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SOME NOTES ON SUBGROUP TOPOLOGIES

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Abstract. In this talk, by reviewing the concept of subgroup topology on a group and its applications to fundamental groups and covering maps, we investigate more results for subgroup topologies. Among presenting some properties, we present a necessary and sufficient condition on a subgroup topology to make a topological group.

1. Introduction and Motivation

The notion of *Subgroup topology* was introduced by Bogley and Sieradski [2] in order to study some properties of the universal path spaces. Also, Wilkins [5] defined a covering topology on the fundamental group on a topological space X, $\pi_1(X)$, in order to study universal covers of geodesic spaces. Moreover, Abdullahi et al. [1] using the concept of subgroup topology introduced and studied some topologies on the fundamental group and used them to classify coverings, semicoverings, and generalized coverings of a topological space.

A collection Σ of subgroups of a group G is called a *neighbourhood family* if for any $H, K \in \Sigma$, there is a subgroup $S \in \Sigma$ such that $S \subseteq H \cap K$. As a result of this property, the collection of all left

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cosets of elements of Σ forms a basis for a topology on G, which is called *the subgroup topology* determined by Σ, and we denote it by G^{Σ}. Bogley and Sieradski [2] focused on some general properties of subgroup topologies and by introducing the intersection $S_{\Sigma} = \cap\{H \mid H \in \Sigma\}$, called the *infinitesimal subgroup* for the neighbourhood family Σ, they showed that the closure of the element $g \in G$ is the coset gS_{Σ}.

Let H be a subgroup of a group G. Then we define Σ^H as follows:

$$\Sigma^H = \{K \leqslant G \mid H \subseteq K\}.$$

It is easy to see that Σ^H is a neighbourhood family. We consider the subgroup topology on G determined by Σ^H and denote it by G^H. Note that the infinitesimal subgroup for the neighbourhood family Σ^H is H. Some famous neighbourhood families of the fundamental group $\pi_1(X, x_0)$ are as follows (see [1]):

Σ^{Span}: The collection of all Spanier subgroups $\pi(\mathcal{U}, x_0)$ of the fundamental group $\pi_1(X, x_0)$ (see [1]).

Σ^{pSpan}: The collection of all path Spanier subgroups of the fundamental group $\pi_1(X, x_0)$ (see [1]).

Σ^{gcov}: The collection of all generalized covering subgroups of $\pi_1(X, x_0)$ (see [3]).

Σ^{wh}: It is known that the whisker topology on the fundamental group, $\pi_1^{wh}(X, x_0)$, is a subgroup topology with respect to $\Sigma^{wh} = \{i_*\pi_1(U, x_0) | \text{U is an open neighborhood of X at } x_0\}$ (see [1]).

Σ_K: For a compact geodesic space X, Wilkins [5] by defining a neighbourhood family of subgroups of $\pi_1(X)$, Σ_K, introduced a subgroup topology on the fundamental group of X, denoted by $\pi_1^{\mathcal{C}}(X)$, which is called covering topology on $\pi_1(X)$.

In Section 2, by reviewing some known results on subgroup topology, we investigate more results for subgroup topologies. Among presenting some properties, we present a necessary and sufficient condition on a subgroup topology to make a topological group.

2. Main results

First, we review some basic properties of subgroup topologies (see Bogley and Sieradski [2]). Let the group G have the subgroup topology determined by a neighbourhood family Σ of subgroups of G. Then the following statements hold.

(a) The infinitesimal subgroup S_{Σ} is a maximal indiscrete subspace of G.

(b) Given $g \in G$, the connected component of g in G is the coset gS_{Σ}.

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(c) The group G is discrete if and only if Σ contains the trivial group.
(d) When the group G is totally disconnected but not discrete, it is perfect in the topological sense that each element of G is an accumulation point of G.
(e) When the group G is not totally disconnected, it consists of non-trivial indiscrete components, which are the cosets gS_Σ of the infinitesimal subgroup.

Also, the following statements are equivalent:
(i) G is Hausdorff;
(ii) G satisfies the T_0 separation property;
(iii) The infinitesimal subgroup S_Σ is trivial;
(iv) G is totally disconnected.

The following result can be obtained easily.

Theorem 2.1. *Let $H \leqslant G$; then G^H is discrete if and only if $H = 1$. Also, G^H is indiscrete if and only if $H = G$.*

Using [2, Theorem 2.9] and the above theorem, we have the following result.

Theorem 2.2. *Let H be a subgroup of G; then the following statements are equivalent:*
(i) *G^H is Hausdorff.*
(ii) *G^H is T_0.*
(iii) *G^H is totally disconnected.*
(iv) *G^H is discrete.*
(v) *H is the trivial subgroup.*

It is pointed out in [2] that although the group G equipped with a subgroup topology may not necessarily be a topological group, in general (it may not even be a quasitopological group), because right translation maps by a fixed element of G need not be continuous, but it has some of properties of topological groups (for more details, see [2, Theorem 2.9]). Wilkins [5, Lemma 5.4] showed that a group G with the subgroup topology determined by a neighbourhood family Σ is a topological group when all subgroups in Σ are normal. Moreover, it is proved in [1, Corollary 2.2] that a group equipped with a subgroup topology is a topological group if and only if all right translation maps are continuous. A *Dedekind group* is a group G such that every subgroup of G is normal. Clearly all abelian groups are Dedekind groups. A non-abelian Dedekind group is called a Hamiltonian group. It is proved that every Hamiltonian group is a direct product of the form $Q_8 \times B \times D$, where Q_8 is the quaternion group, B is an elementary abelian 2-group, and D is a torsion abelian group with all elements

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of odd order (see [4, p.190]). Using these facts, we have the following result.

Theorem 2.3. *Let H be a normal subgroup of G such that the quotient group G/H is a Dedekind group. Then G^H is a topological group.*

In the following theorem, we vastly extend the above result.

Theorem 2.4. *Let G be a group and let Σ be a neighbourhood family on G such that $S_\Sigma \in \Sigma$. Then G^Σ is a topological group if and only if S_Σ is a normal subgroup of G. In particular, G^H is a topological group if and only if H is a normal subgroup of G.*

In the following, we intend to compare two subgroup topologies.

Theorem 2.5. *Let G be a group and let Σ and Σ' be two neighbourhood families on G such that $G^\Sigma = G^{\Sigma'}$. Then $S_\Sigma = S_{\Sigma'}$.*

The following theorem shows that the converse of the above result holds under a condition.

Theorem 2.6. *Let G be a group and let Σ and Σ' be two neighbourhood families on G such that $S_\Sigma \leq S_{\Sigma'}$ and $S_\Sigma \in \Sigma$. Then $G^{\Sigma'} \preccurlyeq G^\Sigma$. In particular, if $H \leq K \leq G$, then $G^K \preccurlyeq G^H$.*

It is known that in every left (right) topological groups, any open subgroup is closed but the converse does not hold, in general. Note that by [1, Proposition 2.4] one can show that the converse holds for subgroup topology if the infinitesimal subgroup is an open subgroup. Hence every closed subgroup of G^H is also open.

In the following theorem, we consider the subgroup topology on the direct product of groups.

Theorem 2.7. *Let G and G' are two groups, and let H and H' be subgroups of G and G', respectively. Then $G^H \times {G'}^{H'} \cong (G \times G')^{H \times H'}$.*

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