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Solving linear two-dimensional Fredholm integral equations system by triangular functions

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Abstract

In this paper we intend to offer a numerical method to solve linear two-dimensional Fredholm integral equations system of the second kind. This method converts the given two-dimensional fredholm integral equations system into a linear system of algebraic equations by using two-dimensional triangular functions. Moreover, we prove the convergence of the method. Finally the proposed method is illustrated by an example and also results are compared with the exact solution by using computer simulations.

Keywords: Two-dimensional Fredholm integral equations system of the second kind (2D-FIES-2), Two-dimensional triangular functions (2D-TFs).

Mathematics Subject Classification (2010): 45B05, 45B99.

1 Introduction

As we know the differential and integral equations are one of the important parts of the analysis theory that play major role in numerical analysis. Recently, introduced a new set of triangular orthogonal functions have been applied for solving Fredholm integral equation system by Babolian et al.[1]. Maleknejad et al.[2] have used two-dimensional triangular functions for solving nonlinear class of mixed Volterra Fredholm integral equations. The aim of this paper is to apply the two-dimensional triangular functions (2D-TFs) for the linear two-dimensional fredholm integral equations system of the second kind (2D-FIES-2).

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2 Solving linear 2D-FIES-2

Definition 2.1. [2] An $(m_1 \times m_2)$-set of two-dimensional triangular functions (2D-TFs) are defined on $\Omega = [0,1] \times [0,1]$ as:

$$T_{i,j}^{1,1}(s,t) = \begin{cases} (1 - \frac{s-ih_1}{h_1})(1 - \frac{t-jh_2}{h_2}), & ih_1 \le s < (i+1)h_1, \\ & jh_2 \le t < (j+1)h_2, \\ 0, & otherwise, \end{cases}$$

$$T_{i,j}^{1,2}(s,t) = \begin{cases} (1 - \frac{s-ih_1}{h_1})(\frac{t-jh_2}{h_2}), & ih_1 \le s < (i+1)h_1, \\ & jh_2 \le t < (j+1)h_2, \\ 0, & otherwise, \end{cases}$$

$$T_{i,j}^{2,1}(s,t) = \begin{cases} (\frac{s-ih_1}{h_1})(1 - \frac{t-jh_2}{h_2}), & ih_1 \le s < (i+1)h_1, \\ & jh_2 \le t < (j+1)h_2, \\ 0, & otherwise, \end{cases}$$

$$T_{i,j}^{2,2}(s,t) = \begin{cases} (\frac{s-ih_1}{h_1})(\frac{t-jh_2}{h_2}), & ih_1 \le s < (i+1)h_1, \\ & jh_2 \le t < (j+1)h_2, \\ 0, & otherwise, \end{cases}$$

where $i = 0, 1, \cdots, m_1 - 1, j = 0, 1, \cdots, m_2 - 1, h_1 = \frac{1}{m_1}, h_2 = \frac{1}{m_2}$. m_1 and m_2 are arbitrary positive integers. On the other hand, if

$$T11(s,t) = [T_{0,0}^{1,1}(s,t), \ldots, T_{0,m_2-1}^{1,1}, T_{1,0}^{1,1}(s,t), \ldots, T_{m_1-1,m_2-1}^{1,1}(s,t)]^T,$$

$$T12(s,t) = [T_{0,0}^{1,2}(s,t), \ldots, T_{0,m_2-1}^{1,2}, T_{1,0}^{1,2}(s,t), \ldots, T_{m_1-1,m_2-1}^{1,2}(s,t)]^T,$$

$$T21(s,t) = [T_{0,0}^{2,1}(s,t), \ldots, T_{0,m_2-1}^{2,1}, T_{1,0}^{2,1}(s,t), \ldots, T_{m_1-1,m_2-1}^{2,1}(s,t)]^T,$$

$$T22(s,t) = [T_{0,0}^{2,2}(s,t), \ldots, T_{0,m_2-1}^{2,2}, T_{1,0}^{2,2}(s,t), \ldots, T_{m_1-1,m_2-1}^{2,2}(s,t)]^T,$$

then $T(s,t)$, can be defined as $T(s,t) = [T11(s,t) \quad T12(s,t) \quad T21(s,t) \quad T22(s,t)]_{4m_1m_2 \times 1}^T$.

We can carry out the following double integration of $T(s,t)$:

$$\int_0^1 \int_0^1 T^T(s,t)T(s,t)\mathrm{d}s\mathrm{d}t = D, \tag{2.1}$$

where D is a $(4m_1m_2 \times 4m_1m_2)$-matrix (see[2]). Let $f(s,t)$ be a function of two variables on $\Omega = [0,1] \times [0,1]$. It can be approximated with respect to 2D-TFs as $f(s,t) \simeq T^T(s,t).F$ where $4m_1m_2$-vector $F = [F1^T \quad F2^T \quad F3^T \quad F4^T]^T$ is called the 2D-TF coefficient vector. Also, let $k(s,t,x,y)$ be a function of four variables on $(\Omega \times \Omega)$. It can be approximated with respect to 2D-TFs as $k(s,t,x,y) \simeq T^T(s,t).K.T(x,y)$, where $T(s,t)$ and $T(x,y)$ are 2D-TF vectors of dimension $4m_1m_2$ and $4m_3m_4$, respectively and K is a $(4m_1m_2 \times 4m_3m_4)$ 2D-TF coefficient matrix. More details about the properties of the triangular functions are given in [2]. In this paper for convergence of the proposed method, we supposed that $m_1 = m_2 = m_3 = m_4 = m$. Now, we introduce the two-dimensional Fredholm integral equations system of the second kind

(2D-FIES-2) in the following form

$$\begin{cases} u_1(x,y) = g_1(x,y) + \sum_{j=1}^{n} \lambda_{1j} \int_0^1 \int_0^1 k_{1j}(x,y,s,t) u_j(s,t) \mathrm{d}s\mathrm{d}t, \\ u_2(x,y) = g_2(x,y) + \sum_{j=1}^{n} \lambda_{2j} \int_0^1 \int_0^1 k_{2j}(x,y,s,t) u_j(s,t) \mathrm{d}s\mathrm{d}t, \\ \quad \vdots \\ u_n(x,y) = g_n(x,y) + \sum_{j=1}^{n} \lambda_{nj} \int_0^1 \int_0^1 k_{nj}(x,y,s,t) u_j(s,t) \mathrm{d}s\mathrm{d}t, \end{cases} \tag{2.2}$$

where $k_{ij}(x,y,s,t)$, $i,j = 1,\dots,n$, are an orbitary kernel function over $(\Omega \times \Omega)$ and $\lambda_{ij} \neq 0$, $i,j = 1,\dots,n$ are real constants and $u_i(x,y)$ and $g_i(x,y)$ are real valued functions for $i = 1,\cdots,n$ and $u_1(x,y), u_2(x,y),\dots,u_n(x,y)$ are the solutions to be determined. For convenience, we consider the ith equation of system (2.2) as

$$u_i(x,y) = g_i(x,y) + \sum_{j=1}^{n} \lambda_{ij} \int_0^1 \int_0^1 k_{ij}(x,y,s,t) u_j(s,t) \mathrm{d}s\mathrm{d}t. \tag{2.3}$$

For solving system (2.2) by using 2D-TFs, first let us expand $u_i(x,y)$, $g_i(x,y)$ and $k_{ij}(x,y,s,t)$ as

$$\begin{aligned} u_i(x,y) &\simeq T^T(x,y).U_i, \\ g_i(x,y) &\simeq T^T(x,y).G_i, \\ k_{ij}(x,y,s,t) &\simeq T^T(x,y).K_{ij}.T(s,t), \end{aligned} \tag{2.4}$$

Substituting the Eqs. (2.4) into Eq. (2.3), we get

$$\begin{aligned} T^T(x,y)U_i &\simeq T^T(x,y)G_i + \sum_{j=1}^{n} \lambda_{ij} \int_0^1 \int_0^1 \left(T^T(x,y)K_{ij}T(s,t)T^T(s,t)U_j\right) \mathrm{d}s\mathrm{d}t \\ &= T^T(x,y)G_i + T^T(x,y) \sum_{j=1}^{n} \lambda_{ij} K_{ij} \left(\int_0^1 \int_0^1 T(s,t)T^T(s,t) \mathrm{d}s\mathrm{d}t \right) U_j. \end{aligned} \tag{2.5}$$

Next, by substituting the Eq. (2.1) into Eq. (2.5), we can write

$$T^T(x,y)U_i \simeq T^T(x,y)G_i + T^T(x,y) \sum_{j=1}^{n} \lambda_{ij} K_{ij} D U_j$$

Thus we have$U_i = G_i + \sum_{j=1}^{n} \lambda_{ij} K_{ij} D U_j$, Then we get the following system

$$\sum_{j=1}^{n} \left(\Delta_{ij} - \lambda_{ij} K_{ij} D\right) U_j = G_i, \qquad \Delta_{ij} = \begin{cases} I & i = j \\ 0 & i \neq j, \end{cases} \tag{2.6}$$

for $i,j = 1,2,\dots,n$ and I is a $4m^2 \times 4m^2$ identity matrix. By solving matrix system (2.6) with Gauss elimination method, we can find U_i for $i = 1,2,\dots,n$. So $u_i(x,y) \simeq T^T(x,y)U_i$.

Theorem 2.2. *(Convergence Analysis) If $k_{ij}(x,y,s,t)$, $i,j = 1,2,\dots,n$ are bounded and continuous, then approximate solution of system (2.2), converges to the exact solution.*

Proof. Suppose that $u_{i,m}(x,y), i = 1,\dots,n$ is an approximate value of the exact solution $u_i(x,y)$. Therefore

$$\begin{aligned} u_{i,m}(x,y) = &\sum_{p=0}^{m-1} \sum_{q=0}^{m-1} c_{p,q}^{i} T_{p,q}^{1,1}(s,t) + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} d_{p,q}^{i} T_{p,q}^{1,2}(s,t) \\ &+ \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} e_{p,q}^{i} T_{p,q}^{2,1}(s,t) + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} l_{p,q}^{i} T_{p,q}^{2,2}(s,t), \end{aligned} \tag{2.7}$$

by using Eqs. (2.3) and (2.7), we can write

$$
\begin{aligned}
\|u_{i,m}(x,y)-u_i(x,y)\| &= \max_{(x,y)\in\Omega} |u_{i,m}(x,y)-u_i(x,y)| \\
&= \max_{(x,y)\in\Omega} |\sum_{j=1}^{n} \lambda_{ij} \int_0^1 \int_0^1 k_{ij}(x,y,s,t)(\sum_{p=0}^{m-1}\sum_{q=0}^{m-1} c_{p,q}^{j} T_{p,q}^{1,1}(s,t) \\
&\quad + \sum_{p=0}^{m-1}\sum_{q=0}^{m-1} d_{p,q}^{j} T_{p,q}^{1,2}(s,t) + \sum_{p=0}^{m-1}\sum_{q=0}^{m-1} e_{p,q}^{j} T_{p,q}^{2,1}(s,t) + \sum_{p=0}^{m-1}\sum_{q=0}^{m-1} l_{p,q}^{j} T_{p,q}^{2,2}(s,t))\mathrm{d}s\mathrm{d}t \\
&\quad - \sum_{j=1}^{n} \lambda_{ij} \int_0^1 \int_0^1 k_{ij}(x,y,s,t) u_j(s,t)\mathrm{d}s\mathrm{d}t| \\
&\leq M \sum_{j=1}^{n} \int_0^1 \int_0^1 \max_{(x,y)\in\Omega} |u_{j,m}(s,t)-u_j(s,t)|\mathrm{d}s\mathrm{d}t \\
&= M \sum_{j=1}^{n} \int_0^1 \int_0^1 \|u_{j,m}(x,y)-u_j(x,y)\|\mathrm{d}s\mathrm{d}t,
\end{aligned}
$$

where

$$M = \max_{0\leq x,y,s,t\leq 1} |\lambda_{ij} k_{ij}(x,y,s,t)| < \infty.$$

Also, we have $\lim_{m\to\infty} u_{j,m}(x,y) = u_j(x,y)$, so $\|u_{j,m}(x,y)-u_j(x,y)\| \to 0$ as $m\to\infty$ for $j=1,\ldots,n$, and since M is bounded, thus

$$\lim_{m\to\infty} \|u_{i,m}(x,y)-u_i(x,y)\| \to 0,$$

so the proof is completed. □

Example 2.3. Consider the system of linear two-dimensional Fredholm integral equations

$$
\begin{cases}
u_1(x,y) = xy - \frac{6}{20}x + \int_0^1\int_0^1 xu_1(s,t)\mathrm{d}s\mathrm{d}t + \int_0^1\int_0^1 xst^4 u_2(s,t)\mathrm{d}s\mathrm{d}t, \\
u_2(x,y) = x^2 - \frac{1}{6}xy - \frac{1}{3}y^2 + \int_0^1\int_0^1 xysu_1(s,t)\mathrm{d}s\mathrm{d}t + \int_0^1\int_0^1 y^2 u_2(s,t)\mathrm{d}s\mathrm{d}t.
\end{cases}
$$

One can easily verify that $(u_1(x,y), u_2(x,y)) = (xy, x^2)$ is an exact solution of the given problem.

The absolute error of $u_i(x,y)$ is $E_i = |u_{i,m}(x,y)-u_i(x,y)|$ for $i=1,2$ with $m=32$, is listed in Table 1. Also Figure 1 illustrate the comparison values between the exact solution and the approximate solution by the presented method. Moreover, Absolute error functions obtained by the present method also shown in Figure 1.

Table 1: Numerical results for Example 2.3, with $m = 32$.

(x,y)	Approximate solution $(u_{1,m}(x,y), u_{2,m}(x,y))$	E_1	E_2
$(0.0, 0.0)$	(0.0000e-00, 0.0000e-00)	0.0000e-00	0.0000e-00
$(0.1, 0.1)$	(1.0024e-02, 1.0160e-02)	2.4369e-05	1.5985e-04
$(0.3, 0.3)$	(9.0073e-02, 9.0266e-02)	7.3107e-05	2.6647e-04
$(0.5, 0.5)$	(2.5012e-01, 2.5009e-01)	1.2184e-04	8.8975e-05
$(0.7, 0.7)$	(4.9017e-01, 4.9041e-01)	1.7058e-04	4.0883e-04
$(0.9, 0.9)$	(8.1022e-01, 8.1044e-01)	2.1932e-04	4.4457e-04

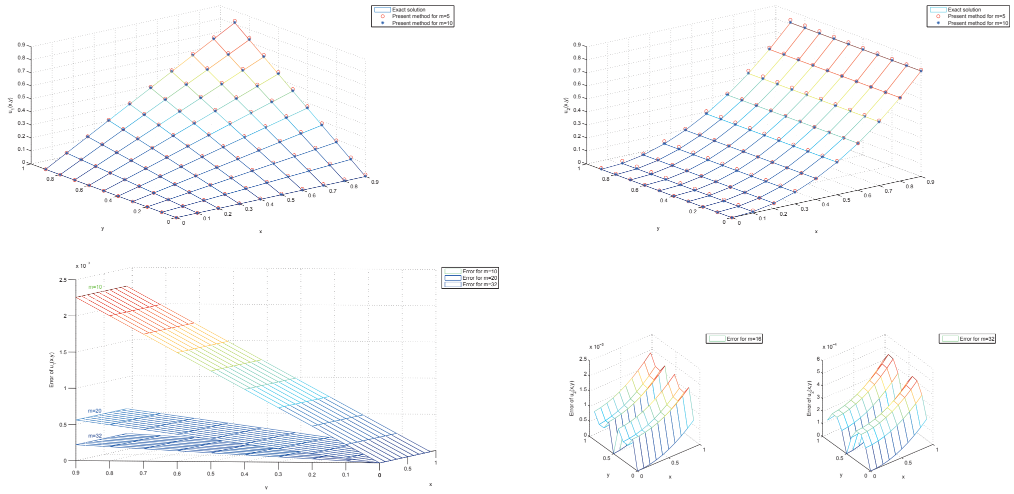

Figure 1: Up: Comparison between the Exact solution and the Approximate solution by the present method for $m = 5$ and 10 . Down:Absolute error $u_1(x, y)$ by the present method for $m = 10, 20, 32$ and absolute error $u_2(x, y)$ by the present method for $m = 16, 32$ for Example 2.3.

Conclusion

In this paper, we introduced TFs method for approximating the solution of the linear 2D-FIES-2. The structural properties of TFs are utilized to reduce the 2D-FIES-2 to a linear system of algebraic equations, without using any integration. In the above presented numerical examples one can see that the proposed method well performs for linear 2D-FIES-2.

References

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