

## SOME NOTES ON SUBGROUP TOPOLOGIES

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**ABSTRACT.** In this talk, by reviewing the concept of subgroup topology on a group and its applications to fundamental groups and covering maps, we investigate more results for subgroup topologies. Among presenting some properties, we present a necessary and sufficient condition on a subgroup topology to make a topological group.

### 1. INTRODUCTION AND MOTIVATION

The notion of *Subgroup topology* was introduced by Bogley and Sieradski [2] in order to study some properties of the universal path spaces. Also, Wilkins [5] defined a covering topology on the fundamental group on a topological space  $X$ ,  $\pi_1(X)$ , in order to study universal covers of geodesic spaces. Moreover, Abdullahi et al. [1] using the concept of subgroup topology introduced and studied some topologies on the fundamental group and used them to classify coverings, semicoverings, and generalized coverings of a topological space.

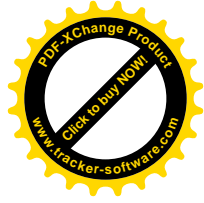
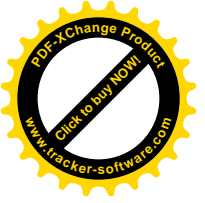
A collection  $\Sigma$  of subgroups of a group  $G$  is called a *neighbourhood family* if for any  $H, K \in \Sigma$ , there is a subgroup  $S \in \Sigma$  such that  $S \subseteq H \cap K$ . As a result of this property, the collection of all left

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cosets of elements of  $\Sigma$  forms a basis for a topology on  $G$ , which is called *the subgroup topology* determined by  $\Sigma$ , and we denote it by  $G^\Sigma$ . Bogley and Sieradski [2] focused on some general properties of subgroup topologies and by introducing the intersection  $S_\Sigma = \cap\{H \mid H \in \Sigma\}$ , called the *infinitesimal subgroup* for the neighbourhood family  $\Sigma$ , they showed that the closure of the element  $g \in G$  is the coset  $gS_\Sigma$ .

Let  $H$  be a subgroup of a group  $G$ . Then we define  $\Sigma^H$  as follows:

$$\Sigma^H = \{K \leq G \mid H \subseteq K\}.$$

It is easy to see that  $\Sigma^H$  is a neighbourhood family. We consider the subgroup topology on  $G$  determined by  $\Sigma^H$  and denote it by  $G^H$ . Note that the infinitesimal subgroup for the neighbourhood family  $\Sigma^H$  is  $H$ . Some famous neighbourhood families of the fundamental group  $\pi_1(X, x_0)$  are as follows (see [1]):

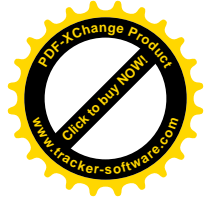
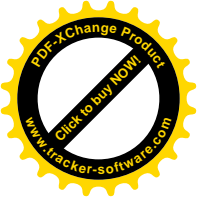
- $\Sigma^{Span}$ : The collection of all Spanier subgroups  $\pi(\mathcal{U}, x_0)$  of the fundamental group  $\pi_1(X, x_0)$  (see [1]).
- $\Sigma^{pSpan}$ : The collection of all path Spanier subgroups of the fundamental group  $\pi_1(X, x_0)$  (see [1]).
- $\Sigma^{cov}$ : The collection of all generalized covering subgroups of  $\pi_1(X, x_0)$  (see [3]).
- $\Sigma^{wh}$ : It is known that the whisker topology on the fundamental group,  $\pi_1^{wh}(X, x_0)$ , is a subgroup topology with respect to  $\Sigma^{wh} = \{i_*\pi_1(U, x_0) \mid U \text{ is an open neighborhood of } X \text{ at } x_0\}$  (see [1]).
- $\Sigma_K$ : For a compact geodesic space  $X$ , Wilkins [5] by defining a neighbourhood family of subgroups of  $\pi_1(X)$ ,  $\Sigma_K$ , introduced a subgroup topology on the fundamental group of  $X$ , denoted by  $\pi_1^c(X)$ , which is called covering topology on  $\pi_1(X)$ .

In Section 2, by reviewing some known results on subgroup topology, we investigate more results for subgroup topologies. Among presenting some properties, we present a necessary and sufficient condition on a subgroup topology to make a topological group.

## 2. MAIN RESULTS

First, we review some basic properties of subgroup topologies (see Bogley and Sieradski [2]). Let the group  $G$  have the subgroup topology determined by a neighbourhood family  $\Sigma$  of subgroups of  $G$ . Then the following statements hold.

- (a) The infinitesimal subgroup  $S_\Sigma$  is a maximal indiscrete subspace of  $G$ .
- (b) Given  $g \in G$ , the connected component of  $g$  in  $G$  is the coset  $gS_\Sigma$ .



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- (c) The group  $G$  is discrete if and only if  $\Sigma$  contains the trivial group.
- (d) When the group  $G$  is totally disconnected but not discrete, it is perfect in the topological sense that each element of  $G$  is an accumulation point of  $G$ .
- (e) When the group  $G$  is not totally disconnected, it consists of non-trivial indiscrete components, which are the cosets  $gS_\Sigma$  of the infinitesimal subgroup.

Also, the following statements are equivalent:

- (i)  $G$  is Hausdorff;
- (ii)  $G$  satisfies the  $T_0$  separation property;
- (iii) The infinitesimal subgroup  $S_\Sigma$  is trivial;
- (iv)  $G$  is totally disconnected.

The following result can be obtained easily.

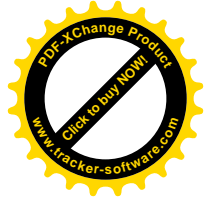
**Theorem 2.1.** *Let  $H \leq G$ ; then  $G^H$  is discrete if and only if  $H = 1$ . Also,  $G^H$  is indiscrete if and only if  $H = G$ .*

Using [2, Theorem 2.9] and the above theorem, we have the following result.

**Theorem 2.2.** *Let  $H$  be a subgroup of  $G$ ; then the following statements are equivalent:*

- (i)  $G^H$  is Hausdorff.
- (ii)  $G^H$  is  $T_0$ .
- (iii)  $G^H$  is totally disconnected.
- (iv)  $G^H$  is discrete.
- (v)  $H$  is the trivial subgroup.

It is pointed out in [2] that although the group  $G$  equipped with a subgroup topology may not necessarily be a topological group, in general (it may not even be a quasitopological group), because right translation maps by a fixed element of  $G$  need not be continuous, but it has some of properties of topological groups (for more details, see [2, Theorem 2.9]). Wilkins [5, Lemma 5.4] showed that a group  $G$  with the subgroup topology determined by a neighbourhood family  $\Sigma$  is a topological group when all subgroups in  $\Sigma$  are normal. Moreover, it is proved in [1, Corollary 2.2] that a group equipped with a subgroup topology is a topological group if and only if all right translation maps are continuous. A *Dedekind group* is a group  $G$  such that every subgroup of  $G$  is normal. Clearly all abelian groups are Dedekind groups. A non-abelian Dedekind group is called a Hamiltonian group. It is proved that every Hamiltonian group is a direct product of the form  $Q_8 \times B \times D$ , where  $Q_8$  is the quaternion group,  $B$  is an elementary abelian 2-group, and  $D$  is a torsion abelian group with all elements



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of odd order (see [4, p.190]). Using these facts, we have the following result.

**Theorem 2.3.** *Let  $H$  be a normal subgroup of  $G$  such that the quotient group  $G/H$  is a Dedekind group. Then  $G^H$  is a topological group.*

In the following theorem, we vastly extend the above result.

**Theorem 2.4.** *Let  $G$  be a group and let  $\Sigma$  be a neighbourhood family on  $G$  such that  $S_\Sigma \in \Sigma$ . Then  $G^\Sigma$  is a topological group if and only if  $S_\Sigma$  is a normal subgroup of  $G$ . In particular,  $G^H$  is a topological group if and only if  $H$  is a normal subgroup of  $G$ .*

In the following, we intend to compare two subgroup topologies.

**Theorem 2.5.** *Let  $G$  be a group and let  $\Sigma$  and  $\Sigma'$  be two neighbourhood families on  $G$  such that  $G^\Sigma = G^{\Sigma'}$ . Then  $S_\Sigma = S_{\Sigma'}$ .*

The following theorem shows that the converse of the above result holds under a condition.

**Theorem 2.6.** *Let  $G$  be a group and let  $\Sigma$  and  $\Sigma'$  be two neighbourhood families on  $G$  such that  $S_\Sigma \leq S_{\Sigma'}$  and  $S_\Sigma \in \Sigma$ . Then  $G^{\Sigma'} \preceq G^\Sigma$ . In particular, if  $H \leq K \leq G$ , then  $G^K \preceq G^H$ .*

It is known that in every left (right) topological groups, any open subgroup is closed but the converse does not hold, in general. Note that by [1, Proposition 2.4] one can show that the converse holds for subgroup topology if the infinitesimal subgroup is an open subgroup. Hence every closed subgroup of  $G^H$  is also open.

In the following theorem, we consider the subgroup topology on the direct product of groups.

**Theorem 2.7.** *Let  $G$  and  $G'$  are two groups, and let  $H$  and  $H'$  be subgroups of  $G$  and  $G'$ , respectively. Then  $G^H \times G'^{H'} \cong (G \times G')^{H \times H'}$ .*

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