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ORIGINAL PAPER

$\phi(L)$-Factorable Operators on $L^p(G)$ for a Locally Compact Abelian Group

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Abstract

Let G be a locally compact abelian group, ϕ be a topological isomorphism on G, and L be a uniform lattice in G. We provide a development of the $L^1(G/\phi(L))$ function-valued product on $L^p(G)$ called $(\phi(L), p)$-bracket product, where $1 < p < \infty$. Among other things, we study $\phi(L)$-factorable operators and we prove Riesz representation type Theorem for $L^p(G)$.

Keywords $(\phi(L), p)$-bracket product · Locally compact abelian group · $\phi(L)$-orthogonality · $\phi(L)$-factorable operator · Riesz representation theorem

Mathematics Subject Classification 43A15 · 43A70

1 Introduction

In this paper, we aim to study the $(\phi(L), p)$-bracket product on a locally compact abelian group (LCA group, for short) G, via a topological isomorphism ϕ on G with respect to a uniform lattice L in G. The bracket product on space $L^2(\mathbb{R}^n)$ has been studied by several authors, see for example [3] and the references therein. Ron and

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Shen in [10] extended bracket products for the shift invariant subspaces of $L^2(\mathbb{R}^n)$. They defined the bracket product of $f, g \in L^2(\mathbb{R}^n)$ by:

$$[f, g](x) = \sum_{\alpha \in 2\pi\mathbb{Z}^n} f(x+\alpha)\overline{g(x+\alpha)}.$$

Then, $[f, g]$ is an element of $L^1(\mathbb{T}^n)$ and we have $\|[f, f]\|_{L^1(\mathbb{T}^n)} = \|f\|^2_{L^2(\mathbb{R}^n)}$, for $f \in L^2(\mathbb{R}^n)$. Cassaza and Lammers [1] improved the bracket product by employing a shift parameter. More precisely, they defined the so-called bracket product as a-inner product by:

$$\langle f, g\rangle_a (t) = \sum_{n\in\mathbb{Z}} f(t-na)\overline{g(t-na)}; \qquad f, g \in L^2(\mathbb{R}), a \in \mathbb{R}^+.$$

They have shown that the relevant a-inner product has a Bessel's inequality, orthogonal sequence and Riesz Representation Theorem for $L^2(\mathbb{R}^n)$. Kamyabi Gol and Raisi Tousi [7] extended this notion to a LCA-group with respect to a uniform lattice via a topological isomorphism ϕ on G. They defined the bracket product $[f, g]_\phi$, associated with ϕ through:

$$[f, g]_\phi(\dot{x}) = \sum_{k\in L} f(x\phi(k^{-1}))\overline{g(x\phi(k^{-1}))}, \quad f, g \in L^2(G),$$

where L is a uniform lattice in G. It is easy to check that $[f, g]_\phi$ in $L^1(G/\phi(L))$. They also defined the norm $\|.\|_\phi$ called ϕ-norm on $L^2(G)$ by $\|f\|_\phi = [f, f]_\phi^{1/2}$, $(f \in L^2(G))$. They studied the modulation and translation $[., .]_\phi$ and the usual inner product of $L^2(G)$. Some of the basic properties of $[., .]_\phi$ (such as, the Cauchy–Schwarz identity, the polarization identity, etc.) are also discussed in [7]. The main aim of this paper is to extend the bracket product notion to $L^p(G)$, for $1 < p < \infty$. In Sect. 2, we first investigate the elementary properties of $[., .]_{\phi,p}$. In particular, we prove Hölder inequality and Triangle inequality. We study the modulation and translation for this bracket product operators which provide some facilities to accurately study this bracket product. Section 3 is devoted to the $\phi(L)$-factorable operators and its consequences. We use this notion to provide the Riesz Representation Theorem for the pair $(L^p(G), L^1(G/\phi(L)))$.

2 Preliminary Results

Let G be a locally compact abelian group equipped with the Haar measure dx, and let $\phi : G \to G$ be a topological isomorphism. For $1 < p < \infty$,, let L_ϕ be the left translation operator on $L^p(G)$ defined by $L_\phi f(x) = (f \circ \phi^{-1})(x)(f \in L^p(G), x \in G)$. Note that by the uniqueness of Haar measure, there exists a positive number $\sigma(\phi)$, such that $\int_G L_\phi f(x)d(x) = \sigma(\phi)\int_G f(x)d(x)$ for all $f \in L^1(G)$. In this case, the

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

map σ is a homomorphism on the group of all isomorphisms on G; see [6]. Let H be a closed subgroup of G with the Haar measure dh. Let G/H be the quotient group with Haar measure $d\dot{x}$. It is known that $\mathrm{d}x$, $\mathrm{d}h$, $\mathrm{d}\dot{x}$ are related to each others under the following identity, which is known as Weil's type Formula:

$$\int_G f(x)\mathrm{d}x = \int_{G/H}\int_H f(xh)\mathrm{d}h\mathrm{d}\dot{x},\ f \in L^1(G). \tag{2.1}$$

This formula shows that for $f \in L^1(G)$, the integral $\int_H f(xh)dh$ exists almost everywhere in x and defines an integrable function on G/H, such that the integral formula holds. In fact, the formula (2.1) should be understood as a one-sided version of Foubini's Theorem for product spaces; see [3].

We recall that the Fourier transform $\widehat{}$: $L^1(G) \longrightarrow C_0(\widehat{G})$, $f \longmapsto \widehat{f}$, is defined by $\widehat{f}(\xi) = \int_G f(x)\overline{\xi(x)}\mathrm{d}x$ for $\xi \in \widehat{G}$, the dual group of G. It is well known that if $f \in L^p(G)(1 \leq p \leq 2)$, then $\widehat{f} \in L^q(\widehat{G})$, where q and p are conjugate exponents, and $\|\widehat{f}\|_q \leq \|f\|_p$ (see [4]).

Throughout this article, we always assume that G is a second countable LCA group. In this case, we always have a uniform lattice in G; see [9]. Suppose that L is a uniform lattice in G, and $\phi : G \longrightarrow G$ is a topological isomorphism. It is well known that $G/\phi(L)$ is a LCA group and it is topologically isomorphic with G/L (for more details, see also [6]).

Let $f, g \in L^p(G)$, $1 < p < \infty$, and q be the conjugate exponent to p. Then, $fg^{p-1} \in L^1(G)$, and hence by Weil's formula, we get:

$$\begin{aligned}
\int_{G/\phi(L)} \sum_{k\in L} |fg^{p-1}(x\phi(k^{-1}))|\mathrm{d}\dot{x} &= \int_{G/\phi(L)} \sum_{\phi(k)\in\phi(L)} |fg^{p-1}(x\phi(k^{-1}))|\mathrm{d}\dot{x} \\
&= \int_G |\ fg^{p-1}(x)\ |\ \mathrm{d}x \\
&\leq \left(\int_G |\ f(x)\ |^p\ \mathrm{d}x\right)^{1/p} \left(\int_G |\ g^{p-1}(x)\ |^q\ \mathrm{d}x\right)^{1/q} \\
&\leq \|f\|_p \|g^{p-1}\|_q.
\end{aligned}$$

Thus, for almost all $\dot{x} \in G/\phi(L)$, the series $\sum_{k\in L} fg^{p-1}(x\phi(k^{-1}))$ converges.

Therefore, each function $g \in L^p(G)$ induces a bounded linear map:

$$\begin{aligned}
\Gamma_g : L^p(G) &\longrightarrow L^1(G/\phi(L)), \\
f &\mapsto \Gamma_g(f) = [f, g]_{\phi,p}
\end{aligned}$$

with $\|\Gamma_g\| = \|g\|_p^{p-1}$, where $[f, g]_{\phi,p}(\dot{x}) = \sum_{k\in L} fg^{p-1}(x\phi(k^{-1}))$.

Note that $\Gamma_g(f) = [f, g]_{\phi,p}$ is $\phi(L)$-periodic and we call $[f, g]_{\phi,p}$ the $(\phi(L), p)$-bracket product of $f, g \in L^p(G)$.

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Consequently, one may define the $(\phi(L), p)$-norm as follows:

$$\begin{cases} \|.\|_{\phi,p} : L^p(G) \longrightarrow L^p(G/\phi(L)), \\ f \mapsto \|f\|_{\phi,p} = (\Gamma_{|f|}(|\ f\ |))^{1/p}, \end{cases}$$

which is an isometry, $\|\|f\|_{\phi,p}\|_p = \|f\|_p$. Indeed, by Weil's Formula for $f \in L^p(G)$, $1 < p < \infty$ we have:

$$\begin{aligned}
\|\|f\|_{\phi,p}\|_p^p &= \int_{G/\phi(L)} \|f\|_{\phi,p}^p(\dot{x})\mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} \Gamma_{|f|}(|\ f\ |)(\dot{x})\mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} [|\ f\ |, |\ f\ |]_{\phi,p}(\dot{x})\mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} \sum_{\phi(k)\in\phi(L)} |f\ ||\ f\ |^{p-1}\ (x\phi(k^{-1}))\mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} \sum_{\phi(k)\in\phi(L)} |\ f\ |^p\ (x\phi(k^{-1}))\mathrm{d}\dot{x} \\
&= \int_G |\ f\ |^p\ (x)\mathrm{d}x \\
&= \|f\|_p^p.
\end{aligned}$$

Now, in the following two examples, we show that our definitions extend the previous ones mentioned earlier.

Example 2.1 Consider $G = \mathbb{R}$, $L = \mathbb{Z}$ in the above definition. Fix $a \in \mathbb{R}^+$. Then, $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, given by $\phi(x) = ax$ is a topological isomorphism and the bounded linear map $\Gamma_g : L^p(\mathbb{R}) \longrightarrow L^1([0,a])$, defines by $\Gamma_g(f)(x) = [f, g]_{\phi,p}(x) = \sum_{n\in\mathbb{Z}} fg^{p-1}(x - na)$ is the a-pointwise inner product of f and g introduced by Casazza and Lammers in [1] for $p = 2$. Moreover, if ϕ is the identity function on $\mathbb{R}$, $p = 2$, then the (ϕ, p)-bracket product is exactly one defined by Ron and Shen [10].

Example 2.2 Let $G = \mathbb{R}^n \times \mathbb{Z}^n \times \mathbb{T}^n \times Z_n$, for $n \in \mathbb{N}$, where Z_n is the finite abelian group $\{\overline{0}, \overline{1}, \overline{2}, ..., \overline{n-1}\}$ of residues module n and $L = \mathbb{Z}^n \times \mathbb{Z}^n \times \{1\} \times Z_n$ a uniform lattice in G. Let A be an invertible $n \times n$ real matrix and fix $l \in \mathbb{Z}^n$. Define $\phi : G \longrightarrow G$ by $\phi(x, m, t, p) = (Ax, l + m, t, p)$, for every $x \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$, $t \in \mathbb{T}^n$, $p \in Z_n$. For $f, g \in L^p(G)$, the $(\phi(L), p)$-bracket product is defined by:

$$\begin{aligned}
\Gamma_g(f)(x) &= [f, g]_{\phi,p}(x) \\
&= \sum_{k\in\mathbb{Z}^n, n\in\mathbb{Z}^n, q\in Z_n} fg^{p-1}((Ax, l + m, t, p) - \phi(k, n, 1, q)) \\
&= \sum_{k\in\mathbb{Z}^n, n\in\mathbb{Z}^n, q\in Z_n} fg^{p-1}(Ax - k, l + m - n, t - 1, p - q).
\end{aligned}$$

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

In the following proposition, we explain some properties of Γ_g.

Proposition 2.3 *Let $f, g \in L^p(G)$ for $1 < p < \infty$ and $c \in \mathbb{C}$. Then, the following properties hold:*

(i) $\Gamma_g(f + h) = \Gamma_g(f) + \Gamma_g(h)$
(ii) $\Gamma_g(cf) = c\Gamma_g(f)$
(iii) $\Gamma_{cg}(f) = c^{p-1}\Gamma_g(f)$
(iv) $\Gamma_{cg}(cf) = c^p\Gamma_g(f)$.

Proof The proof is obvious. □

It is worth to note that the ϕ-norm satisfies the properties of norm. Indeed, for all $\dot{x} \in G/\phi(L)$, $c \in \mathbb{C}$ and $f, g \in L^p(G)$, the equality $\| \|f\|_{\phi,p}\|_p = \|f\|_p$ implies that if $\|f\|_{\phi,p} = 0$, then $f = 0$ *a.e.*. Also $\Gamma_{|cf|}(| cf |) =| c |^p \Gamma_{|f|}(| f |)$, i.e., $\|cf\|_{\phi,p} =| c | \|f\|_{\phi,p}$. For triangular inequality:

$$\|f + g\|_{\phi,p} \le \|f\|_{\phi,p} + \|g\|_{\phi,p},$$

we have:

$$\begin{aligned}
\|f + g\|_{\phi,p}(\dot{x}) &= (\Gamma_{|f+g|}(| f + g |)(\dot{x}))^{1/p} \\
&= ([| f + g |, | f + g |]_{\phi,p}(\dot{x}))^{1/p} \\
&= \left(\sum_{k\in L} | f + g || f + g |^{p-1} (x\phi(k^{-1}))\right)^{1/p} \\
&= \left(\sum_{k\in L} | f + g |^p (x\phi(k^{-1}))\right)^{1/p} \\
&= \|f + g\|_{l^p(L)} \\
&\le \|f\|_{l^p(L)} + \|g\|_{l^p(L)} \\
&= \left(\sum_{k\in L} | f |^p (x\phi(k^{-1}))\right)^{1/p} + (\sum_{k\in L} | f + g |^p (x\phi(k^{-1})))^{1/p} \\
&= ([| f |, | f |]_{\phi,p}(\dot{x}))^{1/p} + ([| g |, | g |]_{\phi,p}(\dot{x}))^{1/p} \\
&= (\Gamma_{|f|}(| f |)(\dot{x}))^{1/p} + (\Gamma_{|g|}(| g |)(\dot{x}))^{1/p} \\
&= \|f\|_{\phi,p}(\dot{x}) + \|g\|_{\phi,p}(\dot{x}).
\end{aligned}$$

The following proposition demonstrates the duality property of $(\phi(L), p)$-bracket product.

Proposition 2.4 *For $f, g \in L^p(G)$ and $1 < p < \infty$. Then:*

$$\int_{G/\phi(L)} \Gamma_g(f)(\dot{x})d\dot{x} =< f, \overline{g^{p-1}} > . \tag{2.2}$$

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Proof By Weil's Formula:

$$\begin{aligned}\int_{G/\phi(L)} \Gamma_g(f)(\dot{x})\mathrm{d}\dot{x} &= \int_{G/\phi(L)} [f, g]_{\phi,p}(\dot{x})\mathrm{d}\dot{x} \\ &= \int_G (f.g^{p-1})(x)\mathrm{d}x \\ &= \langle f, \overline{g^{p-1}}\rangle.\end{aligned}$$

□

Note that if $p = 2$, then we get:

$$\int_{G/\phi(L)} [f, g]_{\phi,p}(\dot{x})\mathrm{d}\dot{x} = \langle f, \overline{g}\rangle_{L^2(G)},$$

which has already appeared in [3].

For the Hölder inequality, we need the following Lemmas.

Lemma 2.5 *Let $f, g \in L^p(G)$ for $1 < p < \infty$, where q is the conjugate exponent to p. Then:*

$$[f, g]_{\phi,p} = [g^{p-1}, f^{p-1}]_{\phi,q}.$$

Proof For any $\dot{x} \in G/\phi(L)$, we have:

$$\begin{aligned}[f, g]_{\phi,p}(\dot{x}) &= \sum_{k\in L} fg^{p-1}(x\phi(k^{-1})) \\ &= \sum_{k\in L} g^{p-1} f^{(p-1)(q-1)}(x\phi(k^{-1})) \\ &= [g^{p-1}, f^{p-1}]_{\phi,q}(\dot{x}).\end{aligned}$$

□

At this point, for $f \in L^p(G)$, we define the $\phi(L)$-pointwise normalization of f as follows:

$$\mathrm{N}_{\phi(L)}(\mathrm{f})(\dot{x}) = \begin{cases} | f(\dot{x}) | / \|f\|_{\phi,p}(\dot{x}) & \|f\|_{\phi,p}(\dot{x}) \neq 0, \\ 0 & \|f\|_{\phi,p}(\dot{x}) = 0. \end{cases}$$

Lemma 2.6 *With the above notations, and non-zeros $f, g \in L^p(G)$, $(1 < p, q < \infty)$, we have:*

(i) $\Gamma_g(N_{\phi(L)}(f)) = \left(\frac{1}{\|f\|_{\phi,p}}\right)\Gamma_g(| f |)$,

where $\|f\|_{\phi,p} \neq 0$.

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(ii) $\Gamma_{N_{\phi(L)}(g)}(N_{\phi(L)}(f)) = \left(\frac{1}{\|f\|_{\phi,p}}\right)\left(\frac{1}{\|g\|_{\phi,p}^{p-1}}\right)\Gamma_{|g|}(|\ f\ |)$,
for $\|f\|_{\phi,p} \neq 0$, $\|g\|_{\phi,p} \neq 0$.

In particular, $\Gamma_{|g|}(|\ f\ |) = 0$ *if and only if:*

$$\Gamma_{N_{\phi(L)}(g)}(N_{\phi(L)}(f)) = 0.$$

(iii) *For* $f \neq 0$ *a.e., we have:*

$$\Gamma_{N_{\phi(L)}(f)}(N_{\phi(L)}(f)) = 1.$$

(iv) *For* $f \neq 0$*, we have,* $\|N_{\phi(L)}(f)\|_{L^p(G)}^p = \mid G/\phi(L) \mid < \infty$, *(* $\mid E \mid$ *denotes the Haar measure of the Borel set* $E \subseteq G$ *).*

(v) $N_{\phi(L)}\big(N_{\phi(L)}(f)\big) = N_{\phi(L)}(f)$.

Proof Proof of (i) is clear. For (ii), we have:

$$\begin{aligned}
\Gamma_{N_{\phi(L)}(g)}(N_{\phi(L)}(f))(\dot{x}) &= \left[N_{\phi(L)}(f), N_{\phi(L)}(g)\right]_{\phi,p}(\dot{x}) \\
&= \left[\frac{\mid f \mid}{\|f\|_{\phi,p}}, \frac{\mid g \mid}{\|g\|_{\phi,p}}\right]_{\phi,p}(\dot{x}) \\
&= \left(\frac{1}{\|f\|_{\phi,p}(\dot{x})}\right)\left(\frac{1}{\|g\|_{\phi,p}^{p-1}(\dot{x})}\right)[\mid f \mid, \mid g \mid]_{\phi,p}(\dot{x}) \\
&= \left(\frac{1}{\|f\|_{\phi,p}}\right)\left(\frac{1}{\|g\|_{\phi,p}^{p-1}}\right)\Gamma_{|g|}(|\ f\ |)(\dot{x}).
\end{aligned}$$

Now, using (ii), the proofs of (iii) and (iv) are obvious. For (v):

$$\begin{aligned}
N_{\phi(L)}\big(N_{\phi(L)}(f)\big)(\dot{x}) &= \mid N_{\phi(L)}(f)(x) \mid / \|N_{\phi(L)}(f)\|_{\phi,p}(\dot{x}) \\
&= \mid N_{\phi(L)}(f)(\dot{x}) \mid \\
&= \mid f(\dot{x}) \mid / \|f\|_{\phi,p}(\dot{x}) \\
&= N_{\phi(L)}(f)(\dot{x}).
\end{aligned}$$

□

Proposition 2.7 *(Hölder's inequality) Let* $f, g \in L^p(G)$ *for* $1 < p, q < \infty$ *where* q *is the conjugate exponent to* p*. Then:*

$$\mid [f, g]_{\phi,p} \mid \leq \| f \|_{\phi,p} \| g^{p-1} \|_{\phi,q}, \tag{2.3}$$

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Proof Put $g^{p-1} = \psi$, then $\psi \in L^q(G)$. Now, we have:

$$\begin{aligned}\parallel \psi \parallel_{\phi,q}^q (\dot{x}) &= \Gamma_{|\psi^{q-1}|}(|\ \psi^{q-1}\ |)(\dot{x}) \\ &= [|\ \psi^{q-1}\ |, |\ \psi^{q-1}\ |]_{\phi,p}(\dot{x}) \\ &= \sum_{k\in L} |\ \psi^{q-1}\ ||\ \psi\ |\ (x\phi(k^{-1})).\end{aligned}$$

If either $\parallel f \parallel_{\phi,p}= 0$ or $\parallel \psi \parallel_{\phi,q}= 0$, then the inequality holds trivially. The same holds when either $\parallel f \parallel_{\phi,p}= \infty$ or $\parallel \psi \parallel_{\phi,q}= \infty$, the result is trivial. Moreover, it is easy to see that if:

$$|\ \Gamma_{|\psi|}(|\ f\ |)\ | \leq \parallel f \parallel_{\phi,p} \parallel \psi \parallel_{\phi,q}$$

holds for a particular f, ψ, then it also holds for all scaler multiples of f and ψ. It is, therefore, it would suffice to prove that (2.3) holds when $\parallel f \parallel_{\phi,p} (\dot{x}) = \parallel \psi \parallel_{\phi,q} (\dot{x}) = 1$, where 1 denotes the constant function of $G/\phi(L)$ onto $\mathbb{C}$. To this end, by [5, Lemma 6.1], we have:

$$\begin{aligned}&|\ f(x\phi(l^{-1}))\ ||\ \psi(x\phi(l^{-1}))\ | \leq 1/p\ |\ f^p(x\phi(l^{-1}))\ | + 1/q\ |\ \psi^q(x\phi(l^{-1}))\ |, \\ &|\ f\ ||\ \psi\ |\ (x\phi(l^{-1})) \leq 1/p\ |\ ff^{p-1}(x\phi(l^{-1}))\ | + 1/q\ |\ \psi^{q-1}\psi(x\phi(l^{-1}))\ |; \\ &\sum_{l\in L} |\ f\ ||\ \psi\ |\ (x\phi(l^{-1})) \leq 1/p \left(\sum_{l\in L} |\ f\ ||\ f^{p-1}\ |\ (x\phi(l^{-1}))\right) \\ &\quad + 1/q \left(\sum_{l\in L} |\ \psi^{q-1}\ ||\ \psi\ |\ (x\phi(l^{-1}))\right).\end{aligned}$$

Thus:

$$\begin{aligned}\left|\ \sum_{l\in L} |\ f\ ||\ \psi\ |\ (x\phi(l^{-1}))\ \right| &\leq 1/p[|\ f\ |, |\ f\ |]_{\phi,p}(\dot{x}) + 1/q[|\ \psi\ |, |\ \psi\ |]_{\phi,q}(\dot{x}) \\ &= 1/p \parallel f \parallel_{\phi,p}^p (\dot{x}) + 1/q \parallel \psi \parallel_{\phi,q}^q (\dot{x}) \\ &= \parallel f \parallel_{\phi,p} (\dot{x}) \parallel \psi \parallel_{\phi,q} (\dot{x}).\end{aligned}$$

Now, put $\psi = g^{p-1}$. We have:

$$|\ [|\ f\ |, |\ g\ |]_{\phi,p}\ | \leq \parallel f \parallel_{\phi,p} \parallel g^{p-1} \parallel_{\phi,q}.$$

General case, if $\|f\|_{\phi,p} \neq 1$ and $\|g\|_{\phi,p} \neq 1$, then using Lemma 2.6, part (ii) can be written as:

$$\Gamma_{\frac{|\ g\ |}{\|g\|_{\phi,p}}}\left(\frac{|\ f\ |}{\|f\|_{\phi,p}}\right) = \left(\frac{1}{\|f\|_{\phi,p}}\right)\left(\frac{1}{\|g\|_{\phi,p}^{p-1}}\right)\Gamma_{|g|}(|\ f\ |).$$

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

Indeed, by Lemma 2.5, we have $\|g\|_{\phi,p} = \|g^{p-1}\|_{\phi,q}^{q/p}$. Hence:

$$\|g\|_{\phi,p}^{p-1} = \|g^{p-1}\|_{\phi,q}. \tag{2.4}$$

□

It is worthwhile to note that using (2.4), we have:

$$|\, [|\, f \,|, |\, g \,|]_{\phi,p} \,| \leq \|\, f \,\|_{\phi,p} \|\, g \,\|_{\phi,p}^{p-1}.$$

Definition 2.8 For $\gamma \in \hat{G}$, we denote the modulation operator on $L^p(G)$ by M_γ, which is defined by $M_\gamma f(x) = \gamma(x) f(x)$ for all $f \in L^p(G)$.

In the next proposition, some properties of the Fourier transform of the $(\phi(L), p)$-bracket product are established.

Proposition 2.9 *Suppose $f, g \in L^p(G)$ and $\gamma \in \phi(L)^\perp (\cong \widehat{G/\phi(L)})$, where $\phi(L)^\perp$ is the* ***annihilator*** *of $\phi(L)$ in $\widehat{G}$. Then:*

(i) $\Gamma_g(M_\gamma f) = \Gamma_{M_{\gamma^{\frac{1}{p-1}}} g}(f)$

(ii) $(\Gamma_g(f))^\wedge(\gamma) = \langle f, \overline{M_{\gamma^{\frac{-1}{p-1}}} g^{p-1}} \rangle = \langle M_{\gamma^{-1}} f, \overline{g^{p-1}} \rangle$, *and*

(iii) $(\Gamma_g(f))^\wedge(\gamma_1\gamma_2) = (\Gamma_{M_{\gamma_1^{-1}} g})^\wedge(\gamma_2) =< M_{\gamma_2}^{-1} f, \overline{M_{\gamma_1} g^{p-1}} >$.

Proof The proof of (i) is clear. For (ii), since $\gamma(\phi(k^{-1})) = 1$ for all $k \in L$, we have:

$$\begin{aligned}
(\Gamma_g(f))^\wedge(\gamma) &= \widehat{[f, g]}_{\phi,p}(\gamma) \\
&= \int_{G/\phi(L)} [f, g]_{\phi,p}(\dot{x}) \gamma^{-1}(\dot{x}) \mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} \sum_{\phi(k)\in\phi(L)} f g^{p-1}(x\phi(k^{-1})) \gamma^{-1}(x\phi(k^{-1})) \mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} \sum_{\phi(k)\in\phi(L)} f(x\phi(k^{-1})) M_{\gamma^{-1}} g^{p-1}(x\phi(k^{-1})) \mathrm{d}\dot{x} \\
&= \int_G f M_{\gamma^{-1}} g^{p-1}(x) \mathrm{d}x \\
&= < f, \overline{M_{\gamma^{-1}} g^{p-1}} > (x).
\end{aligned}$$

Part (iii) is a direct consequence of (ii) and its proof. □

Example 2.10 Let $f, g \in L^p(\mathbb{R}^n)$, the modulation operator on $L^p(\mathbb{R}^n)$ defined by $M_a f(x) = e^{2\pi i a x} f(x)$, where $x \in \mathbb{R}^n$ and $a \in \widehat{\mathbb{R}^n}$. Consider $\mathbb{Z}^n$ as a uniform lattice

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

in $\mathbb{R}^n$. Then:

$$
\begin{aligned}
(\Gamma_g(f))^\wedge(\gamma) &= \widehat{[f,g]}_{\phi,p}(\gamma) \\
&= \int_{[0,a]^n} [f,g]_{\phi,p}(t) e^{-2\pi i \gamma t} \mathrm{d}t \\
&= \int_{[0,a]^n} \sum_{l\in\mathbb{Z}^n} f g^{p-1}(t-al) e^{-2\pi i \gamma t} \mathrm{d}t \\
&= \int_{\mathbb{R}^n} f g^{p-1}(x) e^{-2\pi i \gamma(x)} \mathrm{d}x \\
&= \langle f, \overline{M_{\gamma^{-1}} g^{p-1}} \rangle.
\end{aligned}
$$

Corollary 2.11 *If $\Gamma_g(f) \in L^1(G/\phi(L))$ and $\widehat{\Gamma_g(f)} = 0$, then $\Gamma_g(f) = 0$ a.e. with respect to the Haar measure on $G/\phi(L)$.*

Now, we are going to consider translation operators for $(\phi(L), p)$-bracket product. Note that, since G is LCA group, then the left and right translations coincide . For $y \in G$, the translation operator on $L^1(G/\phi(L))$ is defined by:

$$
T_y \Gamma_g(f)(\dot{x}) = \Gamma_g(f)(y^{-1}\dot{x}).
$$

One can easily check:

$$
T_y \Gamma_g(f) = \Gamma_{T_y g}(T_y f). \tag{2.5}
$$

Indeed:

$$
\begin{aligned}
T_y \Gamma_g f(\dot{x}) &= T_y [f,g]_{\phi,p}(\dot{x}) \\
&= [f,g]_{\phi,p}(y^{-1}\dot{x}) \\
&= \sum_{k\in L} T_y f(x\phi(k^{-1})) T_y g^{p-1}(x\phi(k^{-1})) \\
&= [T_y f, T_y g]_{\phi,p}(\dot{x}) \\
&= \Gamma_{T_y g}(T_y f)(\dot{x}).
\end{aligned}
$$

In the next proposition, we have some properties concerning the translation operator T_y.

Proposition 2.12 *Let $y \in G$ and T_y be the translation operator on $L^1(G/\phi(L))$. Then:*

(i) $\int_{G/\phi(L)} \Gamma_g(T_y f)(\dot{x}) \mathrm{d}\dot{x} = \int_{G/\phi(L)} \Gamma_{T_{y^{-1}} g}(f)(\dot{x}) \mathrm{d}\dot{x}$,
(ii) $\Gamma_g(T_y f) = T_y(\Gamma_{T_{y^{-1}} g}(f))$,
(iii) $\|T_y f\|^p_{\phi,p} = T_y \|f\|^p_{\phi,p}$ *and*
(iv) $(T_y(\Gamma_g(f)))^\wedge(\xi) = (\Gamma_g(f))^\wedge(\xi)\xi^{-1}(y)$, *for* $\xi \in \phi(L)^\perp$.

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

Proof For (i), let $\dot{x} \in G/\phi(L)$. Then, by the Weil's Formula, we have:

$$\begin{aligned}
\int_{G/\phi(L)} \Gamma_g(T_y f)(\dot{x})\mathrm{d}\dot{x} &= \int_{G/\phi(L)} [T_y f, g]_{\phi,p}(\dot{x})\mathrm{d}\dot{x} \\
&= \int_G T_y f.g^{p-1}(x)\mathrm{d}x \\
&= \int_G f(y^{-1}x)g^{p-1}(x)\mathrm{d}x \\
&= \int_G f(x)g^{p-1}(yx)\mathrm{d}x \\
&= \int_G f(x)T_{y^{-1}}g^{p-1}(x)\mathrm{d}x \\
&= \int_{G/\phi(L)} [f, T_{y^{-1}}g]_{\phi,p}(\dot{x})\mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} \Gamma_{T_{y^{-1}}g}(f)(\dot{x})\mathrm{d}\dot{x}.
\end{aligned}$$

Part (ii) and (iii) are obvious by (2.5). For $\xi \in \phi(L)^{\perp}$, we get:

$$\begin{aligned}
(T_y(\Gamma_g(f)))^{\wedge}(\xi) &= (T_y[f, g]_{\phi,p})^{\wedge}(\xi) \\
&= \int_{G/\phi(L)} T_y[f, g]_{\phi,p}(\dot{x})\xi^{-1}(\dot{x})\mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} [f, g]_{\phi,p}(y^{-1}\dot{x})\xi^{-1}(\dot{x})\mathrm{d}\dot{x} \\
&= \xi^{-1}(y)\int_{G/\phi(L)} [f, g]_{\phi,p}(\dot{x})\xi^{-1}(\dot{x})\mathrm{d}\dot{x} \\
&= \widehat{[f, g]}_{\phi,p}(\xi)\xi^{-1}(y^{-1}) \\
&= (\Gamma_g(f))^{\wedge}(\xi)\xi^{-1}(y^{-1}).
\end{aligned}$$

Therefore, part (iv) is proved. □

At this point, we denote the set of all $\phi(L)$-periodic functions in $L^{\infty}(G)$ by $B_{\infty}(G)$, i.e., $B_{\infty}(G) = \{h \in L^{\infty}(G); h(x\phi(k)) = h(x), \, for \; all \; k \in L\}$. It is easy to show that $B_{\infty}(G)$ is a closed subspace of $L^{\infty}(G)$. Moreover, $L^p(G)$ is a Banach $B_{\infty}(G)$-module.

Proposition 2.13 *Let $f, g \in L^p(G)$, $1 < p, q < \infty$, and q is conjugate exponents of p. Then, for all $h \in B_{\infty}(G)$, we have:*

(i) $\Gamma_g(fh) = h(\Gamma_g(f))$,
(ii) $\Gamma_{hg}(f) = h^{p-1}(\Gamma_g(f))$.

In particular, if $h(\dot{x}) \neq 0$ *a.e., then* $\Gamma_g(f) = 0$ *if and only if* $\Gamma_g(fh) = \Gamma_{h^{\frac{1}{p-1}}g}(f) = 0$.

Proof For (i), let $h \in B_\infty(G)$:

$$\begin{aligned}
\Gamma_g(fh)(\dot{x}) &= [fh, g]_{\phi,p}(\dot{x}) \\
&= \sum_{k\in L} fhg^{p-1}(x\phi(k^{-1})) \\
&= \sum_{k\in L} f(x\phi(k^{-1}))g^{p-1}(x\phi(k^{-1}))h(x\phi(k^{-1})) \\
&= \sum_{k\in L} fg^{p-1}(x\phi(k^{-1}))h(\dot{x}) \\
&= h[f, g]_{\phi,p}(\dot{x}) \\
&= h(\Gamma_g(f))(\dot{x}).
\end{aligned}$$

Also for proof of (ii), we have:

$$\begin{aligned}
\Gamma_{hg}(f)(\dot{x}) &= [f, hg]_{\phi,p}(\dot{x}) \\
&= \sum_{k\in L} f(hg)^{p-1}(x\phi(k^{-1})) \\
&= \sum_{k\in L} f(x\phi(k^{-1}))h^{p-1}(x\phi(k^{-1}))g^{p-1}(x\phi(k^{-1})) \\
&= \sum_{k\in L} fg^{p-1}(x\phi(k^{-1}))h^{p-1}(\dot{x}) \\
&= h^{p-1}[f, g]_{\phi,p}(\dot{x}) \\
&= h^{p-1}(\Gamma_g(f))(\dot{x}).
\end{aligned}$$

□

Definition 2.14 Let $f \in L^p(G)$, $g \in L^q(G)$ where $1/p+1/q = 1$ and $1 < p, q < \infty$. For $E \subseteq L^p(G)$, the $\phi(L)$-orthogonal complement of E is defined as:

$$E^{\perp_{\phi,p}} = \{g \in L^q(G); \Gamma_{g^{q-1}}(f) = 0 \ \ a.e. \ \ for\ all \ \ f \in L^p(G)\}.$$

In the next proposition, the relation between the $\phi(L)$-orthogonal complement of E in $L^p(G)$ and its orthogonal complement in $L^q(G)$ is investigated.

Proposition 2.15 *For* $E \subseteq L^p(G)$, *we have* $E^{\perp_{\phi,p}} = \cap_{h\in B_\infty(G)}(hE)^{\perp_{\phi,p}}$.

Proof Let $g \in E^{\perp_{\phi,p}}$. Then, for $h \in B_\infty(G)$ and $f \in E$ by Propositions (2.13) and (2.4), we have:

$$< hf, \overline{g^{p-1}} >= \int_{G/\phi(L)} \Gamma_{g^{q-1}}(hf)(\dot{x})\mathrm{d}\dot{x} = \int_{G/\phi(L)} h(\dot{x})\Gamma_{g^{q-1}}(f)(\dot{x})\mathrm{d}\dot{x} = 0;$$

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

hence, $g \in \cap_{h\in B_\infty(G)}(hE)^{\perp_{\phi,p}}$. Now, for $g \in \cap_{h\in B_\infty(G)}(hE)^{\perp}$, $f \in E$ and $n \in \mathbb{N}$, define $h_n = \Gamma_{g^{q-1}}(f)$, when $|\Gamma_{g^{q-1}}(f)| \leq n$, and $h_n = 0$ otherwise. Then, $h_n \in B_\infty(G)$. Therefore, we have:

$$\begin{aligned} 0 &= |\Gamma_{h_n g^{p-1}}(f)(\dot{x})| \\ &= \int_{G/\phi(L)} |h_n^{p-1}(\dot{x})\Gamma_{g^{q-1}}(f)(\dot{x})| \,\mathrm{d}\dot{x} \\ &= \int_{G/\phi(L)} |h_n^{p-1}(\dot{x})h_n(\dot{x})| \,d\dot{x} \\ &= \int_{G/\phi(L)} |h_n|^p (\dot{x})\mathrm{d}\dot{x}. \end{aligned}$$

Therefore, $|h_n|(\dot{x}) = 0$. Hence, $\Gamma_{g^{q-1}}(f) = 0$ a.e., that is, $g \in E^{\perp_{\phi,p}}$. □

3 $\phi(L)$-Factorable Operators

Let G be an LCA group and E be a subgroup of G or $G/\phi(L)$, in which we suppose that L be a uniform lattice in G, and $\phi : G \longrightarrow G$ is a topological isomorphism. In this section, $\phi(L)$-factorable operators are defined and some of their properties are investigated. Moreover, the relation between $\phi(L)$-factorable operators and $(\phi(L), p)$-bracket product is shown. Finally, the Riesz Representation Theorem for $L^p(G)$ with the $(\phi(L), p)$-bracket product is proven.

Definition 3.1 An operator $U : L^p(G) \longrightarrow L^r(E)$ that $1 \leq r, p \leq \infty$ is called $\phi(L)$-factorable if $U(hf) = hU(f)$, for all $f \in L^p(G)$ and all $\phi(L)$-periodic $h \in L^\infty(G)$, where E is a subgroup of G or $G/\phi(L)$.

In the following, some properties of the $\phi(L)$-factorable operators are examine.

Lemma 3.2 *Let $U_1, U_2 : L^p(G) \longrightarrow L^1(G/\phi(L))$ be two $\phi(L)$-factorable operators. Then, $U_1 = U_2$ if and only if:*

$$\int_{G/\phi(L)} U_1(f)(\dot{x})\mathrm{d}\dot{x} = \int_{G/\phi(L)} U_2(f)(\dot{x})\mathrm{d}\dot{x},$$

for every $f \in L^p(G)$.

Proof The necessary part is obvious. For the converse, by [4, theorem 4.33], it is enough to show that $\widehat{U_1(f)} = \widehat{U_2(f)}$ for all $f \in L^p(G)$. Let $\xi \in (\widehat{G/\phi(L)}) = \phi(L)^\perp$ and

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

$f \in L^p(G)$, since ξ as a function in $L^\infty(G)$ is $\phi(L)$-periodic, we obtain:

$$\begin{aligned}\widehat{U_1(f)}(\xi) &= \int_{G/\phi(L)} U_1(f)(\dot{x})\xi(\dot{x})\mathrm{d}\dot{x}\\ &= \int_{G/\phi(L)} U_1(\xi f)(\dot{x})\mathrm{d}\dot{x}\\ &= \int_{G/\phi(L)} U_2(\xi f)(\dot{x})\mathrm{d}\dot{x}\\ &= \int_{G/\phi(L)} U_2(f)(\dot{x})\xi(\dot{x})\mathrm{d}\dot{x}\\ &= \widehat{U_2(f)}(\xi).\end{aligned}$$

Hence, the Fourier coefficients for $U_1(f)$ and $U_2(f)$ are the same for all $f \in L^p(G)$ and, therefore, $U_1 = U_2$. □

Lemma 3.3 *Let $h \in B_\infty(G)$ and $f \in L^p(G)$ where $1 < p < \infty$. Then,*

$$\int_G |\ hf\ |^p\ (x)\mathrm{d}x = \int_{G/\phi(L)} |\ h(\dot{x})\ |^p\ \|f\|_{\phi,p}^p(\dot{x})\mathrm{d}\dot{x}.$$

Proof Using Weil's Formula, we have:

$$\begin{aligned}\int_G |\ hf\ |^p\ (x)\mathrm{d}x &= \int_{G/\phi(L)} \sum_{\phi(k)\in\phi(L)} |\ h(x\phi(k^{-1}))\ |^p\ f(x\phi(k^{-1}))\ |^p\ \mathrm{d}\dot{x}\\ &= \int_{G/\phi(L)} |\ h(x)\ |^p \sum_{\phi(k)\in\phi(L)} |\ f(x\phi(k^{-1}))\ |^p\ \mathrm{d}\dot{x}\\ &= \int_{G/\phi(L)} |\ h(\dot{x})\ |^p\ \|f\|_{\phi,p}^p(\dot{x})\mathrm{d}\dot{x},\end{aligned}$$

in which $h \in B_\infty(G)$ and $f \in L^p(G)$. □

Note that, if $h \in L^\infty(G)$ and $f \in L^p(G)$, then $|\ hf\ |^p \in L^1(G)$.

Proposition 3.4 *Let U be a $\phi(L)$-factorable linear operator from $L^p(G)$ to $L^p(G/\phi(L))$, $1 < p < \infty$. Then, U is bounded if and only if there is a constant $B > 0$ ($B = \|U\|$), so that for every $f \in L^p(G)$, we have:*

$$|\ U(f)(\dot{x})\ | \le B\|f\|_{\phi,p}(\dot{x}), \quad for \quad a.e.\dot{x} \in G/\phi(L).$$

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

Proof Let $h \in B_\infty(G)$ and $f \in L^p(G)$. By Lemma 3.3:

$$\begin{aligned}
\int_{G/\phi(L)} \mid h(\dot{x}) \mid^p \mid U(f)(\dot{x}) \mid^p \, \mathrm{d}\dot{x} &= \int_{G/\phi(L)} \mid U(hf)(\dot{x}) \mid^p \, \mathrm{d}\dot{x} \\
&= \|U(hf)\|^p_{L^p(G/\phi(L))} \\
&\leq \|U\|^p \int_G \mid hf \mid^p (x)\mathrm{d}x \\
&= \|U\|^p \int_{G/\phi(L)} \mid h(\dot{x}) \mid^p \|f\|^p_{\phi,p}(\dot{x})\mathrm{d}\dot{x}.
\end{aligned}$$

It follows immediately that $\mid U(f)(\dot{x}) \mid^p \leq \|U\|^p \|f\|^p_{\phi,p}(\dot{x})$, a.e. for $\dot{x} \in G/\phi(L)$. Conversely, let $f \in L^p(G)$, we have:

$$\begin{aligned}
\|U(f)\|^p_{\phi,p} &= \int_{G/\phi(L)} \mid U(f)(\dot{x}) \mid^p \, \mathrm{d}\dot{x} \\
&\leq \int_{G/\phi(L)} B^p \|f\|^p_{\phi,p}(\dot{x})\mathrm{d}\dot{x} \\
&= B^p \int_{G/\phi(L)} \|f\|^p_{\phi,p}(\dot{x})\mathrm{d}\dot{x} \\
&= B^p \|f\|^p_p.
\end{aligned}$$

Therefore, the proof is completed. □

Proposition 3.5 *If $U : L^p(G) \longrightarrow L^p(G)$ ($1 < p < \infty$) is a $\phi(L)$-factorable linear operator, then U is bounded if and only if there is a constant $B > 0$ ($B = \|U\|$), so that for every $f \in L^p(G)$, we have:*

$$\|U(f)\|_{\phi,p} \leq B\|f\|_{\phi,p}.$$

Proof For $h \in B_\infty(G)$ and $f \in L^p(G)$, by Proposition 3.4, we get:

$$\begin{aligned}
\int_{G/\phi(L)} \mid h(\dot{x}) \mid^p \|U(f)(\dot{x})\|^p_{\phi,p}(\dot{x})\mathrm{d}\dot{x} &= \int_{G/\phi(L)} \mid h(\dot{x}) \mid^p \Gamma_{|U(f)|} \mid U(f) \mid (\dot{x})\mathrm{d}\dot{x} \\
&= \int_{G/\phi(L)} \|U(hf)\|^p_{\phi,p}(\dot{x})\mathrm{d}\dot{x} \\
&= \|U(hf)(x)\|^p_{L^p(G)} \\
&\leq \|U\|^p \|hf\|^p_{L^p(G)}(x) \\
&= \|U\|^p \int_{G/\phi(L)} \mid h(\dot{x}) \mid^p \|f\|^p_{\phi,p}(\dot{x})\mathrm{d}\dot{x}.
\end{aligned}$$

It follows that $\|U(f)\|^p_{L^p(G)} \leq \|U\|^p \|f\|^p_{\phi,p}$ a.e. with respect to $G/\phi(L)$. □

Theorems 3.6 and 3.8 are of the main theorems in this section which are Riesz representation type theorem for the $(\phi(L), p)$-bracket product in $L^p(G)$.

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

Theorem 3.6 *An operator $U : L^p(G) \longrightarrow L^1(G/\phi(L))$ is a bounded $\phi(L)$-factorable if and only if there exists $g \in L^q(G)$, such that $U(f) = \Gamma_{g^{q-1}}(f)$ for all $f \in L^p(G)$. Moreover, $\|U\| = \|g\|_q$.*

Proof Let $U : L^p(G) \longrightarrow L^1(G/\phi(L))$(where for $1 < p < \infty$) be a bounded $\phi(L)$-factorable operator. Define the linear functional $\Psi : L^p(G) \longrightarrow \mathbb{C}$ by $\Psi(f) = \int_{G/\phi(L)} U(f)(\dot{x})\mathrm{d}\dot{x}$. The isometric isomorphism property $\left(L^p(G)\right)^* \cong L^q(G)$ for $(p \neq \infty)$ implies that there exist $g \in L^q(G)$, such that $\Psi(f) = \int_G fg(x)\mathrm{d}x$ for all $f \in L^p(G)$. Thus:

$$\begin{aligned} \int_{G/\phi(L)} U(f)(\dot{x})\mathrm{d}\dot{x} &= \Psi(f) \\ &= \int_G fg(x)\mathrm{d}x \\ &= \int_{G/\phi(L)} \Gamma_{g^{q-1}}(f)(\dot{x})\mathrm{d}\dot{x}. \end{aligned}$$

By (3.4), $U(f) = \Gamma_{g^{q-1}}(f)$, for all $f \in L^p(G)$.

Moreover, for any $f \in L^p(G)$:

$$\begin{aligned} \|U(f)\|_{L^1(G/\phi(L))} &= \|\Gamma_{g^{q-1}}(f)\|_{L^1(G/\phi(L))} \\ &\leq \|f\|_p \|g\|_q. \end{aligned}$$

Therefore, $\|U\| \leq \|g\|_q$. Now, letting $f =| g^{q-1} |$; hence:

$$\begin{aligned} \|U(| g^{q-1} |)\|_{L^1} &= \int_{G/\phi(L)} | U(| g^{q-1} |)(\dot{x}) | \mathrm{d}\dot{x} \\ &= \int_{G/\phi(L)} | \Gamma_{|g^{q-1}|}(| g^{q-1} |)(\dot{x}) | \mathrm{d}\dot{x} \\ &= \int_{G/\phi(L)} | [| g^{q-1} |, | g^{q-1} |]_{\phi,p}(\dot{x}) | \mathrm{d}\dot{x} \\ &= \int_{G/\phi(L)} | [| g |, | g |]_{\phi,q}(\dot{x}) | \mathrm{d}\dot{x} \\ &= \int_{G/\phi(L)} \|g\|^q_{\phi,q}(\dot{x})\mathrm{d}\dot{x} \\ &= \|g\|^q_q. \end{aligned}$$

Thus:

$$\|g\|^q_q = \|U(| g^{q-1} |)\|_{L^1} \leq \|U\| \|g\|^{q-1}_q,$$

i.e., $\|g\|_q \leq \|U\|$.

For the converse, according of $g \in L^q(G)$, U is bounded. For every $\phi(L)$-periodic $h \in L^\infty(G)$ and $f \in L^p(G)$:

$$U(hf) = \Gamma_{g^{q-1}}(hf) = h(\Gamma_{g^{q-1}}(f)) = hU(f).$$

Therefore, the proof is complete. $\square$

It is worth mentioning that Theorem3.6 for p = 2 gives the Riesz representation theorem expressed in [5, theorem 5.25].

Corollary 3.7 *Let $f, g \in L^p(G)$ $(1 < p < \infty)$. Then, $\Gamma_g(f)$ is $\phi(L)$-factorable.*

Proof The proof yields just using Proposition2.13 and Theorem3.6. $\square$

We say $f \in L^p(G)$ is $\phi(L)$-bounded if there exists $M > 0$, such that $\|f\|_{\phi,p} \leq M$.

Theorem 3.8 *A linear operator $U : L^p(G) \longrightarrow L^p(G/\phi(L))$ $(1 < p < \infty)$ is a bounded $\phi(L)$-factorable if and only if there exists $\phi(L)$-bounded $g \in L^q(G)$, such that $U(f) = \Gamma_{g^{q-1}}(f)$ for all $f \in L^p(G)$. Moreover, $\|U\| = esssup_{\dot{x}\in G/\phi(L)}\|g\|_{\phi,p}(\dot{x})$.*

Proof That is, U be a bounded $\phi(L)$-factorable operator from $L^p(G) \longrightarrow L^p(G/\phi(L))$. Since $G/\phi(L)$ is compact, $L^p(G/\phi(L)) \subseteq L^1(G/\phi(L))$, and so, by Theorem 3.6, there exists $g \in L^q(G)$, such that $U(f) = \Gamma_{g^{q-1}}(f)$ for all $f \in L^p(G)$. Letting $f = g^{q-1}$ and using Proposition 3.4, we get:

$$\begin{aligned} \mid \Gamma_{g^{p-1}}(g^{p-1}) \mid &= \|g^{q-1}\|^p_{\phi,p} \\ &= \mid U(\mid g^{q-1} \mid) \mid \\ &\leq \|U\|\|g^{q-1}\|_{\phi,p}. \end{aligned}$$

Hence, $\|g^{q-1}\|_{\phi,p} \leq \|U\|$ or $\|g\|_{\phi,q} \leq \|U\|$. For the converse, let g be a $\phi(L)$-bounded and $U(f) = \Gamma_{g^{q-1}}(f)$ for some $\phi(L)$-bounded, so $g \in L^q(G)$. Then, by Corollary 3.7, U is $\phi(L)$-factorable. Now, by the assumption, g is $\phi(L)$-bounded and by Theorem3.6, we have:

$$\begin{aligned} \|Uf\|^p_p &= \int_{G/\phi,p} \mid \Gamma_{g^{p-1}}(f) \mid^p (\dot{x})\mathrm{d}\dot{x} \\ &\leq \int_{(G/\phi(L))} \|f\|^p_{\phi,p}\|g\|^p_{\phi,q}(\dot{x})\mathrm{d}\dot{x} \\ &\leq esssup_{\dot{x}\in G/\phi(L)}\|g\|^p_{\phi,p} \int_{G/\phi(L)} \|f\|^p_{\phi,p}(\dot{x})\mathrm{d}\dot{x} \\ &= esssup_{\dot{x}\in G/\phi(L)}\|g\|^p_{\phi,p}\|f\|^p_p, \end{aligned}$$

where $\dot{x} \in G/\phi(L)$. Thus, $\|U\|$ is bounded.

Now, by letting $f = g^{q-1}$, we get $\|U\| = esssup_{\dot{x}\in G/\phi(L)}\|g\|_{\phi,p}(\dot{x})$. This completes the proof. $\square$

Journal: 41980 Article No.: 0537 TYPESET DISK LE CP Disp.:2021/2/4 Pages: 19 Layout: Small-Ex

Theorem 3.9 *For $1 < p < \infty$, let $U : L^p(G) \longrightarrow L^q(G)$, (where $L^q(G)$ is dual of $L^p(G)$), be a bounded $\phi(L)$-factorable operator and U^* be its adjoint. Then, U^* is $\phi(L)$-factorable. Moreover, for $f \in L^p(G)$ and $g \in L^q(G)$, we have:*

$$\Gamma_{g^{q-1}}(U(f)) = \Gamma_{U^*(g)}(f).$$

Proof For $f \in L^p(G)$, $g \in L^q(G)$, and $h \in B_{\infty(G)}$, we have:

$$\begin{aligned}\left\langle U^*(hg), \overline{f^{p-1}}\right\rangle &= \left\langle hg, U(\overline{f^{p-1}})\right\rangle \\ &= \left\langle g, \overline{h}U(\overline{f^{p-1}})\right\rangle \\ &= \left\langle g, U(\overline{hf^{p-1}})\right\rangle \\ &= \left\langle U^*(g), \overline{hf^{p-1}}\right\rangle \\ &= \left\langle hU^*(g), \overline{f^{p-1}}\right\rangle.\end{aligned}$$

Therefore, U^* is $\phi(L)$-factorable. Now, we have:

$$\begin{aligned}\int_{G/\phi(L)} \Gamma_{g^{q-1}}(U(f))(\dot{x})d\dot{x} &= \left\langle U(f), \overline{g^{q-1}}\right\rangle \\ &= \left\langle f, U^*(\overline{g^{q-1}})\right\rangle \\ &= \int_{G/\phi(L)} \Gamma_{U^*(g)}(f)(\dot{x})\mathrm{d}\dot{x}.\end{aligned}$$

Therefore, Lemma 3.2 completes the proof. □

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