),$$
$$(x, y) \mapsto [t_x, t_y].$$

It is obvious that the commutators $[t_x, t_y]$ are independent of the chosen transversal for every $x \in Z_q(G)$ and $y \in G$. Now let $(x_1, y_1) = (x_2, y_2)$ such that $x_1, x_2 \in Z_q(G)$

and $y_1, y_2 \in G$. Then $t_{x_1} = t_{x_2} z_1$ and $t_{y_1} = t_{y_2} z_2$ for some $z_1, z_2 \in Z_q(E)$, so we can easily verify that $[t_{x_1}, t_{y_1}] = [t_{x_2}, t_{y_2}]$. Therefore α is well-defined. We claim that α is bilinear homomorphism. We have

$$\begin{aligned}\alpha(x_1x_2, y) = [t_{x_1x_2}, t_y] &= [t_{x_1}t_{x_2}z_1, t_y] \\ &= [t_{x_1}, t_y][t_{x_2}, t_y] \\ &= \alpha(x_1, y)\alpha(x_2, y)\end{aligned}$$

for every $x_1, x_2 \in Z_q(G),\ z_1 \in Z_q(E)$ and $y \in G$, by relation (3.1). We can similary prove that $\alpha(x, y_1y_2) = \alpha(x, y_1)\alpha(x, y_2)$ for every $x_1 \in Z_q(G)$ and $y_1, y_2 \in G$. Thus α is a bilinear homomorphism.

We claim that the left kernel of α is trivial. Let x be an arbitrary element of the left kernel of α. If u is an arbitrary element of E, then there exists an element of G like g such that $u = t_g z (z \in Z_q(E))$. Therefore, $t_x \in Z(E)$. We also know that $t_{x^q} \in \ker\varphi = Z_q(E)$, since x is an element of left kernel of α, which is a subset of $Z_q(G)$. Now we verify that $t_x t^{-1}_{x^{q-1}} \in Z_q(E)$. Since t_x is a central element of E, $o(t_x)$ divides $o(t_x t^{-1}_{x^{q-1}})$. On the other hand, $o(t_x t^{-1}_{x^{q-1}})$ divides q. Therefore, $t_x \in Z_q(E)$ and so the left kernel of α is trivial.

We claim that the right kernel of α must contain $G'G^q$. Assume that $[g_1, g_2]g^q$ is an element of $G'G^q$ such that $g, g_1, g_2 \in G$. Clearly, $t_{[g_1,g_2]g^q} = t_{[g_1,g_2]} t_{g^q} z$, by relation (3.1) for some $z \in Z_q(E)$. We also have $t_{[g_1,g_2]} = [t_{g_1}, t_{g_2}]z_1$ such that $z_1 \in Z_q(E)$, since φ is a homomorphism. Thus

$$\begin{aligned}[t_y, t_{[g_1,g_2]g^q}] &= [t_y, t_{[g_1,g_2]} t_{g^q} z] \\ &= [t_y, t_{[g_1,g_2]}][t_y, t_{g^q}] \\ &= [t_y, [t_{g_1}, t_{g_2}]z_1][t_y, t_{g^q}] \\ &= [t_y, [t_{g_1}, t_{g_2}]][t_y, t_{g^q}] \\ &= [t_{g_1}, t_{g_2}, t_y]^{-1}[t_y, t_g^q]\end{aligned}$$

for all $y \in Z_q(G)$. By the Hall–Witt identity, we have

$$[t_{g_1}, t_{g_2}, t_y]^{t_{g_2}^{-1}}[t_{g_2}^{-1}, t_y^{-1}, t_{g_1}]^{t_y}[t_y, t_{g_1}^{-1}, t_{g_2}^{-1}]^{t_{g_1}} = 1$$

and so $[t_{g_1}, t_{g_2}, t_y] = 1$. We can conclude that $[t_y, t_g^q] = [t_y, t_{g^q}]$ by relation (3.1). Therefore,

$$[t_y, t_{[g_1,g_2]g^q}] = [t_y, t_g^q].$$

Now we can easily conclude that

$$[t_y, t_{[g_1,g_2]g^q}] = [t_y, t_g]^q = 1,$$

since $[t_y, t_x] \in Z_q(E)$ for every $x \in G$. As a result, $G'G^q$ is a subset of the right kernel of α. The proof is completed. □

The following theorem is a necessary condition for the ν_q-capability of groups.

Theorem 3.1. *Let $q \geq 0$ and $q \neq 1$. If G is a ν_q-capable group and $G/V_q(G)$ is of finite exponent, then the exponent of $V_q^*(G)$ divides the exponent of $G/V_q(G)$.*

Proof. By Lemma 3.3, we have the following bilinear homomorphism

$$\alpha : Z_q(G) \times G \to Z_q(E)$$
$$(x, y) \mapsto [t_x, t_y].$$

Also the left kernel of α is trivial and the right kernel of it contains $G'G^q$. Assume $\exp(G/G'G^q) = e$ and $yG'G^q$ is an arbitrary element of $G/G'G^q$. Then, y^e is an element of the right kernel of α. Hence, $1 = \alpha(x, y^e) = \alpha(x^e, y)$ for all $x \in Z_q(G)$ and $y \in G$. As a consequence, x^e is trivial. The result follows. □

Now we express another proposition about ν_q-capable groups.

Proposition 3.2. *Let G be a ν_q-capable group for $q \geq 0$. Then, the order of z divides* $\exp(G/G'G^q\langle z\rangle)$ *for all $z \in Z_q(G)$.*

Proof. For $q = 1$, the statement of this proposition is trivial. By using Lemma 3.3 for $q \geq 0$ and $q \neq 1$, we know that there is a bilinear homomorphism

$$\alpha : Z_q(G) \times G \to Z_q(E)$$
$$(x, y) \mapsto [t_x, t_y].$$

Also the left kernel of α is trivial and the right kernel of it contains $G'G^q$. Hence,

$$\alpha(x, y^n) = \alpha(x, y)^n = \alpha(x^n, y)$$

for all $x \in Z_q(G)$, $y \in G$, and $n \in \mathbb{Z}$. Clearly, the exponent of $G/(G'G^q\langle z\rangle)$ divides q. Let $\exp(G/G'G^q\langle z\rangle) = e$. Since $1 = \alpha(z, y^e) = \alpha(z^e, y)$ for all $y \in G$, we conclude that z^e is an element of the left kernel of α. Consequently, z^e is trivial, as required. □

Finally let us mention a necessary condition for the p-capability of p-groups.

Corollary 3.2. *Let G be a p-capable finite p-group. Then $Z_p(G)$ is an elementary abelian group.*

Proof. From Proposition 3.2 and the fact that the exponent of the group $G/(G'G^p\langle z\rangle)$ is p, the result follows. □

4. A New Criterion for q-Capable Groups

In this section, we obtain a new criterion for q-capable groups which is more practical by its construction. Given two normal subgroups G and H of some group L, the subgroup of L generated by the commutators $[g, h]$ and the qth powers k^q for $g \in G$, $h \in H$ and $k \in G \cap H$ is denoted by $G\sharp qH$. The following computations play an important role in the main results. Clearly, $Z(G) \wedge^q G$ is abelian. We define $\mu : Z(G) \to G/G'$ by $\mu(x) = xG' \in G/G'$ for all $x \in Z(G)$. Let

$$K_{G/G'} = Z(G) \times_{G/G'} G/G' = \{(x, xG') \mid x \in Z(G)\}$$

and

$$K_G = Z(G) \times_G G = \{(x, x) \mid x \in Z(G)\}.$$

Now, we can consider the following pullbacks

$$\begin{array}{ccc} Z(G) \times_{G/G'} G/G' & \xrightarrow{\pi_2} & G/G' \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle id} \\ Z(G) & \xrightarrow{\mu} & G/G' \end{array}$$

and

$$\begin{array}{ccc} Z(G) \times_G G & \xrightarrow{\pi_2} & G \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle id} \\ Z(G) & \xrightarrow{\subseteq} & G \end{array}$$

Lemma 4.1. *Let G be a group and $q \geq 0$. Then, $Z(G) \otimes^q G$ is an abelian group and*

$$Z(G) \otimes^q G \cong Z(G) \otimes^q (G/G').$$

Proof. [9, Theorem 6(iv)] implies

$$\begin{aligned} Z(G) \otimes^q G &\cong (Z(G)/Z(G)\sharp_q Z(G)) \otimes_{\mathbb{Z}} (G/G\sharp_q G) \\ &\cong (Z(G)/Z(G)^q) \otimes_{\mathbb{Z}} (G/G'G^q) \cong Z(G) \otimes^q (G/G'), \end{aligned}$$

where $\otimes_{\mathbb{Z}}$ denotes the usual tensor product of abelian groups. Therefore, $Z(G) \otimes^q G$ is abelian, as required. □

We are ready to present the following result.

Lemma 4.2. *Let G be a group. Then $Z(G) \wedge^q G \cong Z(G) \wedge^q (G/G')$.*

Proof. We define

$$\begin{aligned} f : Z(G) \wedge^q (G/G') &\to Z(G) \wedge^q G \\ x \wedge yG' &\mapsto x \wedge y \\ \{(x, xG')\} &\mapsto \{(x, x)\}, \end{aligned}$$

where $x \in Z(G), y \in G$ and $xG', yG' \in G/G'$. Let $y = y_1 g_1$ and $g_1 = \prod_{i=1}^{t} [x_i, d_i]$ for all $y_1, y, x_i, d_i \in G$ and $x \in Z(G)$. Then $x \wedge y = x \wedge y_1 g_1 = (x \wedge y_1)^{y_1}(x \wedge g_1)$. It is sufficient to show that $x \wedge g_1$ is trivial.

$$x \wedge g_1 = x \wedge \prod_{i=1}^{t} [x_i, d_i] = (x \wedge [x_1, d_1])^{[x_1,d_1]} \left(x \wedge \prod_{i=2}^{t} [x_i, d_i] \right).$$

By Lemma 4.1, we can easily verify that

$$\begin{aligned} x \wedge [x_1, d_1] &= (x \wedge {}^{x_1}d_1)(x \wedge d_1^{-1}) \\ &= (x \wedge {}^{x_1}d_1)(x \wedge d_1)^{-1} = 1. \end{aligned}$$

By the same process, we conclude that $x \wedge g_1 = 1$ and so $x \wedge y = x \wedge y_1 g_1 = x \wedge y_1$. Therefore, f is well-defined. It is enough to show that f commutes with relations (2.1)–(2.7). In effect

$$f(xx' \wedge yG') = xx' \wedge y = ({}^{x}x' \wedge {}^{x}y)(x \wedge y) = f({}^{x}(x' \wedge yG'))f(x \wedge yG')$$

and

$$f(x \wedge yy'G') = x \wedge yy' = (x \wedge y)({}^{y}x \wedge {}^{y}y') = f(x \wedge yG')f({}^{yG'}(x \wedge y'G')).$$

We define the actions of $K^q_{G/G'}$ and K^q_G on G by ${}^{(x,xG')^q}m = m^{x^q}$ and ${}^{(x,x)^q}m = m^{x^q}$ for all $x \in Z(G)$ and $m \in G$. Let $\overline{x}$ denote the image of $x \in G$ in G/G'. Now we state the following relations.

$$\begin{aligned} f\big({}^{(x,\overline{x})^q}g \wedge {}^{(x,\overline{x})^q}\overline{h}\big) &= f\big({}^{(x,\overline{x}^q)}g \wedge \overline{{}^{(x,\overline{x}^q)}h}\big) \\ &= {}^{x^q}g \wedge {}^{x^q}h \\ &= f(\{(x, \overline{x})\})f(g \wedge \overline{h})f(\{(x, \overline{x})\})^{-1}, \\ f(\{(x, \overline{x})(x', \overline{x'})\}) &= \{(x, x)(x', x')\} \\ &= \{(x, x)\} \prod_{i=1}^{q-1} (x^{-1} \wedge ({}^{x^{1-q+i}}x')^i)\{(x', x')\} \\ &= f(\{(x, \overline{x})\}) \prod_{i=1}^{q-1} f(x^{-1} \wedge \overline{({}^{x^{1-q+i}}x')^i})f(\{x', \overline{x'}\}), \end{aligned}$$

$$\begin{aligned} f(x^q \wedge \overline{x'}^q) &= x^q \wedge x'^q \\ &= [\{(x,x)\}, \{(x',x')\}] \\ &= [f(\{(x,x)\}), f(\{(x',x')\})], \\ f(\{m^n m^{-1}, \overline{{}^n m m^{-1}}\}) &= \{m^n m^{-1}, {}^n m m^{-1}\} \\ &= (m \otimes n)^q \\ &= f(m \wedge \overline{n})^q \end{aligned}$$

and

$$f(x \wedge \overline{x}) = (x \wedge x) = 1$$

for all $x, x', g, m, n \in Z(G)$ and $y, y', h \in G$. We claim that f is an isomorphism. Define the following map

$$f_1 : Z(G) \wedge^q G \to Z(G) \wedge^q (G/G')$$

$$x \wedge y \mapsto x \wedge \overline{y},$$

$$\{(x,x)\} \mapsto \{(x, \overline{x})\}.$$

Clearly, f_1 is a homomorphism. Since $ff_1 = \mathrm{id}_{Z(G)\wedge^q G}$ and $f_1 f = \mathrm{id}_{Z(G)\wedge^q (G/G')}$, we obtain

$$Z(G) \wedge^q G \cong Z(G) \wedge^q (G/G'),$$

which completes the proof. □

The following corollary is an immediate result from Lemma 4.2 and [9, Proposition 7].

Corollary 4.1. *Let G be a group and $q \geq 0$. Then the canonical sequence of homomorphisms*

$$Z(G) \wedge^q (G/G') \xrightarrow{\rho} G \wedge^q G \to G/Z(G) \wedge^q G/Z(G) \to 1$$

is exact.

Now we are ready to define the q-epicenter of groups by the homomorphism ρ, which is introduced in Corollary 4.1.

Definition 4.1. Let G be a group and $q \geq 0$. The q-epicenter of G is denoted by $\mathfrak{Z}_q^*(G)$ and is defined as

$$\{z \in Z(G) | z \wedge a \in \ker \rho \text{ for all } a \in G/G'\}.$$

In the following theorem, we provide a certain characterization for the q-capability of groups.

Theorem 4.1. *Let G be a group and $q \geq 0$. Then $(Z_q^*)^*(G) \subseteq Z_q^{\wedge}(G) = \mathfrak{Z}_q^*(G)$.*

Proof. If $q = 1$, then G is ν_1-capable, since $Z_1(G) = 1$. Hence $(Z_q^*)^*(G)$ is trivial, by Theorem 2.1. We also have $(Z_q^*)^*(G) \subseteq Z_q^{\wedge}(G)$, such that $q \geq 0$ and $q \neq 1$, by Corollary 3.1. Now let $z \in \mathfrak{Z}_q^*(G)$. Then $\rho(z \wedge yG') = z \wedge y = 1_{G \wedge^q G}$ for all $y \in G$. Therefore, $z \in Z_q^{\wedge}(G)$.

Conversely, let $z \in Z_q^{\wedge}(G)$. This means that $z \wedge x$ is trivial, for all $x \in G$. If xG' is an arbitrary element of G/G', then $\rho(z \wedge xG') = 1_{G \wedge^q G}$ and so z is an element of $\mathfrak{Z}_q^*(G)$. As a result, $\mathfrak{Z}_q^*(G) = Z_q^{\wedge}(G)$, as desired. □

The following criterion is an important result of Theorem 4.1 which gives a necessary and sufficient condition for q-capable groups. This result can also be useful to study ν_q-capable groups.

Theorem 4.2. *A group G is q-capable if and only if its q-epicenter $\mathfrak{Z}_q^*(G)$ is trivial for $q \geq 0$.*

Proof. This is an immediate consequence of Theorem 4.1. □

5. The Application of the Criterion

In this section, we obtain some necessary conditions for the q-capability of groups. Our method involves the kernel of the homomorphism ρ in Corollary 4.1. In fact, we will use the concept of q-epicenter which is defined by $\ker \rho$. Finally, we determine all q-capable extra-special p-groups when q is a power of p.

For a given finitely generated group A, we set $\exp(A) = 0$ when A contains an element of infinite order. Note that in this case $\exp(A)$ is divisible by all integers. We know that a q-capable group is capable. Hence, by [10, Proposition 2], we have the following lemma. We denote the image of $x \in G$ in $(G/G'\langle z \rangle)$ by $\overline{x}$.

Lemma 5.1. *Let $q \geq 0$. Consider G to be a q-capable group with a generating set $J(G)$. Let z be a non-trivial q-central element in G and let $z' \neq 1$ be a power of z (it can be equal to z). Then there exists a generator $x \in J(G)$ such that*

(i) *$z' \neq y^{o(\overline{x})}$ for all $y \in Z_q(G)$.*
(ii) *$z' \neq x^n$ such that n is a natural number.*
(iii) *If $G/Z(G)$ is abelian and no element in $J(G)$ has even order, then $z' \neq w^{o(x)}$ for all $w \in G$.*
(iv) *$x(G'G^q\langle z \rangle) \neq a^{o(z')}$ for all $a \in G/(G'G^q\langle z \rangle)$.*

Proof. We know that a q-capable group is capable. Hence, by [10, Proposition 2], there exists a generator $x \in J(G)$ such that

(1) $z' \neq y^{o(\overline{x})}$ for all $y \in Z(G)$.
(2) $z' \neq x^n$ such that n is a natural number.
(3) If $G/Z(G)$ is abelian and no element in $J(G)$ has even order, then $z' \neq w^{o(x)}$ for all $w \in G$.
(4) $x(G'\langle z \rangle) \neq a^{o(z')}$ for all $a \in G/(G'\langle z \rangle)$.

Now, we want to show that x satisfies in the conditions (i)–(iv). Since $Z_q(G) \subseteq Z(G)$, part (1) implies $z' \neq y^{o(\overline{x})}$ for all $y \in Z_q(G)$. So part (1) implies that part (i). If at least one of conditions (i)–(iii) fails, one of the conditions (1)–(3) fails. It is a contradiction. It is sufficient to show that $x(G'G^q\langle z\rangle) \neq a^{o(z')}$ for all $a \in G/(G'G^q\langle z\rangle)$. We denote the image of $x \in G$ in $(G/G'G^q\langle z\rangle)$ by $\widehat{x}$. Suppose on the contrary that there exists an element a of $G/(G^qG'\langle z\rangle)$ such that $x(G'G^q\langle z\rangle) = a^{o(z')}$. As a result, $x = a^{o(z')}mn^qz^l (m \in G', n \in G, l \geq 0)$ and so by Lemma 4.1

$$\begin{aligned} z' \wedge \widehat{x} &= z' \wedge \widehat{a^{o(z')}mn^q}z^l \\ &= (z' \wedge \widehat{a^{o(z')}})(z' \wedge \widehat{n^q})(z' \wedge \widehat{z^l}) \\ &= z' \wedge \widehat{z^l}. \end{aligned}$$

As a result, $\rho(z' \wedge \widehat{x}) = z' \wedge z^l = 1$. Thus, $z' \wedge \widehat{x} \in \ker\rho$ and consequently, $z' \in \mathfrak{Z}_q^*(G) \neq 1$. As a result, G is not q-capable, by the criterion for q-capability. It is a contradiction. Therefore, $x(G'^q\langle z\rangle) \neq a^{o(z')}$ for all $a \in G/(G'G^q\langle z\rangle)$, as required. □

The following result is taken immediately from Lemma 5.1.

Corollary 5.1. *Let $q \geq 0$. If G is a group such that $\frac{G}{G'G^q}$ is a divisible group and $Z_q(G)$ contains a non-trivial element of finite order, then G is not q-capable.*

For a given subgroup A of an abelian group B, we denote the quotient $A \otimes_{\mathbb{Z}} B/\langle a \otimes a | a \in A\rangle$ of the usual tensor product $A \otimes_{\mathbb{Z}} B$ by $A \wedge_{\mathbb{Z}} B$. Let us first look for a condition for q-capability of some certain groups.

Lemma 5.2. *Let H and K be two normal subgroups of the group G such that K is a subgroup of H and $[H, K]$ is trivial. Then*

$$H \wedge^q K \cong H/H^qH' \wedge_{\mathbb{Z}} K/K^qK'.$$

Proof. By [9, Theorem 6(iv)], we have the epimorphism

$$\begin{aligned} \eta : H \otimes^q K &\to H/H^qH' \otimes_{\mathbb{Z}} K/K^qK' \\ h \otimes k &\mapsto hH^qH' \otimes kK'K^q, \\ \{(x_1, x_1)\} &\mapsto x_1H'H^q \otimes x_1K'K^q, \end{aligned}$$

for all $h \in H, k \in K$ and $x_1 \in K$. The restriction of η to $\langle x \otimes x | x \in K\rangle$ induces the following isomorphism

$$\begin{aligned} &H \otimes^q K/\langle x \otimes x \,|\, x \in K\rangle(H/H^qH' \otimes_{\mathbb{Z}} K/K^qK')/\langle xH^qH' \otimes xK'K^q | x \in K\rangle, \\ &(h \otimes k)\langle x \otimes x \,|\, x \in K\rangle \mapsto (hH^qH' \otimes kK'K^q)(\langle xH^qH' \otimes xK'K^q | x \in K\rangle), \\ &\{(x_1, x_1)\}\langle x \otimes x \,|\, x \in K\rangle \mapsto (x_1H^qH' \otimes x_1K'K^q)(\langle xH^qH' \otimes xK'K^q | x \in K\rangle), \end{aligned}$$

for all $h \in H, k \in K$ and $x_1 \in K$. This means that

$$H \wedge^q K \cong H/H^q H' \wedge_{\mathbb{Z}} K/K^q K',$$

as desired. □

Lemma 5.3. *Let G be a group whose derived factor $G/G' \cong B \oplus C$. Then*

$$Z(G) \wedge^q G/G' \cong (Z(G) \wedge^q B) \oplus (Z(G) \wedge^q C).$$

Proof. Clearly, $Z(G) \wedge_{\mathbb{Z}} (G/G') \cong (Z(G) \wedge_{\mathbb{Z}} B) \oplus (Z(G) \wedge_{\mathbb{Z}} C)$. By Lemmas 5.2 and 4.2, we have

$$\begin{aligned} Z(G) \wedge^q (G/G') &\cong Z(G) \wedge^q G \\ &\cong (Z(G)/Z(G)^q) \wedge_{\mathbb{Z}} (G/G'G^q) \\ &\cong ((Z(G)/Z(G)^q) \wedge_{\mathbb{Z}} (B/B^q)) \oplus ((Z(G)/Z(G)^q) \wedge_{\mathbb{Z}} (C/C^q)) \\ &\cong (Z(G) \wedge^q B) \oplus (Z(G) \wedge^q C), \end{aligned}$$

the result follows. □

Theorem 5.1. *Let G be a group whose center Z is a direct sum of cyclic groups. Suppose that Z has finite exponent* $\exp(Z)$ *and* $\ker\rho$ *is trivial. If $G/G' \cong B \oplus C$ such that the summand C of G/G' is a cyclic group, the summand B of G/G' is the image of Z and* $\exp(Z)$ *divides* $\exp(C)$, *then G is q-capable for $q \geq 0$.*

Proof. By Lemma 5.3, we have

$$Z \wedge^q (G/G') \cong (Z \wedge^q B) \oplus (Z \wedge^q C).$$

Now suppose that $Z = \bigoplus_i \langle x_i \rangle$ where $o(x_i) = n_i$. Fix a generator x_i of Z and assume that c is a generator of C of order n. Clearly, $\langle x_i \rangle \wedge^q C$ is isomorphic to $C_{n_i} \wedge^q C_n$ and also $C_{n_i} \wedge^q C_n = C_{n_i} \otimes^q C_n$. Now we can describe the structure of $C_{n_i} \otimes^q C_n$ by [9, Theorem 6(iv)],

$$C_{n_i} \otimes^q C_n \cong \frac{C_{n_i}}{C_{n_i}^q} \otimes_{\mathbb{Z}} \frac{C_n}{C_n^q}.$$

Consequently, $C_{n_i} \wedge^q C_n$ is isomorphic to $\mathbb{Z}_{n_i}$ or $\mathbb{Z}_{\gcd(n_i,q)}$. Thus, if λ is a positive integer not divisible by $\gcd(o(x_i), q)$ or $o(x_i)$, then

$$1 \neq \lambda x_i \wedge c \in C_{n_i} \wedge^q C_n \subseteq Z \wedge^q (G/G').$$

As a result, $\mathfrak{Z}_q^*(G)$ is trivial and so G is q-capable, as required. □

The following lemma is useful in the next results.

Lemma 5.4. *Let $G \cong \bigoplus_{i=1}^t C_i$ such that n_i divides n_{i+1} and $C_i = \langle x_i \rangle$ is a cyclic summand of G of order n_i for all $1 \leq i \leq t$ and $t \geq 1$. Then the order of $x_i \wedge x_j \in G \wedge^q G$ is n_i or* $\gcd(n_i, q)$ *for all $1 \leq i < j \leq t$.*

Proof. For $t = 1$, by using Lemma 5.2, we have

$$\begin{aligned} C_{n_1} \wedge^q C_{n_1} &\cong C_{n_1}/C_{n_1}^q \wedge_{\mathbb{Z}} C_{n_1}/C_{n_1}^q \\ &= (\langle x_1\rangle/\langle x_1^q\rangle) \wedge_{\mathbb{Z}} (\langle x_1\rangle/\langle x_1^q\rangle) = \frac{(\langle x_1\rangle/\langle x_1^q\rangle) \otimes_{\mathbb{Z}} (\langle x_1\rangle/\langle x_1^q\rangle)}{\langle x_1\langle x_1^q\rangle \otimes x_1\langle x_1^q\rangle\rangle} \\ &= \frac{(\langle x_1\langle x_1^q\rangle \otimes x_1\langle x_1^q\rangle)}{\langle x_1\langle x_1^q\rangle \otimes x_1\langle x_1^q\rangle\rangle} = 1. \end{aligned}$$

Now, let $t = 2$. By induction on t and Lemma 5.2, we obtain that

$$\begin{aligned} G \wedge^q G &\cong G/G^q \wedge_{\mathbb{Z}} G/G^q \\ &\cong \left(\bigoplus_{i=1}^{t} C_i/C_i^q \wedge_{\mathbb{Z}} C_i/C_i^q\right) \oplus \left(\bigoplus_{1\le j<i\le t} (C_i/C_i^q \wedge_{\mathbb{Z}} C_j/C_j^q)\right) \\ &\cong \left(\bigoplus_{i=1}^{t} C_i \wedge^q C_i\right) \oplus \left(\bigoplus_{1\le j<i\le t} (C_i \wedge^q C_j)\right). \end{aligned}$$

We also know that $C_i \square C_j$ is trivial, since the intersection of C_i and C_j is trivial. Thus $x_i \wedge x_j \in C_i \wedge^q C_j \cong C_i \wedge_{\mathbb{Z}} C_j$, for all $1 \le i < j \le t$. By similar procedure as in the proof of Theorem 5.1, the order of $x_i \wedge x_j$ is n_i or $\gcd(n_i, q)$ for all $1 \le i < j \le t$. The proof is complete. □

Now we can detect the q-capable groups with some weaker assumptions than what has been declared in [10, Theorem 5].

Theorem 5.2. *Let $q \ge 0$. Consider G to be a group whose q-center Z_q is a direct sum of cyclic groups. Suppose that Z_q has finite exponent $\exp(Z_q)$ and $\ker\rho$ is trivial. Assume that the canonical homomorphism $Z_q \to G/G'$ maps Z_q isomorphically on to a direct summand of G/G'. If at least two cyclic summands of Z_q have exponent equal to the exponent of Z_q, then G is q-capable.*

Proof. We can choose cyclic subgroups $C_i = \langle x_i\rangle$ of order n_i from Z such that $Z_q = \bigoplus_i C_i$, n_{i+1} divides n_i, and n_1 divides q. Since at least two cyclic summands of Z_q have exponent equal to the exponent of Z_q, we can suppose $n_1 = n_2$. We claim that $\mathfrak{Z}_q^*(Z_q)$ is trivial. By [9, Theorem 6(iv)], $Z_q \otimes^q Z_q$ is abelian and so $Z_q \wedge^q Z_q$ is also abelian. Let $z \in \mathfrak{Z}_q^*(Z_q)$ and $z = \sum \lambda_i x_i$ such that λ_is are integer numbers. Then $z \wedge x$ is trivial for all $x \in Z_q$. Therefore,

$$1 = z \wedge x_2 = \left(\sum_i \lambda_i x_i\right) \wedge x_2 = \sum_i (\lambda_i x_i \wedge x_2) = \sum_i \lambda_i (x_i \wedge x_2)$$

and

$$1 = z \wedge x_1 = \left(\sum_i \lambda_i x_i\right) \wedge x_1 = \sum_i (\lambda_i x_i \wedge x_1) = \sum_i \lambda_i (x_i \wedge x_1).$$

From the construction of the direct product similar to the proof of Lemma 5.4,

$$1 = \lambda_1(x_1 \wedge x_2) = \lambda_3(x_3 \wedge x_2) = \cdots$$

and

$$1 = \lambda_2(x_2 \wedge x_1) = \lambda_3(x_3 \wedge x_1) = \cdots.$$

Now, since n_i divides q, Lemma 5.4 implies that n_i divides λ_i. Thus $z = \sum_{i=1}^{n} \lambda_i x_i$ is trivial and hence $\mathfrak{Z}_q^*(Z_q)$ is trivial. But in this case, $G/G' \cong Z_q \oplus A$ for some abelian group A. By Lemma 5.3, we can obtain

$$Z \wedge^q G/G' \cong (Z_q \wedge^q Z) \oplus (Z_q \wedge^q A) \cong (Z_q \wedge^q Z_q) \oplus (Z_q \otimes^q A).$$

In particular, $Z_q \wedge^q Z_q$ is a subgroup of $Z_q \wedge^q G/G'$. Now by considering the definition of $\mathfrak{Z}_q^*(G)$,

$$\mathfrak{Z}_q^*(G) = \{z \in Z(G) \,|\, z \wedge a = 1_{Z(G)\wedge^q G/G'} \text{ for all} a \in G/G'\},$$

$\mathfrak{Z}_q^*(G)$ is a subgroup of $\mathfrak{Z}_q^*(Z_q)$. If $z \in \mathfrak{Z}_q^*(G)$, then $z \wedge a$ is trivial for all $a \in Z_q$. Since $z \in \mathfrak{Z}_q^*(Z_q) = 1$, we conclude that z is trivial and so G is q-capable, by Theorem 4.2. The proof is complete. □

Corollary 5.2. *Let G be an elementary abelian p-group of order p^k and assume that p divides q $(q \geq 0)$. Then G is q-capable for all $k \geq 2$.*

Proof. For $k = 1$, we have G is a cyclic p-group of order p. We know that G is not capable and so G is not q-capable. Now, let $k \geq 2$. Clearly, $\ker(\rho : G \wedge^q G \to G \wedge^q G)$ is trivial. Theorem 5.2 implies that G is q-capable and the proof is complete. □

Assume p is an odd prime number. One of the most interesting groups are extra-special p-groups which have different properties and plays great roles in group theory. A natural question that arises here is what can one say about q-capability of extra-special p-groups? Let E_1 be the extra special p-group of order p^3 and exponent p $(p > 2)$. Using the results of this section, we can also state that E_1 is q-capable.

Lemma 5.5. *Let $E_1 = \langle x, y | x^p = y^p = [x, y]^p = [y, x, x] = [y, x, y] = 1\rangle$. Then $E_1 \wedge^q E_1$ is the group generated by $x \wedge y, x \wedge [x, y], y \wedge [x, y], \{x\}, \{y\}$ with the following relation.*

$$(x \wedge y)^p = (x \wedge [x, y])^p = (y \wedge [x, y])^p = (\{x\})^p = (\{y\})^p = 1.$$

Proof. According to [16, Lemma 2.7], $E_1 \wedge^p E_1$ is generated by the following set

$$\{x \wedge y, [x, y] \wedge x, [x, y] \wedge y, \{x\}, \{y\}\}.$$

Therefore, $\{x^p\} = (\{x\}) \prod_{i=1}^{q-1} (x \wedge x^i)^{q-1}(\{x^{p-1}\})$ and so $\{x\}^p = \{x^p\} = 1$. By [1, Lemma 3.4] we conclude that $(x \wedge y)^p = x^p \wedge y = 1$. Using the same method, we obtain $(x \wedge [x, y])^p = (y \wedge [x, y])^p = (\{y\})^p = 1$. The proof is complete. □

Theorem 5.3. *Let G be an extra-special p-group $(p > 2)$ and q be a power of p. Then, G is q-capable if and only if $G \cong E_1$.*

Proof. Let G be q-capable. Then G is capable. So $G \cong E_1$, by [3, Corollary 8.2]. Conversely, we claim that $G \cong E_1$ is q-capable. By Lemma 5.2, we get

$$Z(G) \wedge^p G = \langle [x, y] \wedge x, [x, y] \wedge y | ([x, y] \wedge x)^p = ([x, y] \wedge y)^p = 1\rangle \cong C_p \times C_p.$$

By Corollary 4.1, we have the following homomorphism ρ,

$$\rho : Z(G) \wedge^p G/G' \to G \wedge^p G$$
$$[x, y] \wedge xG' \mapsto [x, y] \wedge x,$$
$$[x, y] \wedge yG' \mapsto [x, y] \wedge y.$$

If $[x, y]^l \in \mathfrak{Z}_p^*(G)$ for some $l \geq 0$, then $[x, y]^l \wedge a \in \ker \rho$ for all $a \in G/G'$. Therefore, $[x, y]^l \wedge xG'$ is an element of $\ker \rho$. Thus, $[x, y]^l \wedge x = 1$. On the other hand, $([x, y] \wedge x)^l = 1$ and so p divides l. As a result, $[x, y]^l = 1$. Now, we can easily verify that $\mathfrak{Z}_p^*(G)$ is trivial and consequently, G is p-capable. The rest of the proof is straightforward from Lemma 3.2 and Theorem 4.2. The result follows. □

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