Received: 11 January 2022 | Revised: 17 June 2022 | Accepted: 22 June 2022

IET Control Theory & Applications

DOI: 10.1049/cth2.12327

The Institution of Engineering and Technology WILEY

ORIGINAL RESEARCH

Cross backstepping sliding mode control using integral barrier Lyapunov function for cross-strict feedback systems

Seyedeh Zahra Paylakhi | Naser Pariz | Mohammad Bagher Naghibi Sistani | Mohammad Hossein Javidi Dasht Bayaz

Department of Electrical Engineering, Faculty of Engineering, Ferdowsi University of Mashhad, Mashhad, Iran

Correspondence
Naser Pariz, Faculty of Engineering, Ferdowsi University of Mashhad, Mashhad 9178658391 Iran.
Email: n-pariz@um.ac.ir

Abstract

Cross-backstepping control for a type of uncertain non-strict-feedback non-linear systems with time-varying partial state constraints is the main subject of this work. Non-strict-feedback non-linear systems are partitioned into two strict-feedback non-linear subsystems: constrained subsystem and unconstrained subsystem. An integral barrier Lyapunov function (IBLF) is used in each step of the backstepping design for the constrained subsystem to guarantee the boundedness of the fictional or actual state tracking errors. The effect of uncertainty is reduced using a hybrid cross-backstepping sliding mode control (SMC) technique. The algorithm employs a systematic approach to developing control laws for non-linear systems with matched and unmatched uncertainties. The simulation results of the proposed controller are juxtaposed with those of the cross backstepping with the time-varying barrier Lyapunov function (TVBLF). The results demonstrate the overall better performance of the proposed method.

1 | INTRODUCTION

Control for non-linear systems has sparked a lot of interest; hence, numerous control approaches for non-linear systems have been explored. Non-linear systems have been given control methods based on the Lyapunov function, such as sliding mode control (SMC), Lyapunov redesign, and the backstepping method [1–4].

The backstepping method, a recursive procedure for non-linear systems in the strict feedback form employing a Lyapunov function and a systematic design approach, is one of the significant achievements for managing non-linear systems. It has the ability to improve global stability as well as tracking and transient performance [1]. Due to its numerous advantages, adaptive backstepping control is generally utilized in strict-feedback non-linear systems [5–7]. On the other hand, the typical backstepping technique can only be employed to regulate stringent feedback systems, significantly limiting its application in chaotic synchronization control [7].

Cross-strict feedback systems can be regarded as a type of chaotic system. Because of their importance in practice and theory [9–11], when combined with two or more strict feedback subsystems [8], these systems have gotten a lot of attention in the control area.

Gong et al. [8] introduced a new cross backstepping control approach for cross strict-feedback non-linear systems, verifying the system's global stability. Wang et al. [12] proposed a backstepping approach based on n control inputs for cross-strict feedback systems with unknown parameters. Nonetheless, if the number of control inputs is fewer than n, it will not apply to the systems. Li et al. go on to offer robust backstepping synchronization control methods based on this conclusion, making the cross backstepping methodology appropriate for inputs smaller than n [10, 11]. On the other hand, these solutions ignore the issue of state constraints.

Physical restrictions are widely recognized to cause various constraints in many industrial control systems. If we cannot adequately address these restrictions in the controller design, performance will suffer, and the system will become unstable [13].

The barrier Lyapunov function (BLF) was initially included in the backstepping design framework to solve the output constraint of non-linear systems [14]. The complete state restrictions were successfully addressed by coupling the BLF with an adaptive approach for pure feedback non-linear systems [15]. For strict feedback or pure feedback non-linear systems, many BLF-based backstepping control techniques have been developed [16–19].

The state restrictions must fulfil the constant situation, which is the fundamental limitation of the prior study. Nonetheless, in practice, time-varying state restrictions are ubiquitous. The time-varying full state constraints of non-linear systems were efficiently addressed by applying the backstepping approach with the assistance of the time-varying barrier Lyapunov function (TVBLF), with numerous exemplary results presented [20–23]. For strict feedback non-linear systems with time-varying state constraints, the TVBLF-based adaptive neural network control strategy was described in [20].

The current advances in dealing with time-varying full state constraints are mostly focused on non-linear systems with strict feedback or pure feedback structures. Few findings are available for cross-strict feedback non-linear systems with time-varying state constraints. As a result, in order to overcome this problem, an effective control plan must be developed [24].

To manage a class of cross-strict feedback non-linear systems with partial time-varying state restrictions, [7] uses time-varying tan-type barrier Lyapunov function (TBLF)-based adaptive control; however [7] ignores matched and mismatched uncertainty and disturbances, despite the fact that matched and mismatched uncertainties and disturbances are common in actual engineering fields, including power systems, electronic systems, and motor systems [25].

In [26] the tracking control problem for a class of partial state constrained cross-strict feedback non-linear system is studied. In spite of the fact that it is compared with the current studies, matched and mismatched uncertainties are regarded in [26], but the key assumption on the uncertain terms (i.e. D_i, $i = 1, \ldots, 2n$) is that they are bounded by known and positive functions of states $x_1, \ldots, x_i$ (i.e. $|D_i(X_{2n}, \tau, t)| \leq \rho_i \delta_i(X_i)$, $i = 1, \ldots, 2n, \rho_i > 0$).

Researchers have concentrated on control strategies for systems influenced by uncertainty and disturbances in the past few years [1, 27–29]. SMC has received much interest in comparison with other control methods because of its conceptual simplicity, ease of implementation, and resilience to external disturbances and model uncertainty [30–33]

SMC is a non-linear control approach that uses a discontinuous feedback control action to push closed-loop trajectories to the switching manifold in a limited time. As a result, SMC is widely employed in a variety of applications, including motion control, process control, and so on [25]. Only the sliding mode controller, on the other hand, may reject matched uncertainty and disturbance [34].

To combine the features of the sliding mode with the backstepping controller, a backstepping sliding mode controller [34–37] has been developed, which is resilient to both matched and unmatched uncertainty. In reality, the adaptive backstepping algorithm may be improved to create an adaptive sliding output tracking controller, which provides more robustness. By adding the sliding surface described in terms of the error coordinates [35], the adjustment is carried out at the last stage of the algorithm.

For cross-strict feedback systems with time-varying partial state constraints in the presence of matched and unmatched uncertainties, a cross backstepping SMC using an integral barrier Lyapunov function (IBLF) is presented in this study. The suggested technique ensures that the closed-loop system is bounded, and that the outputs are forced to follow the reference signals, ensuring that all states remain inside predetermined compact frames. The suggested method's major objective is to improve performance over cross backstepping with conventional TBLF in the presence of matched and unmatched uncertainties and disturbances to considerably reduce steady-state error. Also, in this paper, the proposed controller is able to reject the uncertainties, which fulfills the more general conditions, that is, the upper bound of the uncertainty in ith, $i = 1, \ldots, 2n$, (nth) channel is made up of not only the states $x_1, \ldots, x_i$, but also $x_{i+1}(u)$, which is viewed as the virtual control input. Therefore, our contribution is to extend the controllers with cross backstepping approach, presented in [26], in order to reject a more general class of uncertainties.

A fourth-order cross-strict feedback non-linear system has been controlled using the suggested technique. In the existence of uncertainties and disturbances, simulation findings show a significant enhancement in tracking performance while maintaining stability.

The following is how the paper is structured: Section 2 explains the problem statement. The focus of Section 3 is on the controller design. The closed-loop system is next subjected to a complete stability study in Section 4. The simulation results and conclusion are presented in Sections 5 and 6, respectively.

2 | PROBLEM STATEMENT

The following cross-strict feedback non-linear system is assumed in this paper:

$$\begin{aligned}
\dot{x}_1 &= f_1(X_1) + \varphi_1^T(X_1)\theta + g_1(X_1)\left[x_3 + \rho_1(X_{2n}, \tau, t)\right]\\
&\vdots\\
\dot{x}_i &= f_i(X_i) + \varphi_i^T(X_i)\theta + g_i(X_i)\left[x_{i+2} + \rho_i(X_{2n}, \tau, t)\right]\\
&\vdots\\
\dot{x}_{2n-1}(t) &= f_{2n-1}(X_{2n-1}(t)) + \varphi_{2n-1}^T(X_{2n-1}(t))\theta\\
&\quad + g_{2n-1}(X_{2n-1}(t))\left[u_1 + \rho_{2n-1}(X_{2n}, \tau, t)\right]\\
\dot{x}_{2n}(t) &= f_{2n}(X_{2n}(t)) + \varphi_{2n}^T(X_{2n}(t))\theta\\
&\quad + g_{2n}(X_{2n}(t))\left[u_2 + \rho_{2n}(X_{2n}, \tau, t)\right]\\
y &= x_1
\end{aligned} \tag{1}$$

where $X_i = (x_1, \ldots, x_i), i = 1, \ldots, 2n, y \in R$, are state vectors and the output, respectively and u_1 and u_2 represent control input of the two subsystems, $f_i(X_i), g_i(X_i) : R^n \rightarrow R$ are known and smooth functions such that $f_i(0) = 0$

and $\forall X_i \in R^i \rightarrow g_i(X_i) \neq 0$. $\theta \in R^p$ represent the vectors of unknown parameters and $\varphi_i(x_i) \in R^{p_i}, i = 1, \ldots, n$ are known smooth functions. The unmatched uncertainties are denoted by $\rho_i(X_{2n}, \tau, t), i = 1, \ldots, 2(n-1)$. The remaining $\rho_{2n-1}(X_{2n}, u, \tau, t)$ and $\rho_{2n}(X_{2n}, u, \tau, t)$ are matched uncertainties. τ is considered an unknown time-varying term.

On the other side, the entire states $X_{2n} = (x_1, \ldots, x_n, \ldots, x_{2n})$ are divided into two portions, one constrained and the other is free. x_{2i}, as the even integer sequence states, are free and x_{2i-1}, as the odd integer sequence states, are bounded through $k_{c_{2i-1}}$(t), that is, $|x_{2i-1}(t)| \leq k_{c_{2i-1}}(t) \forall t > 0$, in which $k_{c_{2i-1}}(t)$ are predefined time-varying continuous positive smooth functions that can be differentiated to $2n$th order.

The investigated system (1) may be separated into two stringent feedback subsystems (2) and (3), one of which is a restricted state system, and the other is a free-state system, based on the nature of the given description.

$$\begin{aligned}
\dot{x}_1 &= f_1(X_1) + \varphi_1^T(X_1)\theta \\
&\quad + g_1(X_1)\left[x_3 + \rho_1(X_{2n}, \tau, t)\right] \\
&\ \ \vdots \\
\dot{x}_{2i-1}(t) &= f_{2i-1}(X_{2i-1}(t)) + \varphi_{2i-1}^T(X_{2i-1}(t))\theta \\
&\quad + g_{2i-1}(X_{2i-1}(t))\left[u_1 + \rho_{2i-1}(X_{2n}, \tau, t)\right] \\
&\ \ \vdots \\
\dot{x}_{2n-1}(t) &= f_{2n-1}(X_{2n-1}(t)) + \varphi_{2n-1}^T(X_{2n-1}(t))\theta \\
&\quad + g_{2n-1}(X_{2n-1}(t))\left[u_1 + \rho_{2n-1}(X_{2n}, \tau, t)\right]
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
\dot{x}_2 &= f_2(X_2) + \varphi_2^T(X_2)\theta + g_2(X_2)[x_4] \\
&\quad + \rho_2(X_{2n}, \tau, t) \\
&\ \ \vdots \\
\dot{x}_{2i}(t) &= f_{2i}(X_{2i}(t)) + \varphi_{2i}^T(X_{2i}(t))\theta \\
&\quad + g_{2i}(X_{2i}(t))\left[u_2 + \rho_{2i}(X_{2n}, \tau, t)\right] \\
&\ \ \vdots \\
\dot{x}_{2n}(t) &= f_{2n}(X_{2n}(t)) + \varphi_{2n}^T(X_{2n}(t))\theta \\
&\quad + g_{2n}(X_{2n}(t))\left[u_2 + \rho_{2n}(X_{2n}, \tau, t)\right]
\end{aligned} \tag{3}$$

The control objective is to create adaptive controllers u_1 and u_2 to ensure that the system output $y(t)$ closely matches the intended reference signal $y_r(t)$. Simultaneously, we must confirm that partial state restrictions are not broken and that all closed-loop signals are bounded. In order to do this, we assume the following system assumption (1).

Assumption 1. All of the partial time-varying state constraints $k_{c_{2i-1}}(t)$ and the desired system output trajectory $y_r(t)$ are continuous, limited, and differentiable up to the nth order. A differentiable continuous function $A(t)$ exists while positive constants $d_{2i-1,j}, Y_j$ satisfy $y_r(t) \leq A(t) < k_{c_1}(t)$, $|y_r^{(j)}(t)| \leq Y_j$, $|k_{c_{2i-1}}^{(j)}(t)| \leq d_{2i-1,j} (i, j = 1, \ldots, n)$ *for* $t \geq 0$.

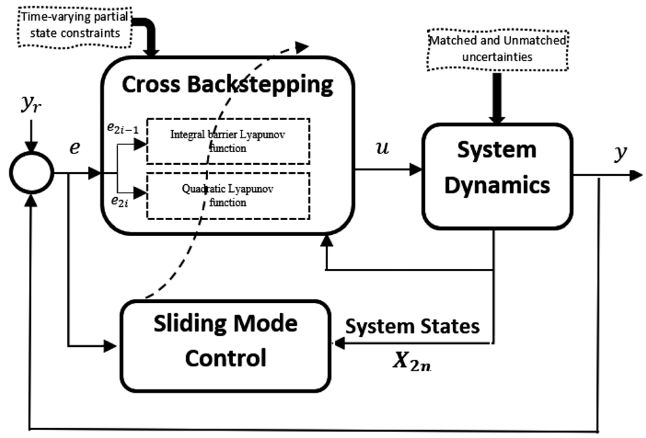

FIGURE 1 Block diagram of the proposed controller

Assumption 2. We assume that the unknown unmatched uncertainty $\rho_i(X_{2n}, \tau, t), i = 1, \ldots, 2(n-1)$, and matched uncertainties $\rho_{2n-1}(X_{2n}, u, \tau, t)$ and $\rho_{2n}(X_{2n}, u, \tau, t)$ satisfy the following inequalities.

$$\begin{aligned}
|\rho_i(X_{2n}, \tau, t)| &\leq M_i(X_i) + \mu_i \left|(x_{i+1})\right| \\
|\rho_{2n-1}(X_{2n}, u, \tau, t)| &\leq M_{2n-1}(X_{2n-1}) + \mu_{2n-1}|u_1| \\
|\rho_{2n}(X_{2n}, u, \tau, t)| &\leq M_{2n}(X_{2n}) + \mu_{2n}|u_2|
\end{aligned} \tag{4}$$

where $\mu_i \in [0, 1), i = 1, \ldots, 2n$ and $M_i(X_i), i = 1, \ldots, 2n$ are the continuous and positive functions.

Remark 1. In Assumption 2, the virtual control input $x_{i+1}$is added to the traditional upper bound $M_i(X_i)$ [1], resulting in a more general upper bound for the uncertainty.

3 | CONTROLLER DESIGN

A block diagram of the proposed controller is portrayed in Figure 1. The specific design process is divided into the following two parts.

3.1 | Control design for constrained subsystem

Because subsystem (2) is a state restricted non-linear system, we utilize an IBLF to guarantee that the state constraints are not broken, and we build the controller u_1 using cross backstepping SMC.

3.1.1 | Cross backstepping control using integral barrier Lyapunov function

The controller is designed in the following steps based on the backstepping design procedure:

Step 1.
The output tracking error formula is

$$e_1 = x_1 - y_r \tag{5}$$

The Lyapunov candidate function is generated using an approximation of the unknown parameter θ.

$$V_1\left(e_1,\tilde{\theta}\right) = \int_0^{e_1} \frac{\delta k_{b_1}^2(t)}{k_{b_1}^2(t)-\delta^2}\,d\delta + \frac{1}{2}\tilde{\theta}^T T^{-1}\tilde{\theta} + \frac{\varepsilon}{n\alpha}e^{-\alpha t} \tag{6}$$

where $k_{b_1}(t) = k_{c_1}(t) - A(t) > 0$ is the error constraint vector and $|e_1| < k_{b_1}(t)$, T is a symmetric and positive definite matrix and $\tilde{\theta} = \theta - \hat{\theta}$.

In the sets $|e_1| < k_{b_1}(t)$, it is known that the candidate Lyapunov function $V_1(e_1,\tilde{\theta})$ is positive definite, continuously differentiable, and radially unbounded; hence, $V_1(e_1,\tilde{\theta})$ is a genuine candidate Lyapunov function.

The time derivative of (6) can be expressed as

$$\begin{aligned}\dot{V}_1\left(e_1,\tilde{\theta}\right) &= \frac{k_{b_1}^2(t)\,e_1\dot{e}_1}{k_{b_1}^2(t)-e_1^2} \\ &+\left(k_{b_1}(t)\log\frac{k_{b_1}^2(t)}{k_{b_1}^2(t)-e_1^2} - \frac{k_{b_1}(t)\,e_1^2}{k_{b_1}^2(t)-e_1^2}\right)\dot{k}_{b_1}(t) \\ &-\tilde{\theta}^T T^{-1}\dot{\hat{\theta}} - \frac{\varepsilon}{n}e^{-\alpha t} \\ &= \frac{k_{b_1}^2(t)\,e_1\left[f_1(x_1)+\Phi_1^T(x_1)\theta + g_1(x_1)\left[x_3+\rho_1(X_{2n},\tau,t)\right]-\dot{y}_r\right]}{k_{b_1}^2-e_1^2} \\ &+\frac{k_{b_1}^2(t)\,e_1\left[\left(\frac{k_{b_1}^2-e_1^2}{k_{b_1}(t)e_1}\log\frac{k_{b_1}^2(t)}{k_{b_1}^2(t)-e_1^2}\right)\dot{k}_{b_1}(t) - \frac{e_1}{k_{b_1}(t)}\dot{k}_{b_1}(t)\right]}{k_{b_1}^2-e_1^2} \\ &-\tilde{\theta}^T T^{-1}\dot{\hat{\theta}} - \frac{\varepsilon}{n}e^{-\alpha t}\end{aligned} \tag{7}$$

To make the system asymptotically stable, $\dot{V}_1(e_1,\tilde{\theta})$ must be semi-definitely negative.

To ensure $\dot{V}_1 \le 0$, we have

$$\begin{aligned}x_{3v}\left(x_1,\hat{\theta}\right) &= \frac{1}{g_1(x_1)}\left[-f_1(x_1) - \Phi_1^T(x_1)\hat{\theta} + \dot{y}_r - \right.\\ &\left(\frac{k_{b_1}^2-e_1^2}{k_{b_1}(t)\,e_1}\log\frac{k_{b_1}^2(t)}{k_{b_1}^2(t)-e_1^2}\right)\dot{k}_{b_1}(t) + \frac{e_1}{k_{b_1}(t)}\dot{k}_{b_1}(t) - c_1e_1 - \\ &\left.\frac{\varepsilon e_1}{nk_{b_1}^2(t)}e^{-\alpha t}\right] - \frac{b_1^2e_1g_1(x_1)}{b_1\left|e_1g_1(x_1)\right| + \frac{\varepsilon}{n}e^{-\alpha t}}\end{aligned} \tag{8}$$

where c_1, ε, and α are positive numbers, becomes

$$\begin{aligned}&\dot{V}_1\left(e_1,\tilde{\theta}\right) \\ &= -\frac{k_{b_1}^2(t)}{(k_{b_1}^2-e_1^2)}c_1e_1^2 - \frac{e_1^2\varepsilon}{(k_{b_1}^2-e_1^2)n}e^{-\alpha t} \\ &-\frac{\varepsilon}{n}e^{-\alpha t} + \tilde{\theta}^T T^{-1}\left(T\varphi_1(x_1)\frac{k_{b_1}^2(t)\,e_1}{(k_{b_1}^2-e_1^2)} - \dot{\hat{\theta}}\right) \\ &-\frac{k_{b_1}^2(t)\,e_1}{(k_{b_1}^2-e_1^2)}g_1(x_1)\frac{b_1^2e_1g_1(x_1)}{b_1\left|e_1g_1(x_1)\right| + \frac{\varepsilon}{n}e^{-\alpha t}} \\ &+\frac{k_{b_1}^2(t)\,e_1}{(k_{b_1}^2-e_1^2)}g_1(x_1)\,\rho_1(X_{2n},\tau,t)\end{aligned} \tag{9}$$

As we know $AB \le |AB| \le |A||B|$ and according to assumption (2) and considering $\left|\frac{b_1^2e_1g_1(x_1)}{b_1|e_1g_1(x_1)|+\frac{\varepsilon}{n}e^{-\alpha t}}\right| \le b_1$, it yields

$$\begin{aligned}&-\frac{k_{b_1}^2(t)\,e_1}{(k_{b_1}^2-e_1^2)}g_1(x_1)\frac{b_1^2e_1g_1(x_1)}{b_1\left|e_{1g_1(x_1)}\right| + \frac{\varepsilon}{n}e^{-\alpha t}} \\ &+\frac{k_{b_1}^2(t)\,e_1}{(k_{b_1}^2-e_1^2)}g_1(x_1)\,\rho_1(X_{2n},\tau,t) \\ &\le\left(\frac{k_{b_1}^2(t)}{(k_{b_1}^2-e_1^2)}\right)\left(\frac{-b_1^2e_1^2g_1^2(x_1)}{b_1\left|e_1g_1(x_1)\right| + \frac{\varepsilon}{n}e^{-\alpha t}}\right. \\ &\left.+\frac{\left(b_1\left|e_1g_1(x_1)\right| + \frac{\varepsilon}{n}e^{-\alpha t}\right)\left|e_1g_1(x_1)\right|\left[M_1 + \mu_1\left|x_{3v}\right| + \mu_1b_1\right]}{b_1\left|e_1g_1(x_1)\right| + \frac{\varepsilon}{n}e^{-\alpha t}}\right)\end{aligned} \tag{10}$$

If b_1 is chosen as

$$b_1 > \frac{M_1(x_1) + \mu_1\left|x_{3v}\right|}{(1-\mu_1)} \tag{11}$$

It is clear that $[M_1(x_1) + \mu_1|x_{3v}| + \mu_1b_1] < b_1$ and because $\frac{\frac{\varepsilon}{n}e^{-\alpha t}b_1|e_1g_1(x_1)|}{b_1|e_1g_1(x_1)|+\frac{\varepsilon}{n}e^{-\alpha t}} < \frac{\varepsilon}{n}e^{-\alpha t}$, thus

$$\begin{aligned}&-\frac{k_{b_1}^2(t)\,e_1}{(k_{b_1}^2-e_1^2)}g_1(x_1)\frac{b_1^2e_1g_1(x_1)}{b_1\left|e_1g_1(x_1)\right| + \frac{\varepsilon}{n}e^{-\alpha t}} \\ &+\frac{k_{b_1}^2(t)\,e_1}{(k_{b_1}^2-e_1^2)}g_1(x_1)\,\rho_1(X_{2n},\tau,t) < \left(\frac{k_{b_1}^2(t)}{(k_{b_1}^2-e_1^2)}\right)\frac{\varepsilon}{n}e^{-\alpha t}\end{aligned} \tag{12}$$

We may simplify (9) by considering the virtual control input (8) and a suitably smooth function b_1 that satisfies (11) as follows:

$$\dot{V}_1\left(e_1,\tilde{\theta}\right) \leq -\frac{k_{b_1}^2(t)}{(k_{b_1}^2-e_1^2)}c_1e_1^2+\tilde{\theta}^T T^{-1} \times\left(T\varphi_1(x_1)\frac{k_{b_1}^2(t)e_1}{(k_{b_1}^2-e_1^2)}-\dot{\hat{\theta}}\right) \tag{13}$$

We select $\dot{\hat{\theta}} = T'_1(x_1)\frac{k_{b_1}^2(t)e_1}{(k_{b_1}^2-e_1^2)}$ to eliminate the last term of (13).

Step *k*.

$$e_{2k-1}=x_{2k-1}\left(X_{2k-1},\hat{\theta},t\right)-x_{(2k-1)v}\left(X_{2k-1},\hat{\theta},t\right) \tag{14}$$

The Lyapunov function is selected as follows ($E_{2k-1}=(e_1,e_2,\dots,e_{2k-1})$):

$$V_{2k-1}\left(E_{2k-1},\tilde{\theta}\right)=V_{2k-3}\left(E_{2k-3},\tilde{\theta}\right)+\int_0^{e_{2k-1}}\frac{\delta k_{b_{2k-1}}^2(t)}{k_{b_{2k-1}}^2(t)-\delta^2}d\delta+\frac{\varepsilon}{n\alpha}e^{-\alpha t} \tag{15}$$

where $k_{b_{2k-1}}(t) = k_{c_{2k-1}}(t)-\alpha_{2k-1}(t)>0$ is the error constraint vector, $\alpha_{2k-1}(t)=\sup\{x_{(2k-1)v}(X_{2k-1},\hat{\theta})\}$ and $|e_{2k-1}|<k_{b_{2k-1}}(t)$.

According to (14) and referring to (2), results

$$\begin{aligned}
&\dot{V}_{2k-1}\left(E_{2k-1},\tilde{\theta}\right)\\
&=\sum_{i=1}^{2k-3}\Bigg[-\frac{k_{b_i}^2(t)}{(k_{b_i}^2-e_i^2)}c_ie_i^2-\frac{e_i^2\varepsilon}{(k_{b_i}^2-e_i^2)n}e^{-\alpha t}\\
&+\frac{k_{b_i}^2(t)e_i}{k_{b_i}^2(t)-e_i^2}g_i(X_i)\rho_i(X_{2n},\tau,t)\\
&-\frac{k_{b_i}^2(t)e_i}{k_{b_i}^2(t)-e_i^2}\frac{b_i^2g_i^2(X_i)e_i}{b_i\left|g_i(X_i)e_i\right|+\frac{\varepsilon}{n}e^{-\alpha t}}\Bigg]\\
&+\tilde{\theta}^TT^{-1}\left[T\sum_{i=1}^{2k-3}\left(\varphi_i(X_i)\frac{k_{b_i}^2(t)e_i}{(k_{b_i}^2-e_i^2)}\right)-\dot{\hat{\theta}}\right]-k\frac{\varepsilon}{n}e^{-\alpha t}\\
&+(k_{b_{2k-1}}(t)\,log\frac{k_{b_{2k-1}}^2(t)}{k_{b_{2k-1}}^2(t)-e_{2k-1}^2}\\
&-\frac{k_{b_{2k-1}}(t)e_{2k-1}^2}{k_{b_{2k-1}}^2(t)-e_{2k-1}^2})\dot{k}_{b_{2k-1}}(t)\\
&+\frac{k_{2k-1}^2(t)e_{2k-1}}{k_{b_{2k-1}}^2(t)-e_{2k-1}^2}\left(f_{2k-1}(X_{2k-1})+\varphi_{2k-1}^T(X_{2k-1})\theta\right.\\
&+g_{2k-1}(X_{2k-1})\left[x_{2k+1}+\rho_{2k-1}(X_{2n},\tau,t)\right]\\
&\left.-\dot{x}_{(2k-1)v}\left(X_{2k-1},\hat{\theta},t\right)\right)
\end{aligned} \tag{16}$$

The virtual control input $x_{(2k+1)v}(X_{(2k+1)},\hat{\theta})$ is designed as

$$\begin{aligned}
&x_{(2k+1)v}\left(X_{(2k+1)},\hat{\theta}\right)\\
&=\frac{1}{g_{2k-1}(X_{2k-1})}\\
&\times\left[\begin{array}{c}
-f_{2k-1}(X_{2k-1})-\varphi_{2k-1}^T(X_{2k-1})\hat{\theta}\\
-\left(\frac{k_{b_{2k-1}}^2-e_{2k-1}^2}{k_{b_{2k-1}}(t)e_{2k-1}}log\frac{k_{b_{2k-1}}^2(t)}{k_{b_{2k-1}}^2(t)-e_{2k-1}^2}\right)\dot{k}_{b_{2k-1}}\\
+\frac{e_{2k-1}}{k_{b_{2k-1}}}\dot{k}_{b_{2k-1}}(t)-c_{2k-1}e_{2k-1}\\
-g_{2k-3}(X_{2k-3})\frac{(k_{b_{2k-1}}^2(t)-e_{2k-1}^2)}{(k_{b_{2k-3}}^2(t)-e_{2k-3}^2)}e_{2k-3}\\
-\frac{e_{2k-1}\varepsilon}{k_{b_{2k-1}}^2(t)n}e^{-\alpha t}+\dot{x}_{(2k-1)v}\left(X_{2k-1},\hat{\theta},t\right)
\end{array}\right]\\
&-\frac{b_{2k-1}^2e_{2k-1}g_{2k-1}(X_{2k-1})}{b_{2k-1}\left|e_{2k-1}g_{2k-1}(X_{2k-1})\right|+\frac{\varepsilon}{n}e^{-\alpha t}}
\end{aligned} \tag{17}$$

where $c_{2k-1}>0$.

By selecting $b_{2k-1}>\frac{M_{2k-1}(X_{2k-1})+\mu_{2k-1}|x_{(2k+1)v}|}{(1-\mu_{2k-1})}$ and by selecting $\dot{\hat{\theta}}=T\sum_{i=1}^{2k-1}(\varphi_i(X_i)\frac{k_{b_i}^2(t)e_i}{(k_{b_i}^2-e_i^2)})$, we have

$$\dot{V}_{2k-1}\left(E_{2k-1},\tilde{\theta}\right)\leq-\sum_{i=1}^{2k-1}-\frac{k_{b_i}^2(t)}{(k_{b_i}^2-e_i^2)}c_ie_i^2 \tag{18}$$

Step *n*.

$$e_{2n-1}=x_{2n-1}\left(X_{2n-1},\hat{\theta},t\right)-x_{(2n-1)v}\left(X_{(2n-1)},\hat{\theta},t\right) \tag{19}$$

Considering the Lyapunov function as follows:

$$\begin{aligned}
&V_{2n-1}\left(E_{2n-1},\tilde{\theta},t\right)\\
&=V_{2n-3}\left(E_{2n-3},\tilde{\theta}\right)+\int_0^{e_{2n-1}}\frac{\delta k_{b_{2n-1}}^2(t)}{k_{b_{2n-1}}^2(t)-\delta^2}d\delta+\frac{\varepsilon}{n\alpha}e^{-\alpha t}
\end{aligned} \tag{20}$$

we have

$$\begin{aligned}
&\dot{V}_{2n-1}\left(E_{2n-1},\tilde{\theta},t\right)\\
&=\sum_{i=1}^{2n-3}\Bigg[-\frac{k_{b_i}^2(t)}{(k_{b_i}^2-e_i^2)}c_ie_i^2-\frac{e_i^2\varepsilon}{(k_{b_i}^2-e_i^2)n}e^{-\alpha t}
\end{aligned}$$

$$+\frac{k_{b_i}^2(t)\,e_i}{k_{b_i}^2(t)-e_i^2}g_i(X_i)\,\rho_i(X_{2n},\tau,t)$$

$$\left.-\frac{k_{b_i}^2(t)\,e_i}{k_{b_i}^2(t)-e_i^2}g_i(X_i)\frac{b_i^2 e_i}{b_i|e_i|+\frac{\varepsilon}{n}e^{-\alpha t}}\right]$$

$$+\tilde{\theta}^T T^{-1}\left[T\sum_{i=1}^{2n-3}\left(\varphi_i(X_i)\frac{k_{b_i}^2(t)\,e_i}{(k_{b_i}^2-e_i^2)}\right)-\dot{\hat{\theta}}\right]-\frac{\varepsilon}{n}e^{-\alpha t}$$

$$+(k_{b_{2n-1}}(t)\,log\frac{k_{b_{2n-1}}^2(t)}{k_{b_{2n-1}}^2(t)-e_{2n-1}^2}$$

$$-\frac{k_{b_{2n-1}}(t)\,e_{2n-1}^2}{k_{b_{2n-1}}^2(t)-e_{2n-1}^2})\dot{k}_{b_{2n-1}}(t)$$

$$+\frac{k_{2n-1}^2(t)\,e_{2n-1}}{k_{b_{2n-1}}^2(t)-e_{2n-1}^2}[f_{2n-1}(X_{2n-1}(t))+\varphi_{2n-1}^T(X_{2n-1}(t))\,\theta$$

$$+g_{2n-1}(X_{2n-1}(t))\,[u_1+\rho_{2n-1}(X_{2n},\tau,t)]$$

$$-\dot{x}_{(2n-1)v}\left(X_{2n-1},\hat{\theta},t\right)] \tag{21}$$

The real control input $u_1 = x_{(2n+1)v}(X_{2n\pm1},\hat{\theta})$ is obtained as follows:

$$u_1=\frac{1}{g_{2n-1}(X_{2n-1})}\times\left[\begin{array}{c}-f_{2n-1}(X_{2n-1}(t))-\varphi_{2n-1}^T(X_{2n-1}(t))\,\hat{\theta}\\ -\left(\frac{k_{b_{2n-1}}^2-e_{2n-1}^2}{k_{b_{2n-1}}(t)\,e_{2n-1}}log\frac{k_{b_{2n-1}}^2(t)}{k_{b_{2n-1}}^2(t)-e_{2n-1}^2}\right)\dot{k}_{b_{2n-1}}\\ +\frac{e_{2n-1}}{k_{b_{2n-1}}}\dot{k}_{b_{2n-1}}(t)-\frac{e_{2n-1}\varepsilon}{k_{b_{2n-1}}^2(t)\,n}e^{-\alpha t}\\ -g_{2n-3}(X_{2n-3}(t))\frac{(k_{b_{2n-1}}^2-e_{2n-1}^2)}{(k_{b_{2n-3}}^2-e_{2n-3}^2)}e_{2n-3}\\ +\dot{x}_{(2n-1)v}\left(X_{2n-1},\hat{\theta},t\right)-c_{2n-1}e_{2n-1}\end{array}\right]-\frac{b_{2n-1}^2 g_{2n-1}(X_{2n-1})\,e_{2n-1}}{b_{2n-1}\left|g_{2n-1}(X_{2n-1})\,e_{2n-1}\right|+\frac{\varepsilon}{n}e^{-\alpha t}} \tag{22}$$

where $c_{2n-1} > 0$.

By selecting $b_{2n-1} > \frac{M_{2n-1}(X_{2n-1})+\mu_{2n-1}|u_1|}{(1-\mu_{2n-1})}$ and $\dot{\hat{\theta}} = T\sum_{i=1}^{2n-1}(\varphi_i(X_i)\frac{k_{b_i}^2(t)e_i}{(k_{b_i}^2-e_i^2)})$, we have

$$\dot{V}_{2n-1}\left(E_{2n-1},\tilde{\theta},t\right)\le\sum_{\substack{i=1\\ i\in odd\ N}}^{2n-1}\left[-\frac{k_{b_i}^2(t)}{(k_{b_i}^2-e_i^2)}c_i e_i^2\right] \tag{23}$$

Remark 2. L'Hopital's rule leads us to

$$\lim_{e_i\to0}\frac{k_{b_i}^2-e_i^2}{k_{b_i}(t)\,e_i}log\frac{k_{b_i}^2(t)}{k_{b_i}^2(t)-e_i^2}=0 \tag{24}$$

Hence, the control laws stabilizing the virtual position have been clearly established.

3.1.2 | Cross backstepping sliding mode control using integral barrier Lyapunov function

As we mentioned, cross backstepping control is somewhat sensitive to parametric uncertainties. So, the mixture of backstepping design and SMC could be a possible optional plot for non-linear systems with uncertainties. The adaptive backstepping method may be improved to provide an adaptive sliding output tracking controller for enhanced robustness. At the end of the procedure, the adjustment is made by adding the following sliding surface described in terms of the error coordinates.

$$s_1=k_1\,e_1+\cdots+k_{2n-3}e_{2n-3}+e_{2n-1} \tag{25}$$

where $k_{2i-1} > 0,\ i = 1,\ldots,n$, are real numbers. Additionally, the Lyapunov function is modified as follows:

$$V_{2n-1}\left(E_{2n-1},\tilde{\theta},t\right)=\sum_{i=1}^{2n-3}\left[\int_0^{e_{2n-3}}\frac{\delta k_{b_{2n-3}}^2(t)}{k_{b_{2n-3}}^2(t)-\delta^2}d\delta\right]+\int_0^{s_1}\frac{\delta k_{b_{2n-1}}^2(t)}{k_{b_{2n-1}}^2(t)-\delta^2}d\delta+\frac{1}{2}\tilde{\theta}^T T^{-1}\tilde{\theta}+\frac{\varepsilon}{n\alpha}\mathrm{e}^{-\alpha t} \tag{26}$$

The time derivative of (26) can be expressed as

$$\dot{V}_{2n-1}\left(E_{2n-1},\tilde{\theta},t\right)$$

$$=\sum_{i=1}^{2n-3}\left[-\frac{k_{b_i}^2(t)}{(k_{b_i}^2-e_i^2)}c_i e_i^2-\frac{k_{b_i}^2(t)}{(k_{b_i}^2-e_i^2)}\frac{\varepsilon}{n}\mathrm{e}^{-at}\right.$$

$$+\frac{k_{b_i}^2(t)\,e_i}{k_{b_i}^2(t)-e_i^2}g_i(X_i)\,\rho_i(X_i,\tau,t)$$

$$\left.-\frac{k_{b_i}^2(t)\,e_i}{k_{b_i}^2(t)-e_i^2}\frac{b_i^2 g_i^2(X_i)\,e_i}{b_i\left|g_i(X_i)\,e_i\right|+\frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}\right]$$

$$+\tilde{\theta}^T T^{-1}\left[T\sum_{i=1}^{2n-3}\left(\varphi_i(X_i)\frac{k_{b_i}^2(t)\,e_i}{(k_{b_i}^2-e_i^2)}\right)-\dot{\hat{\theta}}\right]-\frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}$$

$$+\left(k_{b_{2n-1}}(t)\,log\frac{k_{b_{2n-1}}^2(t)}{k_{b_{2n-1}}^2(t)-s_1^2}-\frac{k_{b_{2n-1}}(t)\,s_1^2}{k_{b_{2n-1}}^2(t)-s_1^2}\right)\dot{k}_{b_{2n-1}}(t)$$

$$+\frac{k_{2n-1}^2(t)\,s_1}{k_{b_{2n-1}}^2(t)-s_1^2}\Bigg[f_{2n-1}(X_{2n-1}(t))+\varphi_{2n-1}^T(X_{2n-1}(t))\,\theta$$
$$+g_{2n-1}(X_{2n-1}(t))\,[u_1+\rho_{2n-1}(X_{2n},\tau,t)]$$
$$-\dot{x}_{(2n-1)v}\left(X_{2n-1},\hat{\theta},t\right)+\sum_{i=1}^{2n-3}[k_i\dot{e}_i]\Bigg] \tag{27}$$

The real control input $u_1 = x_{(2n+1)v}(X_{2n-1},\hat{\theta})$ is designed as follows:

$$u_1=\frac{1}{g_{2n-1}(X_{2n-1})}\times\left[\begin{array}{l}-f_{2n-1}(X_{2n-1}(t))-\varphi_{2n-1}^T(X_{2n-1}(t))\,\hat{\theta}\\ -\left(\frac{k_{b_{2n-1}}^2-s_1^2}{k_{b_{2n-1}}(t)\,s}\log\frac{k_{b_{2n-1}}^2(t)}{k_{b_{2n-1}}^2(t)-s_1^2}\right)\dot{k}_{b_{2n-1}}\\ +\frac{s}{k_{b_{2n-1}}}\dot{k}_{b_{2n-1}}(t)-c_{2n-1}s_1-\frac{s_1\varepsilon}{k_{b_{2n-1}}^2(t)\,n}e^{-\alpha t}\\ -g_{2n-3}(X_{2n-3}(t))\frac{(k_{b_{2n-1}}^2-s_1^2)}{(k_{b_{2n-3}}^2-e_{2n-3}^2)}e_{2n-3}\\ +\dot{x}_{(2n-1)v}\left(X_{2n-1},\hat{\theta},t\right)\\ k_i[g_i(X_i)\,e_{i+2}-g_{i-2}(X_{i-2})\frac{(k_{b_i}^2(t)-e_i^2)}{(k_{b_{i-2}}^2(t)-e_{i-2}^2)}e_{i-2}\\ -c_ie_i-\varphi_i^T(X_i)\,\hat{\theta}-\frac{e_i\varepsilon}{k_{b_i}^2(t)\,n}e^{-\alpha t}\\ -\sum_{i=1}^{2n-3}\left[-\left(\frac{k_{b_i}^2-e_i^2}{k_{b_i}(t)\,e_i}log\frac{k_{b_i}^2(t)}{k_{b_i}^2(t)-e_i^2}\right)\dot{k}_{b_i}\right]\\ +\frac{e_i}{k_{b_i}}\dot{k}_{b_i}(t)-\frac{g_i^2(X_i)\,h_i^2e_i}{h_i\left|g_i(X_i)\,e_i\right|+\frac{\varepsilon}{n}e^{-\alpha t}})]\\ -Ksgn(s_1)\end{array}\right]$$
$$-\frac{h_{2n-1}^2s_1g_{2n-1}(X_{2n-1})}{h_{2n-1}\left|s_1g_{2n-1}(X_{2n-1})\right|+\frac{\varepsilon}{n}e^{-\alpha t}} \tag{28}$$

The $sgn(x)$ is a symbolic function and expressed explicitly as

$$sgn\ (x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases} \tag{29}$$

where $c_{2n-1}>0$.

By selecting $h_{2n-1}>\frac{M_{2n-1}(X_{2n-1})+\mu_{2n-1}|u_1|}{(1-\mu_{2n-1})}$ and $\dot{\hat{\theta}}=T\sum_{i=1}^{2n-3}(\varphi_i(X_i)\frac{k_{b_i}^2(t)e_i}{(k_{b_i}^2-e_i^2)})+(\varphi_{2n-1}(X_{2n-1})\frac{s_1k_{b_{2n-1}}^2(t)}{(k_{b_{2n-1}}^2-s_1^2)})$, we have

$$\dot{V}_{2n-1}\left(E_{2n-1},\tilde{\theta},t\right)\le-\sum_{i=1}^{2n-3}\frac{k_{b_i}^2(t)}{(k_{b_i}^2-e_i^2)}c_ie_i^2$$
$$-\frac{k_{b_{2n-1}}^2(t)}{(k_{b_{2n-1}}^2-e_{2n-1}^2)}c_{2n-1}s_1^2-\frac{k_{b_{2n-1}}^2(t)\,s_1}{k_{b_{2n-1}}^2(t)-s_1^2}Ksgn(s) \tag{30}$$

Remark 3. From (28), there appears to be a possibility of $u_1=x_{(2n+1)v}(X_{2n+1},\hat{\theta})$ becoming unbounded if $e_i=k_{b_i}(t)$ at some t. This problem is addressed in [38], where it is formally demonstrated that, in the closed loop, the error signals $|e_i|$ never reach $k_{b_i}(t)\forall t\ge 0$ given certain initial and feasible conditions. As a result, the control $u_1(t)$ will not become unbounded because of the terms $(k_{b_i}^2(t)-e_i^2)$ in the denominator.

3.2 | Control design for unconstrained subsystem

It is justifiable to employ the conventional quadratic Lyapunov function to examine the stability of subsystem (3) and construct the controller u_2 based on cross backstepping SMC because it is a free state non-linear system.

3.2.1 | Cross backstepping control using quadratic Lyapunov function

The detailed design procedure is listed as follows:

Step 1.

We define the error signal of e_2 as

$$e_2=x_2 \tag{31}$$

And the Lyapunov candidate function is

$$V_2\left(e_2,\tilde{\theta}\right)=\frac{1}{2}e_2^2+\frac{\varepsilon}{na}e^{-at}+\frac{1}{2}\tilde{\theta}^TT^{-1}\tilde{\theta} \tag{32}$$

The time derivative of (32) can be obtained as follows:

$$\dot{V}_2\left(e_2,\tilde{\theta}\right)=e_2\,\dot{e}_2-\tilde{\theta}^TT^{-1}\dot{\hat{\theta}}-\frac{\varepsilon}{n}e^{-at}$$
$$=e_2\left(f_2(X_2)+\varphi_2^T(X_2)\,\theta+g_2(X_2)\,[x_4]\right.$$
$$\left.+\rho_2(X_{2n},\tau,t)\right)$$
$$-\tilde{\theta}^TT^{-1}\dot{\hat{\theta}}-\frac{\varepsilon}{n}e^{-at} \tag{33}$$

Selecting virtual control input as

$$x_{4v}\left(X_4,\hat{\theta}\right)=\frac{-1}{g_2\left(X_2\right)}\left[f_2\left(X_2\right)+\Phi_2^T\left(X_2\right)\hat{\theta}+c_2e_2\right]-\frac{h_2^2g_2\left(X_2\right)e_2}{h_2\left|g_2\left(X_2\right)e_2\right|+\frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}\tag{34}$$

where $c_2>0$, becomes

$$\dot{V}_2\left(e_2,\tilde{\theta}\right)=-c_2e_2^2+\tilde{\theta}^TT^{-1}\left(T\varphi_2\left(X_2\right)e_2-\dot{\hat{\theta}}\right)-\frac{\varepsilon}{n}\mathrm{e}^{-at}-\frac{h_2^2e_2^2g_2^2\left(x_2\right)}{h_2\left|e_2g_2\left(x_2\right)\right|+\frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}+e_2g_2\left(x_2\right)\rho_2\left(X_{2n},\tau,t\right)\tag{35}$$

By selecting $h_2>\frac{M_2(X_2)+\mu_2|x_{4v}|}{(1-\mu_2)}$ and $\dot{\hat{\theta}}=T\varphi_2(X_2)e_2$, Equation (35) is simplified to

$$\dot{V}_2\left(E_2,\tilde{\theta},t\right)\leq-c_2e_2^2\tag{36}$$

Step *k*.

We define the error signal of e_{2k} as

$$e_{2k}=x_{2k}\left(X_{2k},\hat{\theta},t\right)-x_{2kv}\left(X_{2k},\hat{\theta},t\right)\tag{37}$$

And the Lyapunov function is chosen as

$$V_{2k}\left(E_{2k},\tilde{\theta}\right)=V_{2k-2}\left(E_{2k-2},\tilde{\theta}\right)+\frac{1}{2}e_{2k}^2+\frac{\varepsilon}{na}\mathrm{e}^{-at}\tag{38}$$

The time derivative of (38) can be expressed as

$$\begin{aligned}\dot{V}_{2k}\left(E_{2k},\tilde{\theta}\right)=\sum_{i=1}^{2k-2}\Bigg[&-c_ie_i^2+e_i\rho_i\left(X_{2n},\tau,t\right)\\&-e_ig_i\left(X_i\right)\frac{h_i^2e_i}{h_i\left|e_i\right|+\frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}\Bigg]\\&-k\frac{\varepsilon}{n}\mathrm{e}^{-at}+\tilde{\theta}^TT^{-1}\left[T\sum_{i=1}^{2k-2}\left(\varphi_i\left(X_i\right)e_i\right)-\dot{\hat{\theta}}\right]\\&+e_{2k}\Big(f_{2k}\left(X_{2k}\right)+\varphi_{2k}^T\left(X_{2k}\right)\theta+g_{2k}\left(X_k\right)\\&\times\left[x_{2k+2}+\rho_{2k}\left(X_{2n},\tau,t\right)\right]-\dot{x}_{(2k)v}\left(X_{2k},\hat{\theta},t\right)\Big)\end{aligned}\tag{39}$$

The virtual control input $x_{(2k+2)v}(X_{(2k+2)},\hat{\theta})$ is obtained as follows:

$$\begin{aligned}x_{(2k+2)v}\left(X_{2k+2},\hat{\theta}\right)&=\frac{1}{g_{2k}\left(X_k\right)}\begin{bmatrix}-f_{2k}\left(X_{2k}\right)-\varphi_{2k}^T\left(X_{2k}\right)\hat{\theta}-c_{2k}e_{2k}\\-g_{2k-2}\left(X_{2k-2}\right)e_{2k-2}+\dot{x}_{2kv}\left(X_{2k},\hat{\theta},t\right)\end{bmatrix}\\&-\frac{h_{2k}^2e_{2k}g_{2k}\left(X_{2k}\right)}{h_{2k}\left|e_{2k}g_{2k}\left(X_{2k}\right)\right|+\frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}\end{aligned}\tag{40}$$

where $c_{2k}>0$.

Selecting $h_{2k}>\frac{M_{2k}(X_{2k})+\mu_{2k}|x_{(2k+2)v}|}{(1-\mu_{2k})}$ and $\dot{\hat{\theta}}=T\sum_{i=1}^{2k}(\varphi_i(X_i)e_i)$, results

$$\dot{V}_{2k}\left(E_{2k},\tilde{\theta}\right)\leq-\sum_{i=1}^{2k}-c_ie_i^2\tag{41}$$

Step *n*.

Considering the error signal as

$$e_{2n}=x_{2n}\left(X_{2n},\hat{\theta},t\right)-x_{(2n)v}\left(X_{(2n)},\hat{\theta},t\right)\tag{42}$$

And choosing the Lyapunov function as follows:

$$V_{2n}\left(E_{2n},\tilde{\theta}\right)=V_{2n-2}\left(E_{2n-2},\tilde{\theta}\right)+\frac{1}{2}e_{2n}^2+\frac{\varepsilon}{na}\mathrm{e}^{-at}\tag{43}$$

we have

$$\begin{aligned}\dot{V}_{2n}\left(E_{2n},\tilde{\theta}\right)=\sum_{i=1}^{2n-2}\Bigg[&-c_ie_i^2+e_ig_i\left(X_i\right)\rho_i\left(X_{2n},\tau,t\right)\\&-e_ig_i\left(X_i\right)\frac{h_i^2e_ig_i\left(X_i\right)}{h_i\left|e_ig_i\left(X_i\right)\right|+\frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}\Bigg]-\varepsilon\mathrm{e}^{-at}\\&+\tilde{\theta}^TT^{-1}\left[T\sum_{i=1}^{2n-2}\left(\varphi_i\left(X_i\right)e_i\right)-\dot{\hat{\theta}}\right]\\&+e_{2n}\Big(f_{2n}\left(X_{2n}\right)+\varphi_{2n}^T\left(X_{2n}\right)\theta+g_{2n}\left(X_{2n}\right)\\&\times\left[u_2\left(X_{2n},\hat{\theta}\right)+\rho_{2n}\left(X_{2n},\tau,t\right)\right]\\&-\dot{x}_{(2n)v}\left(X_{2n},\hat{\theta},t\right)\Big)\end{aligned}\tag{44}$$

So, the virtual control input $u_2=x_{(2n+2)v}\left(X_{(2n)},\hat{\theta}\right)$ would be

$$u_2\left(X_{2n},\hat{\theta}\right)=\frac{1}{g_{2n}\left(X_{2n}\right)}\times\begin{bmatrix}-f_{2n}\left(X_{2n}\right)-\varphi_{2n}^T\left(X_{2n}\right)\hat{\theta}-c_{2n}e_{2n}\\-g_{2n-2}\left(X_{2n-2}\right)e_{2n-2}+\dot{x}_{2nv}\left(X_{2n},\hat{\theta},t\right)\end{bmatrix}$$

$$-\frac{b_{2n}^2 g_{2n}(X_{2n})\, e_{2n}}{b_{2n}\left|g_{2n}(X_{2n})\, e_{2n}\right| + \frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}} \tag{45}$$

where $c_{2n} > 0$.

By selecting $b_{2n} > \frac{M_{2n}(X_{2n})+\mu_{2n}|u_2(X_{2n},\hat{\theta})|}{(1-\mu_{2n})}$ and $\dot{\hat{\theta}} = T\sum_{i=1}^{2n}(\varphi_i(X_i)e_i)$, we have

$$\dot{V}_{2n}\left(E_{2n},\tilde{\theta}\right) \le -\sum_{i=1}^{2n} -c_i e_i^2 \tag{46}$$

3.2.2 | Cross backstepping sliding mode control using quadratic Lyapunov function

$$s_2 = k_2\, e_2 + \cdots + k_{2n-2}e_{2n-2} + e_{2n} \tag{47}$$

where $k_{2i} > 0,\ i = 1, \ldots, n$, are real numbers.

Similar to the previous section, the Lyapunov function is modified as follows:

$$V_{2n}\left(E_{2n},\tilde{\theta}\right) = \frac{1}{2}\sum_{i=1}^{2n-2} e_i^2 + \frac{1}{2}\tilde{\theta}^T T^{-1}\tilde{\theta} + \frac{1}{2}s_2^2 + \frac{\varepsilon}{na}\mathrm{e}^{-at} \tag{48}$$

we have

$$\begin{aligned}\dot{V}_{2n}\left(E_{2n},\tilde{\theta}\right) = &\sum_{i=1}^{2n-2}\Bigg[-c_i e_i^2 + e_i g_i(X_i)\,\rho_i(X_{2n},\tau,t) \\ &-\frac{b_i^2 e_i^2 g_i^2(X_i)}{b_i\left|e_i g_i(X_i)\right| + \frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}\Bigg] \\ &-\varepsilon\mathrm{e}^{-at} + \tilde{\theta}^T T^{-1}\left[T\sum_{i=1}^{2n-2}(\varphi_i(X_i)\,e_i) - \dot{\hat{\theta}}\right] \\ &+s_2\Big[f_{2n}(X_{2n}(t)) + \varphi_{2n}^T(X_{2n}(t))\,\theta \\ &+g_{2n}(X_{2n}(t))\left[u_2 + \rho_{2n}(X_{2n},\tau,t)\right] \\ &-\dot{x}_{(2n)v}\left(X_{2n},\hat{\theta},t\right) + \sum_{i=1}^{2n-2}[k_i\dot{e}_i]\Big]\end{aligned} \tag{49}$$

We design the real control input $u_2 = x_{(2n+2)v}(X_{2n},\hat{\theta})$ as follows:

$$\begin{aligned}u_2 = \frac{1}{g_{2n}(X_{2n})} \times &\begin{bmatrix}-f_{2n}(X_{2n}(t)) - \varphi_{2n}^T(X_{2n}(t))\,\hat{\theta} - c_{2n}s_2 \\ -g_{2n-2}(X_{2n-2}(t))\, e_{2n-2} + \dot{x}_{(2n)v}\left(X_{2n},\hat{\theta},t\right) \\ -\sum_{i=1}^{2n-2}k_i\begin{bmatrix}g_i(X_i)\, e_{i+2} - c_i e_i - g_{i-2}(X_i)\, e_{i-2} \\ -\varphi_i^T(X_i)\,\hat{\theta} - \frac{g_i^2(X_i)\, b_i^2 e_i}{b_i\left|g_i(X_i)\, e_i\right| + \frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}\end{bmatrix} \\ -Ksgn(s_2)\end{bmatrix} \\ &-\frac{b_{2n}^2 g_{2n}(X_{2n})\, s_2}{b_{2n}\left|g_{2n}(X_{2n})\, s_2\right| + \frac{\varepsilon}{n}\mathrm{e}^{-\alpha t}}\end{aligned} \tag{50}$$

where $c_{2n} > 0$, becomes

By selecting $b_{2n} > \frac{M_{2n}(X_{2n})+\mu_{2n}|x_{(2n+2)v}|}{(1-\mu_{2n})}$ and $\dot{\hat{\theta}} = T\sum_{i=1}^{2n-2}(\varphi_i(X_i)e_i) + \varphi_{2n}(X_{2n})s$, it results

$$\dot{V}_{2n}\left(E_{2n},\tilde{\theta}\right) \le \sum_{i=1}^{2n-2} -c_i e_i^2 - c_i s^2 - s_2 Ksgn(s_2) \tag{51}$$

4 | STABILITY ANALYSIS

Theorem 1 summarizes the findings of the preceding section, and the proof backs it up.

Theorem 1.

The non-linear system (1) is subject to unknown parameter vector θ, with unmatched uncertainties $\rho_i(X_{2n},\tau,t), i = 1, \ldots, 2(n-1)$ and matched uncertainties $\rho_{2n-1}(X_{2n},u,\tau,t)$ and $\rho_{2n}(X_{2n},u,\tau,t)$, without the loss of generality. Assume that these uncertainties satisfy Assumption (2) and that inequality (4) holds. The suggested controllers, (28) and (50), guarantee that

1. The output y ($y = x_1$) globally and asymptotically tracks a reference signal y_r.
2. The other states (x_2, ..., x_n, ...,x_{2n}) and therefore, the control input is bounded.
3. The partial state constraints are not violated.

Proof. From step n, it is concluded that for the Lyapunov function $V(E,\tilde{\theta},t) = V_{2n-1}(E_{2n-1},\tilde{\theta}) + V_{2n}(E_{2n},\tilde{\theta})$

Hence, differentiation under the control inputs u_1 and u_2 along the closed-loop system (1) results in

$$\begin{aligned}\dot{V}\left(E,\tilde{\theta},t\right) \le &-\sum_{i=1}^{2n-3}\frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)}c_i e_i^2 \\ &-\frac{k_{b_{2n-1}}^2(t)}{(k_{b_{2n-1}}^2 - e_{2n-1}^2)}c_{2n-1}s_1^2 - \frac{k_{b_{2n-1}}^2(t)\, s_1}{k_{b_{2n-1}}^2(t) - s_1^2}Ksgn(s_1) \\ &-\sum_{i=1}^{2n-2}c_i e_i^2 - c_i s_2^2 - s_2 Ksgn(s_2) < 0\end{aligned} \tag{52}$$

Next, it is illustrated that how $E\{e_i(i = 1, 2, \ldots, 2n-2)\}$ is regulated as well as its boundedness. As a result, both sides of (52) are integrated in terms of t, yielding

$$V\left(E, \tilde{\theta}, t\right) \le V\left(E(0), \tilde{\theta}(0), 0\right) - \int_0^t \left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right) dt - \int_0^t \left(\frac{k_{b_{2n-1}}^2(t)}{(k_{b_{2n-1}}^2 - e_{2n-1}^2)} c_{2n-1} s_1^2\right) dt - \int_0^t \left(\frac{k_{b_{2n-1}}^2(t)\, s_1}{k_{b_{2n-1}}^2(t) - s_1^2} Ksgn(s_1)\right) dt - \int_0^t \left(\sum_{i=1}^{2n-2} c_i e_i^2\right) dt - \int_0^t \left(c_i s_2^2\right) dt - \int_0^t (s_2 Ksgn(s_2))\, dt \tag{53}$$

Considering $V(E, \tilde{\theta}, t) > 0$, so we can write:

$$\int_0^t \left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right) dt + \int_0^t \left(\frac{k_{b_{2n-1}}^2(t)}{(k_{b_{2n-1}}^2 - e_{2n-1}^2)} c_{2n-1} s_1^2\right) dt + \int_0^t \left(\frac{k_{b_{2n-1}}^2(t)\, s_1}{k_{b_{2n-1}}^2(t) - s_1^2} Ksgn(s_1)\right) dt + \int_0^t \left(\sum_{i=1}^{2n-2} c_i e_i^2\right) dt + \int_0^t \left(c_i s_2^2\right) dt + \int_0^t (s_2 Ksgn(s_2))\, dt \le V\left(E(0), \tilde{\theta}(0), 0\right) \tag{54}$$

when $t \to \infty$, we have

$$\lim_{t\to\infty} \Big(\int_0^t \left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right) dt + \int_0^t \left(\frac{k_{b_{2n-1}}^2(t)}{(k_{b_{2n-1}}^2 - e_{2n-1}^2)} c_{2n-1} s_1^2\right) dt + \int_0^t \left(\frac{k_{b_{2n-1}}^2(t)\, s_1}{k_{b_{2n-1}}^2(t) - s_1^2} Ksgn(s_1)\right) dt + \int_0^t \left(\sum_{i=1}^{2n-2} c_i e_i^2\right) dt + \int_0^t \left(c_i s_2^2\right) dt + \int_0^t (s_2 Ksgn(s_2))\, dt \Big) \le V\left(E(0), \tilde{\theta}(0), 0\right) \tag{55}$$

Having the fact that $V(E(0), \tilde{\theta}(0), 0) < \infty$, thus

$$\lim_{t\to\infty} \Big(\int_0^t \left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right) dt + \int_0^t \left(\frac{k_{b_{2n-1}}^2(t)}{(k_{b_{2n-1}}^2 - e_{2n-1}^2)} c_{2n-1} s_1^2\right) dt + \int_0^t \left(\frac{k_{b_{2n-1}}^2(t)\, s_1}{k_{b_{2n-1}}^2(t) - s_1^2} Ksgn(s_1)\right) dt + \int_0^t \left(\sum_{i=1}^{2n} c_i e_i^2\right) dt + \int_0^t \left(c_i s_2^2\right) dt + \int_0^t (s_2 Ksgn(s_2))\, dt \Big) < \infty \tag{56}$$

So

$$\begin{aligned} &\lim_{t\to\infty} \int_0^t \left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right) dt < \infty \\ &\lim_{t\to\infty} \int_0^t \left(\sum_{i=1}^{2n} c_i e_i^2\right) dt < \infty \end{aligned} \tag{57}$$

Equation (57) shows that $\lim_{t\to\infty} \int_0^t \left(\sum_{i=1}^{2n} c_i e_i^2\right) dt$ and $\lim_{t\to\infty} \int_0^t \left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right) dt$ are upper-bounded functions. Besides, $\frac{d(\sum_{i=1}^{2n-2} c_i e_i^2)}{dt} = 2\sum_{i=1}^{2n-2} c_i e_i \dot{e}_i$ and $\frac{d\left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right)}{dt} = 2\sum_{i=1}^{2n-3} \left[\frac{-k_{b_i}\dot{k}_{b_i} e_i^2 + k_{b_i}^2(t) e_i \dot{e}_i}{(k_{b_i}^2 - e_i^2)^2} c_i e_i^2 + \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i \dot{e}_i\right]$

As a result, the boundedness of all signals and derivatives confirms the boundedness of $\dot{e}_i$; hence, the boundedness of $\frac{d\left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right)}{dt}$ and $\frac{d\left(\sum_{i=1}^{2n-2} c_i e_i^2\right)}{dt}$.

The boundedness of $\frac{d\left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right)}{dt}$ and $\frac{d\left(\sum_{i=1}^{2n-2} c_i e_i^2\right)}{dt}$ effects the uniform continuity of $\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2$ and $\sum_{i=1}^{2n-2} c_i e_i^2$. Since $\lim_{t\to\infty} \int_0^t \left(\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2\right) dt$ and $\lim_{t\to\infty} \int_0^t \left(\sum_{i=1}^{2n-2} c_i e_i^2\right) dt$ are limited and $\sum_{i=1}^{2n-3} \frac{k_{b_i}^2(t)}{(k_{b_i}^2 - e_i^2)} c_i e_i^2$ and $\sum_{i=1}^{2n-2} c_i e_i^2$ are uniformly continuous, based on Barbalat's

Lemma, it is concluded that

$$\begin{aligned}&\lim_{t\to\infty}\sum_{i=1}^{2n-2} c_i e_i^2 = 0\\&\lim_{t\to\infty}\sum_{i=1}^{2n-3}\frac{k_{b_i}^2(t)}{(k_{b_i}^2-e_i^2)}c_i e_i^2 = 0\end{aligned} \tag{58}$$

Since $c_i,\ k_{b_i}(t)(i = 1,\dots,n,\dots,2n) > 0$ (are non-zero), (58) is equivalent to

$$\begin{cases}\lim_{t\to\infty} e_1 = \lim_{t\to\infty}(x_1 - y_r) = 0\\ \lim_{t\to\infty} e_2 = \lim_{t\to\infty} x_2 = 0\\ \lim_{t\to\infty} e_k = \lim_{t\to\infty}\left(x_k - x_{kv}\left(X_{k-1},\hat{\theta},t\right)\right) = 0, k = 3,,2n\end{cases} \tag{59}$$

From Equation (59), it is implied that $y = x_1$ asymptotically tracks y_r ($\lim_{t\to\infty} x_1 = y_r$), proving the results of section one of theorem 1. In addition, from $\lim_{t\to\infty} x_1 = y_r$, we can conclude the boundedness of x_1. Furthermore, based on (59), x_2 is bounded and $\lim_{t\to\infty} x_k = \lim_{t\to\infty} x_{kv}(X_{k-1},\hat{\theta},t), k = 3,\dots,2n$. So, by proving the boundedness of x_{kv}, it is confirmed that x_k is bounded.

Starting from $k = 3$, we have

$$\lim_{t\to\infty} x_3 = \lim_{t\to\infty} x_{3v}\left(X_2,\hat{\theta},t\right) \tag{60}$$

It is needed to prove that $\hat{\theta}$ is bounded based on the definition of x_k and considering that x_1 and x_2 are bounded.

Referring to (53), we have

$$V\left(E,\tilde{\theta},t\right) \prec V\left(E(0),\tilde{\theta}(0),0\right) \tag{61}$$

According to Equation (61), the globally boundedness of $V(E,\tilde{\theta},t)$ is bounded and, therefore, $E\{e_i(i = 1,2,\dots,2n)\}$ and $\tilde{\theta}$ are verified. Also, the boundedness of $\tilde{\theta}$ proves boundedness of $\hat{\theta}$. Consequently, all signals are bounded. Therefore, x_{3v} and, thus, x_3 are bounded. We can extend the same reasoning for $i = 4,5,\dots,2n$.

As it has been verified in step k that $x_{k-1}$and $\hat{\theta}$ are bounded, so the boundedness of x_k could be confirmed by the boundedness of x_{kv}. Finally, the boundedness of the control inputs is confirmed by the fact that all signals are bounded proving the part two of the theorem.

Since $x_1 = e_1 + y_r$, then $|x_1| < k_{c_1}$ holds as long as $k_{b_1} = k_{c_1} - A$. As proved above, $x_{3v}(x_1,\hat{\theta})$ is bounded. That is, $|x_{3v}(X_3,\hat{\theta})| \le \alpha_3$. Then $|x_3(X_3,\hat{\theta})| \le |x_{3v}(X_3,\hat{\theta})| + |e_3| \le \alpha_3 + k_{b_3}$. This implies that $|x_3(X_3,\hat{\theta})| < k_{c_3}$ as long as $k_{b_3} = k_{c_3} - \alpha_3$. As proved above, $x_{(2k-1)v}(X_{(2k-1)},\hat{\theta})$ are all. So, it can be progressively proven that $|x_{(2k-1)}(X_{(2k-1)},\hat{\theta})| < k_{c_{(2k-1)}}$ for $k = 3,\dots,n$, as long as $k_{b_{2k-1}} = k_{c_{2k-1}} - \alpha_{2k-1}$. As a result, it is concluded that the partial state constraints are always met during operation.

5 | NUMERICAL EXAMPLE

Simulation studies on hyperchaotic systems (1) are carried out in this part to validate the efficiency of the suggested control mechanism.

Consider the following strict-feedback form system:

$$\begin{cases}\dot{x}_1 = x_1^2 + x_1^2\sin(x_1)\theta_1 + 2\left[x_3 + \rho_1(X_4,\tau,t)\right]\\ \dot{x}_2 = x_1 + 2x_2^2 + \left[x_1x_2, x_2\cos(x_1)\right]\theta_2\\ \quad + \left[x_4 + \rho_2(X_4,\tau,t)\right]\\ \dot{x}_3 = x_3^2 + 2x_2^2 + x_3^2\cos(x_3)\theta_3 + (1+e^{x_3})[u_1 + \rho_3(X_4,\tau,t)]\\ \dot{x}_4 = x_1 + x_3 + 2x_4^2 + \left[x_4, x_1\cos(x_3)\right]\theta_4\\ \quad + \left(5 + x_2e^{2x_4}\right)u_2 + \left(5 + x_2e^{2x_4}\right)\rho_4(X_4,\tau,t)\\ y = x_1\end{cases} \tag{62}$$

With the unmatched uncertainties $\rho_1(X_2,\tau,t) = x_1^2\sin(0.01t)$ (satisfying $|\rho_1(X_4,\tau,t)| \le x_1^2$), $\rho_2(X_2,\tau,t) = 3x_2\sin(10\pi t)$ (satisfying $\rho_2(X_4,\tau,t) \le 3|x_2|$) and matched uncertainties $\rho_3(X_4,\tau,t) = 2x_3^2\sin(0.03t)$ (satisfying $|\rho_3(X_4,\tau,t)| \le 2x_3^2$), $\rho_4(X_4,\tau,t) = 2x_4\sin(10\pi t)$ (satisfying $\rho_4(X_4,\tau,t) \le 2|x_4|$).

Assuming that x_1 and x_3 are restricted by $|x_1| < k_{c_1}$, $|x_3| < k_{c_3}$ with $k_{c_1} = 6 + \sin(2\pi t)$ and $k_{c_3} = 7 + \sin(2\pi t)$. The other states x_2 and x_4 are free. The initial states of the system are selected as $x_1(0) = 0.35$, $x_2(0) = 0.75$, $x_3(0) = 0.55$, $x_4(0) = 1.25$.

Also, the value of θ is assumed to be $\theta = [1\ 1.5\ 1\ 3]^T$. The control objective enforces the output $y(t)$ to closely track the desired reference $y_r = \sin(\pi t)(|y_r| \le 1)$. Simultaneously, the constraints in the predefined region on the partial states x_1 and x_3 are enforced.

According to the design procedure described in Section 3, we have

$$f_1(x_1) = x_1^2 \quad f_2(X_2) = x_1 + 2x_2^2$$

$$f_3(X_3) = x_3^2 + 2x_2^2 \quad f_4(X_4) = x_1 + x_3 + 2x_4^2$$

$$g_1(x_1) = 2 \quad g_2(X_2) = 1$$

$$g_3(X_3) = 1 + e^{x_3} \quad g_4(X_4) = 5 + x_2e^{2x_4}$$

$$\varphi_1(x_1) = x_1^2\sin(x_1),\ \varphi_2(X_2) = \left[x_1x_2, x_2\cos(x_1)\right]$$

$$\varphi_3(X_3) = x_3^2\cos(x_3),\ \varphi_4(X_4) = \left[x_4, x_1\cos(x_3)\right]$$

$$A = 1,\ k_{b_1} = 5 + \sin(2\pi t)$$

$$h_1 > \frac{M_1(x_1) + \mu_1|x_2|}{(1-\mu_1)} = x_1^2 \Rightarrow h_1 = 1.25x_1^2$$

$$h_2 > \frac{M_2(X_2) + \mu_2|x_3|}{(1-\mu_2)} = 3|x_2| \Rightarrow h_2 = 4|x_2|$$

$$h_3 > \frac{M_3(X_3) + \mu_1 |u_1|}{(1-\mu_1)} = 2x_3^2 \Rightarrow h_3 = 2.25x_3^2$$

$$h_4 > \frac{M_1(X_4) + \mu_1 |u_2|}{(1-\mu_1)} = 2|x_4| \Rightarrow h_4 = 3|x_4|$$

Using MATLAB, we can get the maximum value of $x_{(3)v}(X_2, \hat{\theta})$, that is $\alpha_3(t) = 6.012$. So $k_{b_3}(t) = k_{c_3}(t) - \alpha_3(t) = 0.988 + \sin(2\pi t)$.

Then, the stabilizing controller becomes:

$$u_1 = \frac{1}{1+e^{x_3}} \left[\begin{array}{c} -f_3(X_3(t)) - \varphi_3^T(X_3(t))\hat{\theta} \\ -\left(\frac{k_{b_3}^2 - s_1^2}{k_{b_3}(t)\, s_1} log \frac{k_{b_3}^2(t)}{k_{b_3}^2(t) - s_1^2} \right) \dot{k}_{b_3}(t) \\ + \frac{s_1}{k_{b_3}} \dot{k}_{b_3}(t) - c_3 s_1 - \frac{s_1 \varepsilon}{k_{b_3}^2(t)\, n} e^{-\alpha t} \\ -g_1(X_1(t)) \frac{(k_{b_3}^2 - s_1^2)}{(k_{b_1}^2 - e_1^2)} e_1 \\ + \dot{x}_{3v}\left(X_{2n-1}, \hat{\theta}, t\right) \\ -k_1 [2e_3 - c_1 e_1 - \varphi_1^T(X_1)\hat{\theta} - \frac{e_1 \varepsilon}{k_{b_1}^2(t)\, n} e^{-\alpha t} \\ -\left(\frac{k_{b_1}^2 - e_1^2}{k_{b_1}(t)\, e_1} log \frac{k_{b_1}^2(t)}{k_{b_1}^2(t) - e_1^2} \right) \dot{k}_{b_1}(t) \\ + \frac{e_1}{k_{b_1}} \dot{k}_{b_1}(t) - \frac{g_1^2(X_1)\, h_1^2 e_1}{h_1 |g_1(X_1)\, e_1| + \frac{\varepsilon}{n} e^{-\alpha t}})] \\ -Ksgn(s_1) \end{array} \right] - \frac{h_3^2 s_1 g_3(X_3)}{h_3 |s_1 g_3(X_3)| + \frac{\varepsilon}{n} e^{-\alpha t}} \tag{63}$$

$$u_2 = \frac{1}{5 + x_2 e^{2x_4}} \times \left[\begin{array}{c} -f_4(X_4(t)) - \varphi_4^T(X_4(t))\hat{\theta} - c_4 s_2 \\ -e_2 + \dot{x}_{(4)v}\left(X_{2n}, \hat{\theta}, t\right) \\ -k_2 \left[e_4 - c_2 e_2 - \varphi_2^T(X_i)\hat{\theta} - \frac{h_2^2 g_2^2 e_2}{h_2 |g_2 e_2| + \frac{\varepsilon}{n} e^{-\alpha t}} \right] \\ -Ksgn(s_2) \end{array} \right] - \frac{h_4^2 g_4(X_4)\, s_2}{h_4 |g_4(X_4)\, s_2| + \frac{\varepsilon}{n} e^{-\alpha t}} \tag{64}$$

In the simulation, the parameters are selected as $\{c_1 = 10,\ c_2 = 10,\ c_3 = 20,\ c_4 = 20,\ k_1 = 10,\ k_2 = 10,\ \alpha = 0.001,\ \varepsilon = 10,$

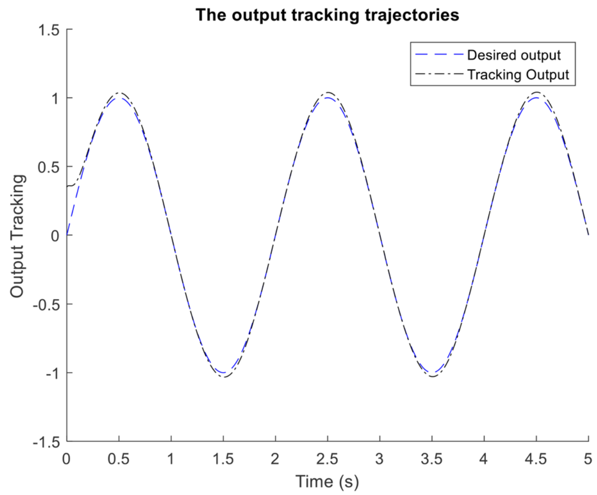

FIGURE 2 The trajectories of the output tracking using the proposed method

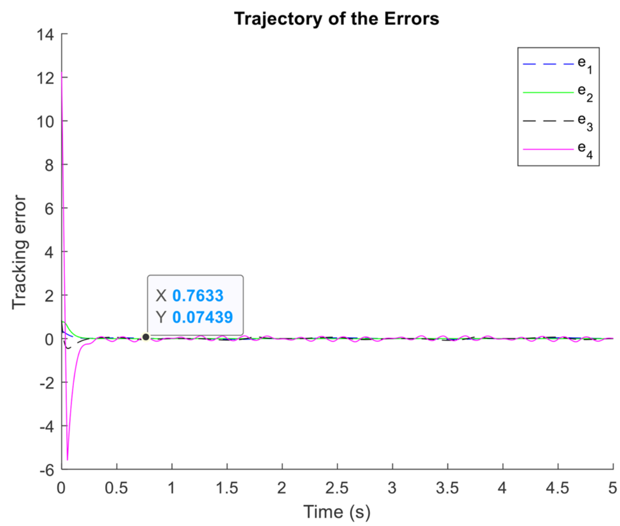

FIGURE 3 The trajectories of the errors using the proposed method

$K = 0.1\}$ and T is a 4×4 identity matrix, to achieve a relatively acceptable tracking behaviour for the designed control system.

Figures 2–6 show the simulation results. Figure 2 shows that the system output y effectively tracks the desired trajectory $y_r(t)$ and Figure 3 shows the state errors validating a good tracking performance. Figures 4 and 5 show the trajectories of the states x_1 and x_3 and their constraints and the trajectories of the free states, respectively. These figures show that the partial time-varying state constraints are kept in their limits. Following Figure 6, the boundedness of control input signals u_1 and u_2 can be seen.

Eventually, in order to demonstrate the effectiveness of the recommended method, a comparative simulation study between the schemes in [7] and this paper is carried out under the same control objective. As previously mentioned in [7], cross backstepping using conventional TBLF is employed to manage a

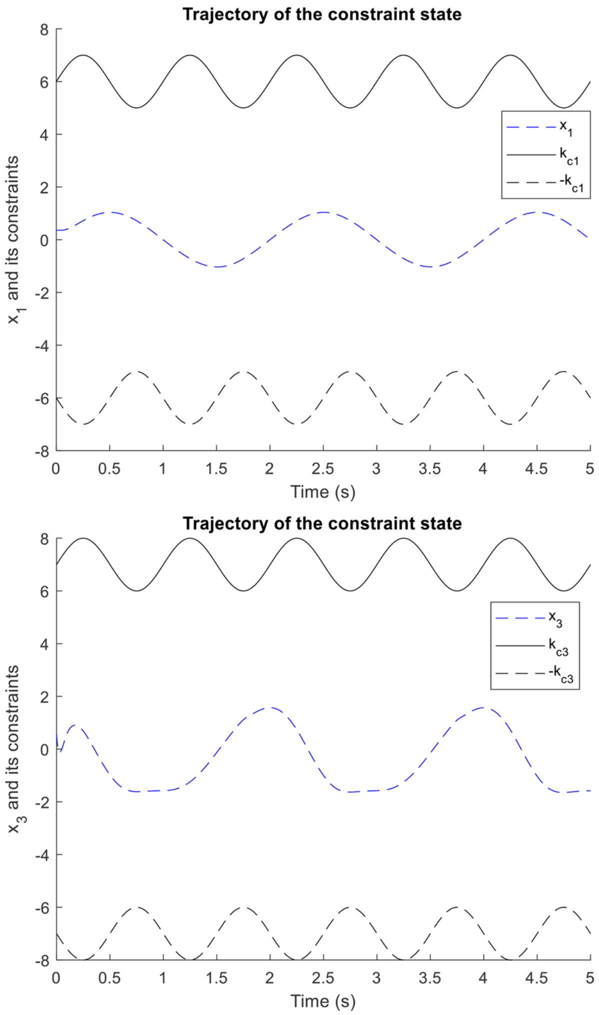

FIGURE 4 The trajectories of the constrained states using the proposed method

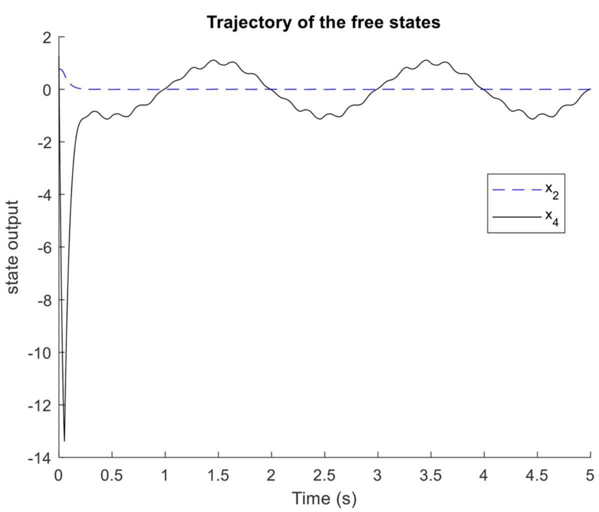

FIGURE 5 The trajectories of the free states using the proposed method

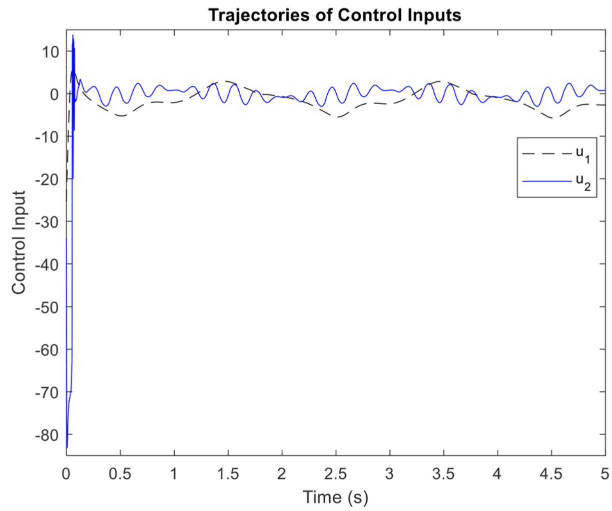

FIGURE 6 The trajectories of the control inputs using the proposed method

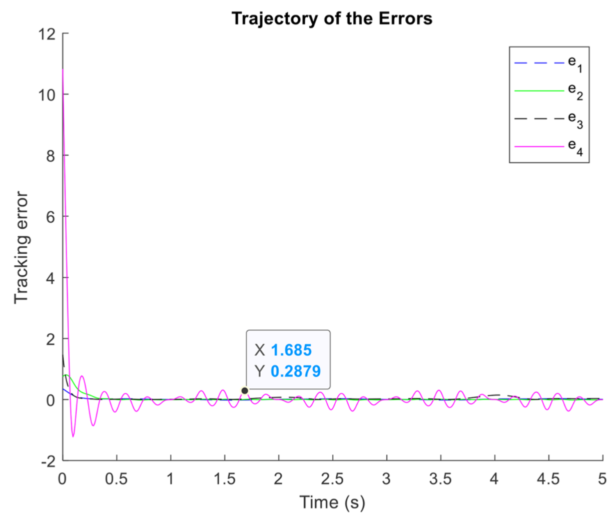

FIGURE 7 The trajectories of the control inputs using cross backstepping with TBLF. TBLF, tan-type barrier Lyapunov function.

class of cross-strict feedback non-linear systems with partial time-varying state constraints.

In order to judge fairly, the control parameters are adopted precisely the same as the proposed method. Figures 7 and 8 show the trajectories of the errors and control inputs using cross backstepping with conventional TBLF, respectively. The simulation results indicate that tracking performance improves significantly in the presence of uncertainties and disturbances while maintaining stability.

Comparing Figures 3 and 7, the superiority of the proposed approach is confirmed since it considerably reduces maximum error (more than 75% reduction) and DC and AC steady-state error. As can be observed, the suggested approach produces

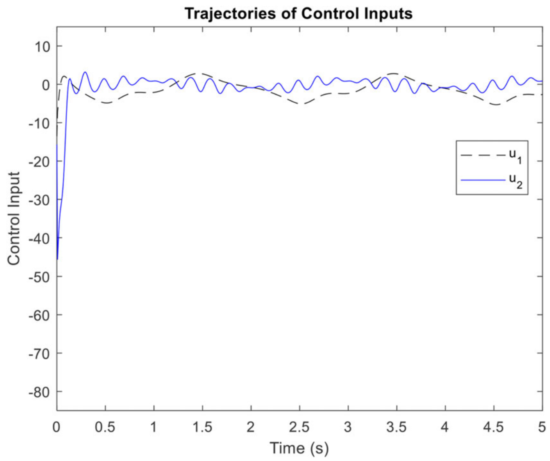

FIGURE 8 The trajectories of the control inputs using cross backstepping with TBLF

superior tracking performance than cross backstepping with traditional TBLF.

6 | CONCLUSION

For a class of cross-strict feedback non-linear systems with unknown parameters, matched and unmatched uncertainty, an adaptive partial time-varying state constrained control architecture is suggested in this study. One constrained and one unconstrained subsystem are created from cross-strict non-linear systems. To ensure the boundedness of the fictitious or actual state tracking defects, the IBLF is used in each stage of the backstepping design for the constrained subsystems. In order to address uncertainty, SMC is combined with backstepping. The suggested control system's stability is also examined using the Lyapunov technique. Implementing the recommended controller on a four-order cross-strict non-linear system validates its efficacy. The proposed control strategy is shown to guarantee that the time-varying partial state requirements are not violated and that the closed loop signals stay bounded, with promising output tracking performance. Finally, the results are compared to traditional TBLF cross backstepping. The results show that the proposed controller has a superior overall performance. More advancement of our control scheme for fractional-order cross-strict feedback non-linear systems with time-varying partial state constraints is suggested for future studies.

CONFLICT OF INTEREST

I, Naser Pariz, declare that there is no conflict of interest in the publication of this article, and that there is no conflict of interest with any other author or institution for the publication of this article.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

REFERENCES

1. Jalalabadi, E., Paylakhi, S.Z., Rahimi-kian, A., Moshiri, B.: Integral backstepping Lyapunov redesign control of uncertain nonlinear systems. IET Control Theory Appl. (2021)
2. Coban, R.: Backstepping integral sliding mode control of an electromechanical system. Automatika 58(3), 266–272 (2017)
3. Li, J., Zhang, Q., Yan, X., Spurgeon, S.K.: Integral sliding mode control for Markovian jump T–S fuzzy descriptor systems based on the super-twisting algorithm. IET Control Theory Appl. 11(8), 1134–1143 (2017)
4. Gomez, M.A., Christopher, C.D., Fridman, L.: Lyapunov redesign for a class of uncertain systems with delays. Nonlinear Anal. Hybrid Syst. 43, 101096 (2021)
5. Luo, X., Wu, X., Guan, X.: Brief paper: adaptive backstepping fault-tolerant control for unmatched nonlinear systems against actuator dead-zone. IET Control Theory Appl. 4(5), 879–888 (2010)
6. Paylakhi, S.Z., Pariz, N., Bayaz, M.H.J.D., Naghibi Sistani, M.B.: Distributed secondary control of islanded microgrids using modified integral backstepping technique. Int. J. Dynam. Control (2022)
7. Wang, C.X., Qi, L., Liu, J.Y., Yu, J.L.: Barrier Lyapunov function based adaptive cross backstepping control for nonlinear systems with time-varying partial state constraints. Int. J. Control Autom. Syst. 18(7), 1771–1781 (2020)
8. Gong, Q., Tian, Y.: A constructive design method for nonlinear cross strict feedback systems. Acta Autom. Sin. 26(4), 447–453 (2000)
9. Wang, J., Gao, J.F., Ma, X.K.: Synchronization control of cross-strict feedback hyperchaotic system based on cross active backstepping design. Phys. Lett. A 369(5–6), 452–457 (2007)
10. Li, H.Y., Hu, Y.A.: Robust sliding-mode backstepping design for synchronization control of cross-strict feedback hyperchaotic systems with unmatched uncertainties. Commun. Nonlinear Sci. Numer. Simul. 16(10), 3904–3913 (2011)
11. Hu, Y.A., Li, H.Y., Huang, H.: Prescribed performance-based backstepping. Nonlinear Dyn. 76(1), 103–113 (2014)/bib>
12. Wang, J., Gao, J., Ma, X.: Synchronization control of cross-strict feedback hyperchaotic system based on cross active backstepping design. Phys. Lett. A 369(5–6), 452–457 (2007)
13. Liu, Y., Zhu, Q., Zhao, N., Wang, L.: Adaptive fuzzy backstepping control for nonstrict feedback nonlinear systems with time-varying state constraints and backlash-like hysteresis. Inf. Sci. 574, 606–624 (2021)
14. Tee, K.P., Ge, S.S., Tay, E.H.: Barrier Lyapunov functions for the control of output-constrained nonlinear systems. Automatica 45, 918–927 (2009)
15. Liu, Y.J., Tong, S.C.: Barrier Lyapunov functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints. Automatica 64, 70–75 (2016)
16. Li, Y.M., Liu, Y.J., Tong, S.C.: Observer-based neuro-adaptive optimized control of strict-feedback nonlinear systems with state constraints. IEEE Trans. Neural Netw. Learn. Syst. 1–15 (2021) https://doi.org/10.1109/TNNLS
17. Liu, L., Liu, Y.J., Chen, A.Q., Tong, S.C., Chen, C.L.P.: Integral barrier lyapunov function-based adaptive control for switched nonlinear systems. Sci. China Inf. Sci. 63(3), 132 203:1–132 203:14 (2020)
18. Liu, L., Chen, A.Q., Liu, Y.J.: Adaptive fuzzy output-feedback control for switched uncertain nonlinear systems with full-state constraints. IEEE Trans. Cybern. 1–12 (2021) https://doi.org/10.1109/TCYB.3050510
19. Tang, F., Niu, B., Wang, H., Zhang, L., Zhao, X.: Adaptive fuzzy tracking control of switched MIMO nonlinear systems with full state constraints and unknown control directions. IEEE Trans. Circuits Syst. II: Express Briefs. 69(6), 2912–2916 (2022)
20. Liu, Y.J., Ma, L., Liu, L., Tong, S.C., Chen, C.L.P.: Adaptive neural network learning controller design for a class of nonlinear systems with time-varying state constraints. IEEE Trans. Neural Netw. Learn. Syst. 31(1), 66–75 (2019)
21. Du, P.H., Sun, K., Zhao, S.Y., Liang, H.J.: Observer-based adaptive fuzzy control for time-varying state constrained strict-feedback nonlinear systems with dead-zone. Int. J. Fuzzy Syst. 21(3), 733–744 (2019)
22. Li, D.J., Lu, S.M., Liu, L.: Adaptive NN cross backstepping control for nonlinear systems with partial time-varying state constraints and its

applications to hyper-chaotic systems. IEEE Trans. Syst. Man Cybern. Syst. 51(5), 2821–2832 (2021)
23. Gao, T.T., Liu, Y.J., Liu, L., Li, D.P.: Adaptive neural network-based control for a class of nonlinear pure-feedback systems with time-varying full state constraints. IEEE/CAA J. Autom. Sin. 5(5), 923–933 (2018)
24. Liu, Y., Zhu, Q., Zhao, N., Wang, L.: Adaptive fuzzy backstepping control for nonstrict feedback nonlinear systems with time-varying state constraints and backlash-like hysteresis. Inf. Sci. 574, 606–624 (2021)
25. Hou, L., Wang, L., Wang, H.: SMC for systems with matched and mismatched uncertainties and disturbances based on NDOB. Zidonghua Xuebao/Acta Autom. Sin. 43(7), 1257–1264 (2017)
26. Wang, C., Qi, L., Yu, X., Yu, J.: Adaptive cross backstepping control for a class of non-strict feedback nonlinear systems with partial state constraints. Int. J. Adapt. Control Signal Process. 35(6), 915–940 (2021)
27. Sinha, R., Harrison, J., Richards, S.M., Pavone, M.: Adaptive Robust Model Predictive Control with Matched and Unmatched Uncertainty. arXiv preprint arXiv:2104.08261, (2021)
28. Thanh, H.L.N.N., Mung, N.X., Nguyen, N.P., Phuong, N.T.: Perturbation observer-based robust control using a multiple sliding surfaces for nonlinear systems with influences of matched and unmatched uncertainties. Mathematics 8(8), 1371 (2020)
29. Ordaz, P., Ordaz, M., Cuvas, C., Santos, O.: Reduction of matched and unmatched uncertainties for a class of nonlinear perturbed systems via robust control. Int. J. Robust Nonlinear Control 29(8), 2510–2524 (2019)
30. Yang, J., Li, S.H., Su, J.Y., Yu, X.H.: Continuous nonsingular terminal sliding mode control for systems with mismatched disturbances. Automatica 49(4), 2287 (2013)
31. Li, X.B., Ma, L., Ding, S.H.: A new second-order sliding mode control and its application to inverted pendulum. Acta Automat. Sin. 41(1), 193–202 (2015)
32. Tamhane, B., Mujumdar, A., Kurode, S.: Mismatched disturbance estimation and compensation for seeker system using higher order sliding mode control. In: Proceedings of the 13th International Workshop Variable Structure Systems, Nantes, France, pp. 1–6 (2014).
33. Zhang, H., Xu, N., Zong, G., Alkhateeb, A.F.: Adaptive fuzzy hierarchical sliding mode control of uncertain under-actuated switched nonlinear systems with actuator faults. Int. J. Syst. Sci. 52(8) 1499–1514 (2021)
34. Adhikary, N., Mahanta, C.: Integral backstepping sliding mode control for underactuated systems: Swing-up and stabilization of the Cart–Pendulum System. ISA Trans. 52(6), 870–880 (2013)
35. Koshkouei, A.J., Zinober, A.S., Burnham, K.J.: Adaptive sliding mode backstepping control of nonlinear systems with unmatched uncertainty. Asian J. Control 6(4), 447–453 (2004)
36. Lu, C.-H., Hwang, Y.-R., Shen, Y.-T.: Backstepping sliding mode tracking control of a vane-type air motor table motion system. ISA Trans. 50(2), 278–286 (2011)
37. Bo, D., Bo, Z., Yuanchun, L.: Backstepping sliding mode decentralized control for reconfigurable manipulators based on ESO. In: 2011 30th Chinese Control Conference (CCC), pp. 3985–3989 (2011)
38. Tee, K.P., Ge, S.S.: Control of nonlinear systems with full state constraint using a barrier Lyapunov function. In: Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference, pp. 8618–8623. IEEE (2009)

How to cite this article: Paylakhi, S.Z., Pariz, N., Naghibi Sistani, M.B., Javidi Dasht Bayaz, M.H.: Cross backstepping sliding mode control using integral barrier Lyapunov function for cross-strict feedback systems. IET Control Theory Appl. 1–15 (2022). https://doi.org/10.1049/cth2.12327