FGM Copula based Analysis of Wireless Communication Performances for Multi-User Channels

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Abstract. The dependence of fading coefficients of wireless communication channels on each other affects communication performances such as Outage probability (OP), coverage region, energy efficiency, and secrecy capacity, possibly being constructive or destructive. In this paper, the outage probability as one of the most important wireless communication performances is investigated by using Copula theory. For this purpose, a wireless three-user multiple access channel (MAC) with Rayleigh fading and independent sources is considered and the outage probability in positive and negative dependence cases is compared. The results show that a negative dependence structure reduces the outage probability (compared to the independent state), but a positive dependence structure increases it.

Keywords: multiple access channel , Rayleigh fading, dependence, Copula theory, outage probability

1 Introduction

According to the surveys, there are more than four billion wireless subscribers worldwide, and the demand for wireless communication services is increasing daily, so we need to improve the methods of using available spectrum sources. One way to increase spectral efficiency is to use multi-user and multi-antenna communication systems in wireless communications. Disturbances caused by radio propagation environments always affect the performance of wireless communication systems. Fading is one of the disturbances of wireless propagation environments. In a wireless fading channel, the channel coefficients are interdependent random variables for which different distributions have been proposed so far [2, 19, 23]. In recent years, there have been many studies about fading channels; in many of these studies, the fading coefficients of wireless communication channels have been assumed to be independent of each other.

It is essential to investigate the performance of wireless communication systems under the influence of the dependence of channel fading coefficients.

Copula theory can be used as a valuable and powerful tool for modeling the dependence between random variables. This theory was first introduced by Sklar in 1959 [21]. Copulas are functions that relate the multivariate distribution function to its marginal distribution functions. Copula theory is widely used in statistics, economics, image processing, machine learning, Internet of Things (IoT), and engineering [4, 16, 18, 20, 24-25]. Also, recently, Copula theory is

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used to evaluate the effect of dependence of wireless channel coefficients on wireless communication performances, including outage probability, coverage region, energy efficiency, and secrecy capacity [8-9, 17]. In [8], by using Copula theory, the effect of the dependence between the Rayleigh fading channel coefficients on the outage probability and the coverage region, two important communication performances, has been evaluated. In [9], the investigated channel is a doubly dirty fading MAC with non-causally known side information at transmitters; closed-form expressions for the outage probability and the coverage region have been obtained using the Copula theory. In [17], the channel coefficients have been considered interdependent, and a general closed-form expression has been obtained for the outage probability assuming an arbitrary fading distribution.

the Farlie-Gumbel-Morgenstern (FGM) Copulas, first studied by Eyraud, Farlie, Gumble, and Morgenstern [5-6, 10, 15], are a well-known family of Copulas and have many properties[3, 7, 11-13]. This family of Copulas has a simple form, and the dependence parameter of these Copulas includes positive and negative values and zero. Also, FGM Copulas are the simplest to calculate joint distributions. Due to these properties, these Copulas are suitable for analyzing wireless channels with dependent coefficients.

In this paper, we study a wireless three-user fading MAC with independent sources and coherent receiver (the receiver knows the channel coefficients). We consider the channel coefficients to be dependent on each other to investigate the effect of the dependence of channel coefficients on wireless communication performances. First, we obtain a closed-form expression for the outage probability using the FGM Copula; then, according to this closed-form expression, we investigate the effect of the dependence of the channel coefficients on the outage probability. To this end, we compare the outage probability in dependent and independent cases and evaluate the impact of positive and negative dependencies on the outage probability.

The structure of this paper is as follows: Copula theory is described in sections 2. Communication channel is described in sections 3. The outage probability is obtained in Section 4. Numerical results are in section 5 and the paper is concluded in section 6.

2 Copula Theory

In this section, we briefly review some definitions and theorems of the Coppola theory that are used in the following sections[16].

Definition 2.1 *A d-dimensional Copula is a functin* $C: [0,1]^d \to [0,1]$ *subject to:*

- C is a grounded function, that is:

$$C(u_1, \dots, u_d) = 0; \text{ if any } u_j = 0, j \in \{1, \dots, d\}$$

- The marginals of C are uniform, that is:

$$C(1, \dots, 1, u_j, 1, \dots, 1) = u_j; \quad \forall \quad j \in \{1, \dots, d\}$$

- C is d-increasing on $[0,1]^d$, that is:

$$\sum_{i_1=1}^{2} \dots \sum_{i_d=1}^{2} (-1)^{i_1+\dots+i_d} C(u_{1i_1}, \dots, u_{di_d}) \geq 0$$

For all $0 \le u_{j1} \le u_{j2} \le 1$ and $j \in \{1, \dots, d\}$.

Theorem 2.2 *Suppose F is a multivariate joint cumulative distribution function (CDF) with marginals $F_1, \dots, F_d$, then there exists a Copula, C, such that:*

$$F(x_1, \dots, x_d) = C\big(F_1(x_1), \dots, F_d(x_d)\big) \tag{2.1}$$

If F_i; $\forall i \in \{1, \dots, d\}$ is continuous, then C is unique, otherwise C is uniquely determined only on $Ran(F_1) \times \dots \times Ran(F_d)$. Conversely, consider a Copula, C, and univariate CDF's $F_1, \dots, F_d$, Then F defined in (2.1) is a multivariate CDF with marginals $F_1, \dots, F_d$.

Corollary 2.3 *The joint probability density function (PDF) corresponding to $F(x_1, \dots, x_d)$ is:*

$$f(x_1, \dots, x_d) = f_1(x_1) \dots f_d(x_d) c\big(F_1(x_1), \dots, F_d(x_d)\big) \tag{2.2}$$

Where $f_i(x_i); i \in \{1, \dots, d\}$ are the marginal PDFs of $f(x_1, \dots, x_d)$ and c is the Copula density function.

The density function of Copula $C(u_1, \dots, u_d)$ is given as:

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d} \tag{2.3}$$

Definition 2.4 *A d-dimensional FGM Copula is defined as [13]:*

$$C(u_1, \dots, u_d) = \left(\prod_{j=1}^{d} u_j\right)\left(1 + \sum_{k=2}^{d} \sum_{1 \le j_1 < \dots < j_k \le d} \theta_{j_1 \dots j_k} \overline{u}_{j1} \dots \overline{u}_{jk}\right) \tag{2.4}$$

Where $(u_1, \dots, u_d) \in [0,1]^d$ and $\overline{u}_j = 1 - u_j, j \in \{1, \dots, d\}$.

3 Communication Channel

We study a wireless three-user fading MAC with independent sources and coherent receiver that the channel coefficients are dependent on each other (Figure 1).

The received signal is:

$$Y_D = h_{1D} X_1 + h_{2D} X_2 + h_{3D} X_3 + Z_D \tag{3.1}$$

Where the signals sent by the first, second, and third transmitters are X_1, X_2 and X_3, respectively.

$h_{iD}; i \in \{1,2,3\}$ are the fading coefficients of the channel between the transmitter i and the receiver. Z_D is independent identically distributed (i.i.d) Additive White Gaussian Noise (AWGN) with zero mean and variance N.

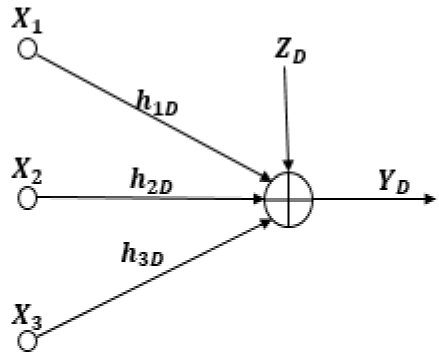

Figure 1: A three-user wireless Rayleigh fading MAC

The capacity region of a three-transmitter wireless MAC with block fading and coherent receiver is (extension of the capacity region of two-user MAC with independent sources [1, 14]):

$$R_1 \leq \frac{1}{2} log_2 \left(1 + \frac{P_1|h_1|^2}{N}\right)$$

$$R_2 \leq \frac{1}{2} log_2 \left(1 + \frac{P_2|h_2|^2}{N}\right)$$

$$R_3 \leq \frac{1}{2} log_2 \left(1 + \frac{P_3|h_3|^2}{N}\right)$$

$$R_1 + R_2 \leq \frac{1}{2} log_2 \left(1 + \frac{P_1|h_1|^2 + P_2|h_2|^2}{N}\right)$$

$$R_1 + R_3 \leq \frac{1}{2} log_2 \left(1 + \frac{P_1|h_1|^2 + P_3|h_3|^2}{N}\right)$$

$$R_2 + R_3 \leq \frac{1}{2} log_2 \left(1 + \frac{P_2|h_2|^2 + P_3|h_3|^2}{N}\right)$$

$$R_1 + R_2 + R_3 \leq \frac{1}{2} log_2 \left(1 + \frac{P_1|h_1|^2 + P_2|h_2|^2 + P_3|h_3|^2}{N}\right) \quad (3.2)$$

Where R_1, R_2 and R_3 are the desired transmission rates of the first, second, and third transmitters, respectively.

4 Outage Probability

The outage probability of a three-user wireless Rayleigh correlated fading MAC is:

$$P_{out} = 1 - P^c \tag{4.1}$$

$$\begin{aligned} P^c = {} & \mathcal{A}_1 + \theta_{12}(\mathcal{A}_1 - 2\mathcal{A}_2 - 2\mathcal{A}_3 + 4\mathcal{A}_4) + \theta_{13}(\mathcal{A}_1 - 2\mathfrak{B}_1 - 2\mathcal{A}_3 + 4\mathfrak{B}_3) \\ & + \theta_{23}(\mathcal{A}_1 - 2\mathfrak{B}_1 - 2\mathcal{A}_2 + 4\mathfrak{B}_2) \\ & + \theta_{123}(\mathcal{A}_1 - 2\mathcal{A}_2 - 2\mathcal{A}_3 + 4\mathcal{A}_4 - 2\mathfrak{B}_1 + 4\mathfrak{B}_2 + 4\mathfrak{B}_3 - 8\mathfrak{B}_4) \end{aligned} \tag{4.2}$$

Where in the equation (4.2) we have:

$$\mathcal{A}_1 = \frac{(\overline{v}_1)^2 e^{-\frac{\mathcal{T}}{\overline{v}_1}}}{(\overline{v}_1 - \overline{v}_2)(\overline{v}_1 - \overline{v}_3)} \tag{4.3}$$

$$\mathcal{A}_2 = \frac{(\overline{v}_1)^2 e^{-\frac{\mathcal{T}}{\overline{v}_1}}}{(2\overline{v}_1 - \overline{v}_2)(\overline{v}_1 - \overline{v}_3)} \tag{4.4}$$

$$\mathcal{A}_3 = \frac{(\overline{v}_1)^2 e^{-\frac{2\mathcal{T}}{\overline{v}_1}}}{2(\overline{v}_1 - 2\overline{v}_2)(\overline{v}_1 - 2\overline{v}_3)} \tag{4.5}$$

$$\mathcal{A}_4 = \frac{(\overline{v}_1)^2 e^{-\frac{2\mathcal{T}}{\overline{v}_1}}}{4(\overline{v}_1 - \overline{v}_2)(\overline{v}_1 - 2\overline{v}_3)} \tag{4.6}$$

$$\mathfrak{B}_1 = \frac{(\overline{v}_1)^2 e^{-\frac{\mathcal{T}}{\overline{v}_1}}}{(\overline{v}_1 - \overline{v}_2)(2\overline{v}_1 - \overline{v}_3)} \tag{4.7}$$

$$\mathfrak{B}_2 = \frac{(\overline{v}_1)^2 e^{-\frac{\mathcal{T}}{\overline{v}_1}}}{(2\overline{v}_1 - \overline{v}_2)(2\overline{v}_1 - \overline{v}_3)} \tag{4.8}$$

$$\mathfrak{B}_3 = \frac{(\overline{v}_1)^2 e^{-\frac{2\mathcal{T}}{\overline{v}_1}}}{4(\overline{v}_1 - 2\overline{v}_2)(\overline{v}_1 - \overline{v}_3)} \tag{4.9}$$

$$\mathfrak{B}_4 = \frac{(\overline{v}_1)^2 e^{-\frac{2\mathcal{T}}{\overline{v}_1}}}{8(\overline{v}_1 - \overline{v}_2)(\overline{v}_1 - \overline{v}_3)} \tag{4.10}$$

In the above equations, $\overline{v}_1$, $\overline{v}_2$, and $\overline{v}_3$ are the average signal-to-noise ratios (SNR) at transmitters t_1, t_2, and t_3, respectively and $\mathcal{T} = 2^{2R_0} - 1$ that R_0 represents the total required threshold information rates.

Proof. Outage probability is the probability that the information rate is greater than the random capacity of the channel or less than a required threshold information rate. According to this definition and considering that any of the inequalities in equation (3.2) can be used to calculate the outage probability, we have:

$$P_{out} = Pr(R_1 + R_2 + R_3 \leq R_0) \quad (4.11)$$

$$= 1 - Pr(R_1 + R_2 + R_3 > R_0) \quad (4.12)$$

$$= 1 - P^c \quad (4.13)$$

Where P^c is the complement of the outage probability.

$$P^c = Pr\left(\frac{1}{2}\log_2\left(1 + \frac{P_1|h_1|^2 + P_2|h_2|^2 + P_3|h_3|^2}{N}\right) > R_0\right) \quad (4.14)$$

$$= Pr\left(\frac{P_1|h_1|^2 + P_2|h_2|^2 + P_3|h_3|^2}{N} > 2^{2R_0} - 1\right) \quad (4.15)$$

$$= Pr(\nu_1 + \nu_2 + \nu_3 > \mathcal{T}) \quad (4.16)$$

$$= \int_0^\infty \int_0^\infty \int_{\mathcal{T}-\nu_2-\nu_3}^\infty f(\nu_1, \nu_2, \nu_3) d\nu_1 d\nu_2 d\nu_3 \quad (4.17)$$

Where ν_1, ν_2 and ν_3 are SNRs at transmitters t_1, t_2, and t_3, respectively and we have:

$$\nu_i = \frac{P_i|h_i|^2}{N}; i \in \{1,2,3\} \quad (4.18)$$

$f(\nu_1, \nu_2, \nu_3)$ in equation (4.17) is the joint PDF of ν_1, ν_2 and ν_3, and we can calculate it according to equation (2.2).

$$f(\nu_1, \nu_2, \nu_3) = f_1(\nu_1) f_2(\nu_2) f_3(\nu_3) c\big(F_1(\nu_1), F_2(\nu_2), F_3(x_3)\big) \quad (4.19)$$

Where $f(\nu_i); i \in \{1,2,3\}$ and $F(\nu_i); i \in \{1,2,3\}$ are PDFs and CDFs of $\nu_i; i \in \{1,2,3\}$, respectively and $c\big(F_1(\nu_1), F_2(\nu_2), F_3(x_3)\big)$ is the density function of three-dimensional FGM Copula.

The channel coefficients, $h_i; i \in \{1,2,3\}$, have a Rayleigh distribution, so $|h_i|^2; i \in \{1,2,3\}$ and consequently $\nu_i; i \in \{1,2,3\}$ have an exponential distribution.

$$f(\nu_i) = \frac{1}{\bar{\nu}_i} exp\left(-\frac{\nu_i}{\bar{\nu}_i}\right); i \in \{1,2,3\} \quad (4.20)$$

$$F(\nu_i) = 1 - exp\left(-\frac{\nu_i}{\bar{\nu}_i}\right); i \in \{1,2,3\} \quad (4.21)$$

Now we calculate $c\big(F_1(\nu_1), F_2(\nu_2), F_3(x_3)\big)$. Considering $d = 3$ in equation (2.4), three-dimensional FGM Copula is obtained as:

$$C(u_1,u_2,u_3)=u_1u_2u_3(1+\theta_{12}\overline{u}_1\overline{u}_2+\theta_{13}\overline{u}_1\overline{u}_3+\theta_{23}\overline{u}_2\overline{u}_3+\theta_{123}\overline{u}_1\overline{u}_2\overline{u}_3) \qquad (4.22)$$

Where $\overline{u}_i=1-u_i; i\in\{1,2,3\}$ and θ_{12}, θ_{13}, θ_{23} and θ_{123} are the FGM Copula parameters. According to (2.3) and (4.22), the density function of the three-dimensional FGM Copula is:

$$\begin{aligned}c(u_1,u_2,u_3)=1&+\theta_{12}(1-2u_1)(1-2u_2)+\theta_{13}(1-2u_1)(1-2u_3)\\&+\theta_{23}(1-2u_2)(1-2u_3)+\theta_{123}(1-2u_1)(1-2u_2)(1-2u_3)\end{aligned} \qquad (4.23)$$

Now according to (4.19), (4.20) and (4.23), $f(\nu_1,\nu_2,\nu_3)$ is obtained as follows:

$$\begin{aligned}f(\nu_1,\nu_2,\nu_3)=\frac{e^{-\frac{\nu_1}{\overline{\nu}_1}-\frac{\nu_2}{\overline{\nu}_2}-\frac{\nu_3}{\overline{\nu}_3}}}{\overline{\nu}_1\overline{\nu}_2\overline{\nu}_3}\Bigg[&1+\theta_{12}\left(1-2e^{-\frac{\nu_1}{\overline{\nu}_1}}\right)\left(1-2e^{-\frac{\nu_2}{\overline{\nu}_2}}\right)\\&+\theta_{13}\left(1-2e^{-\frac{\nu_1}{\overline{\nu}_1}}\right)\left(1-2e^{-\frac{\nu_3}{\overline{\nu}_3}}\right)+\theta_{23}\left(1-2e^{-\frac{\nu_2}{\overline{\nu}_2}}\right)\left(1-2e^{-\frac{\nu_3}{\overline{\nu}_3}}\right)\\&+\theta_{123}\left(1-2e^{-\frac{\nu_1}{\overline{\nu}_1}}\right)\left(1-2e^{-\frac{\nu_2}{\overline{\nu}_2}}\right)\left(1-2e^{-\frac{\nu_3}{\overline{\nu}_3}}\right)\Bigg]\end{aligned} \qquad (4.24)$$

By putting (4.24) in (4.17), it is easy to calculate the triple integral and the outage probability is obtained as (4.2)-(4.10) and the proof is complete. □

5 Numerical Results

Numerical results are presented in this section. According to these results, we can investigate the effect of positive and negative dependencies on the outage probability performance.

In Figure 2, the outage probability is plotted in terms of average SNR. According to this figure, as the SNR increases, the channel condition improves, so the outage probability decreases. Also, we see that the negative dependence structure reduces the outage probability compared to the independent case, which means an improvement in the outage probability performance.

Conversely, we see that the positive dependence structure increases the outage probability compared to the independent case, that is, positive dependence has a detrimental effect on the outage probability performance.

6 Conclusion

In this paper, wireless three-user MAC with independent sources and Rayleigh fading was investigated. Using the FGM Copula, a closed form expression for the outage probability was obtained. Then we analyzed the impact of positive and negative dependencies on the outage probability performance. According to the obtained results, it is clear that negative dependence, compared to the independent state, reduces the outage probability, while positive dependency increases the outage probability compared to the non-dependent case.

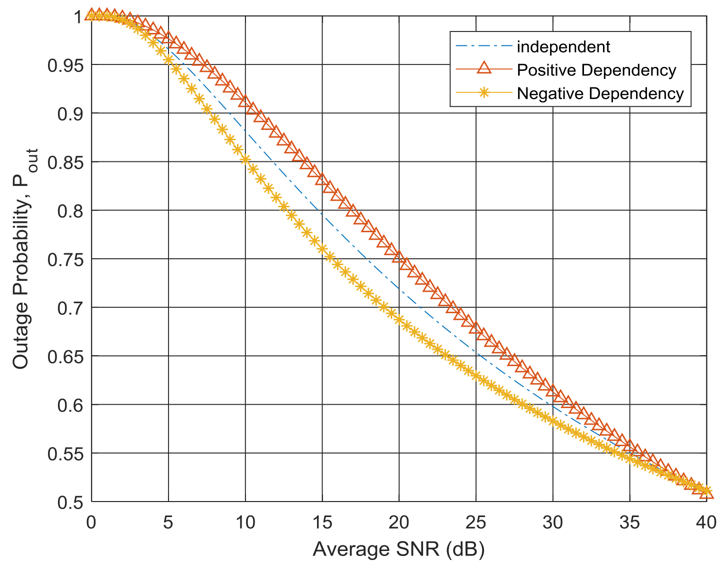

Figure 2: Outage probability versus average SNR

References

[1] Ahlswede, R. (1973). Multi-way communication channels.

[2] Atapattu, S., Tellambura, C., & Jiang, H. (2011). A mixture gamma distribution to model the SNR of wireless channels. IEEE transactions on wireless communications, 10(12), 4193-4203.

[3] Cambanis, S. (1977). Some properties and generalizations of multivariate Eyraud-Gumbel-Morgenstern distributions. Journal of Multivariate Analysis, 7(4), 551-559.

[4] Cherubini, U., Luciano, E., & Vecchiato, W. (2004). Copula methods in finance. John Wiley & Sons.

[5] Eyraud, H. (1936). Les principes de la mesure des correlations. Ann. Univ. Lyon, III. Ser., Sect. A, 1(30-47), 111.

[6] Farlie, D. J. (1960). The performance of some correlation coefficients for a general bivariate distribution. Biometrika, 47(3/4), 307-323.

[7] Genest, C., & Favre, A.-C. (2007). Everything you always wanted to know about copula modeling but were afraid to ask. Journal of hydrologic engineering, 12(4), 347-368.

[8] Ghadi, F. R., & Hodtani, G. A. (2020). Copula function-based analysis of outage probability and coverage region for wireless multiple access communications with correlated fading channels. IET Communications, 14(11), 1804-1810.

[9] Ghadi, F. R., Hodtani, G. A., & López-Martínez, F. J. (2021). The role of correlation in the doubly dirty fading mac with side information at the transmitters. IEEE Wireless Communications Letters, 10(9), 2070-2074.

[10] Gumbel, E. J. (1960). Bivariate exponential distributions. Journal of the American Statistical Association, 55(292), 698-707.

[11] Johnson, N. L., & Kott, S. (1975). On some generalized farlie-gumbel-morgenstern distributions. Communications in Statistics-Theory and Methods, 4(5), 415-427.

[12] Kotz, S., Balakrishnan, N., & Johnson, N. L. (2004). Continuous multivariate distributions, Volume 1: Models and applications (Vol. 1). John Wiley & Sons.

[13] Kotz, S., & Drouet, D. (2001). Correlation and dependence. World Scientific.

[14] Liao, H. (1972). Multiple Access Channels Ph. D Thesis, Department of Electrical Engineering, University of Hawaii, Honolulu].

[15] Morgenstern, D. (1956). Einfache beispiele zweidimensionaler verteilungen. Mitt, Math, Statist., 8, 234-235.

[16] Nelsen, R. B. (2007). An introduction to copulas. Springer Science & Business Media.

[17] Rostamighadi, F., & Abed Hodtani, G. (2022). Copula-based Analysis of Interference Channels: Outage Probability. Iran Workshop on Communication and Information theory,

[18] Salvadori, G., De Michele, C., Kottegoda, N., & Rosso, R. (2007). Extremes in nature, Water Sci. and Technol. Libr. Ser., vol. 56. In: Springer, Dordrecht, Netherlands.

[19] Selim, B., Alhussein, O., Muhaidat, S., Karagiannidis, G. K., & Liang, J. (2015). Modeling and analysis of wireless channels via the mixture of Gaussian distribution. IEEE Transactions on Vehicular technology, 65(10), 8309-8321.

[20] Shemyakin, A., & Kniazev, A. (2017). Introduction to Bayesian estimation and copula models of dependence. John Wiley & Sons.

[21] Sklar, M. (1959). Fonctions de repartition an dimensions et leurs marges. Publ. inst. statist. univ. Paris, 8, 229-231.

[22] Tse, D., & Viswanath, P. (2005). Fundamentals of wireless communication. Cambridge university press.

[23] Zeng, X., Ren, J., Wang, Z., Marshall, S., & Durrani, T. (2014). Copulas for statistical signal processing (Part I): Extensions and generalization. Signal processing, 94, 691-702.

[24] Zheng, C., Egan, M., Clavier, L., Peters, G. W., & Gorce, J.-M. (2019). Copula-based interference models for IoT wireless networks. ICC 2019-2019 IEEE International Conference on Communications (ICC),