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n-Capability of A-Groups

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Abstract

Following P. Hall a soluble group whose Sylow subgroups are all abelian is called A-group. The purpose of this article is to give a new and shorter proof for a criterion on the capability of A-groups of order p^2q, where p and q are distinct primes. Subsequently we give a sufficient condition for n-capability of groups having the property that their center and derived subgroups have trivial intersection, like the groups with trivial Frattini subgroup and A-groups. An interesting necessary and sufficient condition for capability of the A-groups of square free order will be also given.

Keywords: n-capable group, Sylow subgroup, Frattini subgroup

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1. Introduction and Preliminaries

In 1938, Baer [1] initiated a systematic investigation of the question which conditions a group G must fulfill in order to be the group of inner automorphisms of some group E ($G \cong E/Z(E)$). Following M. Hall and Senior [8] such a group G is called *capable*. Baer classified capable groups that are direct sums of cyclic groups. His characterization of finitely generated abelian groups that are capable is given in the following theorem.

Theorem 1.1. [1]. Let G be a finitely generated abelian group written as $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, such that each integer n_i+1 is divisible by n_i, where $\mathbb{Z}_0 = \mathbb{Z}$, the infinite cyclic group. Then G is capable if and only if $k \geqslant 2$ and $n_{k-1} = n_k$.

In 1940, P. Hall [6] introduced the concept of isoclinism of groups which is one of the most significant methods for classification of groups. He showed that

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capable groups play an important role in characterizing finite p-groups. Capable groups also have a fascinating property that does not hold for an arbitrary group. Extraspecial p-groups show that there is no general upper bound on the index of the center of a finite group in terms of the order of its derived subgroup. However, Isaacs [11] proved that if G is a finite capable group, then $|G/Z(G)|$ is bounded above by a function of $|G^{'}|$. Later, Podoski and Szegedy [13] extended the result and gave the following explicit bound for the index of the center in a capable group as follows:

Theorem 1.2. If G is a capable group and $|G'| = n$, then $|G/Z(G)| \leq n^{2\log_2 n}$.

Also, they showed that for a finite capable group we have $[G : Z(G)] \leq |G'|^2$, whenever G' is a cyclic group.

Let us consider soluble groups whose Sylow subgroups are all abelian. Following P. Hall such groups are called A-groups. A-groups were investigated thoroughly by P. Hall and D. R. Taunt in [7] and [16]. In an A-group G, the intersection of the commutator subgroup $G^{'}$ and the center $Z(G)$ is trivial. In 2011, Rashid *et al.* [14] found a criterion for capability of A-groups of order p^2q, where p and q are distinct primes. They determine the explicit structure of the Schur multiplier and exterior center to obtain a necessary condition for capability of these groups. Then, they show that this condition is sufficient using the presentation of the groups of order p^3q and also p^2q with the condition that $p < q$. In this paper, we obtain the same necessary condition by a different process and then show that the sufficiency of the condition will follow using the above property of A-groups. Moreover, we show that capability and n-capability coincide for the groups of order p^2q. Also, we can obtain a sufficient condition for n-capability of the groups having the property that the intersection of the center subgroup and the commutator subgroup is trivial such as A-groups and groups with trivial Frattini subgroup. Finally, we give a criterion for capability of the A-groups of square free order.

The following famous results will be used in the article.

Theorem 1.3. [6] Let J be a generating system for G and $\Delta_J = \cap_{x\in J}\langle x\rangle$. Then $\triangle_J \subseteq \varphi(Z(E))$ for every central extension (E, φ) of G.

The join of all subgroups Δ_J, where J varies over all generating system of G will be denoted by $\Delta(G)$. It follows from Theorem 1.3 that a capable group G must satisfy $\Delta(G) = 1$. However, this condition is not sufficient for G in order to be capable. Here is another necessary condition for a group to be capable.

Theorem 1.4. [2, Proposition 1.2] If G is capable and the commutator factor group G/G' of G is of finite exponent, then also $Z(G)$ is bounded and the exponent of $Z(G)$ divides that of G/G'.

Corollary 1.5. [5, Proposition 1] Let G be a finitely generated capable group. Then every central element z in G has order dividing $exp((G/\langle z\rangle)^{ab})$.

In 1979, Beyl *et al.* [2] studied capable groups by focusing on a characteristic subgroup $Z^*(G)$, called the epicenter of G and it is defined to be the intersection of all normal subgroups N of G such that G/N is capable. In fact, they established a necessary and sufficient condition for a group to be capable in terms of the epicenter.

Theorem 1.6. [2] A group G is capable if and only if $Z^*(G) = 1$.

Obviously, the class of all capable groups is neither subgroup closed nor under homomorphic image. But this class is closed under direct product [2, Proposition 6.1]. It follows that $Z^*(\prod_{i\in I} G_i) \subseteq \prod_{i\in I} Z^*(G_i)$. One should also notice that the inclusion is proper in general. Beyl *et al.* [2] gave a sufficient condition forcing equality as follows.

Theorem 1.7. Let $G = \Pi_{i\in I} G_i$. Assume that for $i \neq j$ the maps $\upsilon_i \otimes 1 : Z^*(G_i) \otimes G_j/G'_j \longrightarrow G_i/G'_i \otimes G_j/G'_j$ are zero, where υ_i is the natural map $Z^*(G_i) \longrightarrow G_i \longrightarrow G_i/G^{'}_i$. Then $Z^*(G) = \Pi_{i\in I} Z^*(G_i)$.

It follows immediately from Theorem 1.7 that a finite nilpotent group is capable if and only if all of whose Sylow subgroups are capable. Hence, if G is a nilpotent capable group and its order as a product of powers of distinct primes to be $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha t}$, then α_i should be greater than 1, for $1 \leq i \leq t$, since cyclic groups are not capable.

2. Main Results

2.1 Capable A-Groups of Order p^2q

The first step shows that a capable group of order p^2q can not be nilpotent.

Lemma 2.1. Let G be a capable group of order p^nq, for some positive integer n and distinct primes p and q. Then G is not a nilpotent group.

Proof. It is straightforward. □

We recall that a nilpotent group of order p^2q is an abelian group, because all of whose Sylow subgroups are abelian.

Lemma 2.2. Let G be a capable group of order p^nq, for some positive integer n and distinct primes p and q. Then the prime q does not divide the order of $Z(G)$.

Proof. If q divides $|Z(G)|$, then $G/Z(G)$ is a finite p-group and so G will be nilpotent. It is a contradiction. □

The next lemma determines the order of the center of a nonnilpotent group of order p^2q.

Lemma 2.3. Let G be a nonnilpotent group of order p^2q. Then the center of G is trivial or of order p.

Proof. Using Lemma 2.2, the order of the center is not divisible by the prime q. Also, if $|Z(G)| = p^2$, then $G/Z(G)$ is a cyclic group and hence G is an abelian group. It is impossible and the result follows. $\square$

It is obvious that if $Z(G) = 1$, then G is capable. Hence, Lemmas 2.1 and 2.3 imply that we should investigate capable groups among nonnilpotent groups in which $|G/Z(G)| = pq$.

Lemma 2.4. Let H be a nonnilpotent group of order pq such that $p > q$. Then $|H'| = p$.

Proof. Let H be a nonnilpotent group of order pq. Then $Z(H) = 1$ and so H is a capable group. Using [13, Theorem 7], we have $|H/Z(H)| \leq |H'|^2$. Thus $|H'| = p$, since $p > q$. $\square$

Lemma 2.5. Let G be a capable group of order p^2q with nontrivial center subgroup. Then $p < q$.

Proof. Let $p > q$. Since $G/Z(G)$ is a nonnilpotent group of order pq, then $|(G/Z(G))^{'}| = p$ by Lemma 2.4. Let z be an element of G such that $Z(G) = \langle z \rangle$. Using Corollary 1.5, the central element z in G has order dividing $exp((G/\langle z \rangle)^{ab})$. It implies that $p|q$ and this is a contradiction. $\square$

Lemma 2.5 illustrates that groups of order p^2q with $p > q$ and nontrivial center are not capable.

Lemma 2.6. Let G be a capable group of the order p^2q with nontrivial center subgroup. Then the commutator subgroup of G is of order q.

Proof. We see at once that G is a nonnilpotent group whose center has order p with $p < q$. Using Lemma 2.4, we have $|(G/Z(G))^{'}| = q$. Now, since the intersection of the commutator subgroup $G^{'}$ and the center $Z(G)$ in the A-group G is trivial, then the result follows. $\square$

Rashid *et al.* [14] using the Schur multiplier and exterior center of groups of order p^2q show that capable groups with nontrivial center can not have the derived factor group isomorphic to a cyclic group of order p^2. In the following, we prove this assertion according to the property of capability directly and without the usage of Schur multiplier.

Lemma 2.7. Let G be a capable group of order p^2q with $|Z(G)| = p$. Then $G^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.

Proof. Let G be a group with the above property. Then Lemma 2.6 implies that $|G'| = q$ and so $|G/G'| = p^2$. Let $G/G' \cong \mathbb{Z}_{p^2}$. Consider x and y in G such that $G/G' = \langle xG' \rangle$ and $G' = \langle y \rangle$. Obviously G can be generated by x and y. First, we assume that $x^{p^2} \neq 1$. Since $x^{p^2} \in G'$, there exists a natural number t such that $x^{p^2} = y^t \neq 1$. Then $\langle x \rangle \cap \langle y \rangle \neq 1$ and so $\triangle(G) \neq 1$. But this implies that G is not capable which contradics to the assumption. Therefore $x^{p^2} = y^q = 1$. The generator xG' of G/G' has order p^2 and so $x^p \notin G'$. On the other hand, since $G' \cap Z(G) = 1$, we have $|G/G'Z(G)| = p$. It follows that $x^p \in Z(G)$. Now, since G is capable, then there exists a group H such that $G \cong H/Z(H)$. We denote $\bar{H} := H/Z(H)$. Under the recent isomorphism, there are corresponding elements x_1 and y_1 of H such that $\bar{x_1}^{p^2} = \bar{y_1}^q = \bar{1}$. We shall show that $x_1^p \in Z(H)$ which is a contradiction. Now $x^p \in Z(G)$ implies that $\bar{x_1}^p \in Z(\bar{H})$. Hence $[\bar{x_1}^p, \bar{y_1}] = \bar{1}$ and so there exists $z_0 \in Z(H)$ such that $x_1^p y_1 = y_1 x_1^p z_0$. This equality deduces that $z_0^p = 1$, since $x_1^{p^2} \in Z(H)$. Similarly, we conclude that $z_0^q = 1$, since $y_1^q \in Z(H)$. Now $z_0^p = z_0^q = 1$ implies that $z_0 = 1$ and hence $x_1^p y_1 = y_1 x_1^p$. Thus $x_1^p \in Z(H)$. Therefore $G^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. □

The next theorem was proved by Rashid *et al.* using the presentation of groups of order p^3q and also p^2q with the condition that $p < q$. Here, we intend to give an easy proof for it using the important property of A-groups mentioned in Section 1.

Theorem 2.8. Let G be a group of order p^2q such that $G^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. Then G is capable.

Proof. The isomorphism in the assumption implies that G^{ab} is capable. Now, the factor groups $G/Z(G)$ and G/G' are capable. Thus $Z^*(G) \subseteq G' \cap Z(G)$, by the comment mentioned after Corollary 1.5. On the other hand, the intersection of the commutator subgroup G' and the center $Z(G)$ in the A-group G is trivial. Therefore, G must be a capable group using Theorem 1.6. □

Finally, we can also conclude a criterion for capability of the groups of order p^2q as follows.

Theorem 2.9. Let G be a group of the order p^2q. Then G is capable if and only if either $Z(G) = 1$ or $G^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ and $p < q$.

The alternating group A_4 of order 4 is an example of capable group with $Z(G) = 1$ and the group G defined by the presentation $G = \langle a, b, c | a^q = b^p = c^p = 1, bab^{-1} = a^i, ac = ca, bc = cb \rangle$, where $i^p \equiv 1 \ (mod\ q)$ and $p | q - 1$, can be considered as an example for the second part of Theorem 2.9 (see [4, Section 59] or [14]).

2.2 n-Capability of A-Groups

Burns and Ellis in [3] and Moghaddam and Kayvanfar in [12] independently introduced the concept of an n-capable group as follows:

Definition 2.10. *A group G is said to be n-capable if there exists a group H such that $G \cong H/Z_n(H)$.*

Obviously 1-capability is capability and G is n-capable ($n \geq 2$) if and only if it is an inner automorphism group of an $(n-1)$-capable group, that is, n-capability implies the 1-capability for a group. The existence of a capable group which is not 2-capable is explained in [3]. Hassanzadeh and the third author [9] referred to an example of P. Hall to construct n-capable groups which are not $(n+1)$-capable.

Burns and Ellis [3] proved the following theorem.

Theorem 2.11. A finitely generated abelian group G is n-capable if and only if it is capable.

We shall also show this coincidence for the groups of order p^2q. For this purpose we need the next theorem.

Theorem 2.12. [12, Theorem 2.2] Let N_i be a normal subgroup of G, and G/N_i an n-capable factor group of G ($i \in I$). If $N = \cap_{i \in I} N_i$, then G/N is n-capable.

Theorem 2.13. Let G be a group of order p^2q. Then G is capable if and only if it is n-capable for all $n \in \mathbb{N}$.

Proof. Clearly G is capable if it is an n-capable group, for some $n \in \mathbb{N}$. Conversely, let G be a capable group of order p^2q. Then either $Z(G) = 1$ or $G/G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. Trivially, G is n-capable for every $n \in \mathbb{N}$ if $Z(G) = 1$. If $Z(G) \neq 1$, then $G/Z(G)$ is a non-abelian group of order pq. Hence $Z(G/Z(G)) = 1$ and $G/Z(G)$ is n-capable for every $n \in \mathbb{N}$. Also, $G/G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ is an n-capable group for every $n \in \mathbb{N}$, by Theorems 1.1 and 2.11. Therefore $G/(G' \cap Z(G))$ is n-capable using Theorem 2.12. Finally, $G' \cap Z(G) = 1$ implies that G is an n-capable group. □

In the following we will give a sufficient condition for n-capability of a group G for which $G^{'} \cap Z(G) = 1$.

Theorem 2.14. Let G be a group such that $G^{'} \cap Z(G) = 1$. If $G/G^{'}$ is an n-capable group, then so is G.

Proof. Since $G' \cap Z(G) = 1$, it follows that $G' \cap Z_n(G) = 1$ by induction on n. Now, n-capability of $G/Z_n(G)$ and $G/G^{'}$ implies that G is n-capable using Theorem 2.12. □

Theorem 2.15. Let G be a finite group all of whose Sylow subgroups are abelian. If $G/G^{'}$ is an n-capable group, then so is G.

Proof. Since all of Sylow subgroups of G are abelian, we have $G' \cap Z(G) = 1$ (see [15, 10.1.7]). The result follows using Theorem 2.14. □

Example 2.16. *Let G be the semidirect product of the normal cyclic subgroup of order q and the elementary abelian p-subgroup of rank $t \geqslant 2.$, i.e., $G = (\oplus_1^t \mathbb{Z}_p) \ltimes \mathbb{Z}_q$, where p and q are two distinct prime. Using Theorem 2.15, one can see that G is an n-capable group since all Sylow subgroups of G are abelian and $G/G' \cong \oplus_1^t \mathbb{Z}_p$ is n-capable.*

Theorem 2.17. Let G be a group with trivial Frattini subgroup. If $G/G^{'}$ is an n-capable group, then so is G.

Proof. By the assumption, we have $G' \cap Z(G) = 1$. Now the proof is clear similar to Theorem 2.14. □

2.3 Capable A-Groups of Square-Free Order

Groups of square free order are part of A-groups. In this section, we intend to obtain a criterion for capability of groups of order n, when n is square-free i.e. $n = p_1 p_2 \ldots p_k$ where the p_i's are distinct primes. For this, we need the next lemma.

Lemma 2.18. [10, 7.9. Corollary] Let G be a group such that $G/Z(G)$ is a finite π-group, then $G^{'}$ is a finite π-group.

Theorem 2.19. Let $|G| = p_1 \ldots p_m$, where $p_1, \ldots, p_m$ are distinct primes. Then G is capable if and only if $Z(G) = 1$.

Proof. Let $|Z(G)| = p_{i_1} \ldots p_{i_t}$, where $1 \leq i_j \leq m$ and $1 \leq j \leq t$. Put $\pi = \{p_{i_1}, \ldots, p_{i_t}\}$. It yields $G/Z(G)$ is a $\pi^{'}$-group. Using Lemma 2.18, the commutator subgroup $G^{'}$ is also a $\pi^{'}$-group. Suppose that $p_{u_1} \ldots p_{u_r}$ where $1 \leq u_s \leq m$ and $1 \leq s \leq r$ be $\pi^{'}$-numbers which do not divide the order of $G^{'}$. Using Couchy's theorem there are subgroups H_{u_s} of order p_{u_s} in G. We can now construct the subgroup $H = G^{'} H_{u_1} \ldots H_{u_r}$ in which $G^{'} \subseteq H$ and $|H| = \dfrac{|G|}{|Z(G)|}$. Therefore we have $G = H \times Z(G)$, since $H \trianglelefteq G$ and $H \cap Z(G) = 1$. Here is the non capable subgroup $Z(G)$ and so G is not capable, by applying Theorem 1.7. □

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