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A GENERALIZATION OF COVERING MAPS WITH RESPECT TO A NEIGHBORHOOD FAMILY OF SUBGROUPS OF THE FUNDAMENTAL GROUP

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Abstract. In this talk, we define a generalized covering map for a topological space X with respect to a neighborhood family of subgroups of the fundamental group $\pi_1(X, x_0)$. By considering some famous subgroups of the fundamental group $\pi_1(X, x_0)$ as neighborhood families, we compare these generalized covering maps with each other according to their neighborhood families. In particular, we compare these generalized covering maps with the most famous notion, coverings, semicoverings, and generalized coverings.

1. Introduction and motivation

A continuous map $p : \widetilde{X} \to X$ is called a covering map if every point of X has an open neighborhood that is evenly covered by p. Some researchers extended the notion of covering maps to some notion such as rigid coverings, semicoverings [3], and generalized coverings [4, 6]. These generalizations focus on keeping some properties of covering maps and eliminating the evenly covered property. Brazas [3] defined semicoverings by removing the evenly covered property and keeping

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local homeomorphism and the unique path and homotopy lifting properties. For generalized coverings, the local homeomorphism is replaced with the unique lifting property (see [4, 6]). In each case, one of the interesting problems is to classify covering, semicovering, and generalized covering spaces with respect to subgroups of the fundamental group. A subgroup H of the fundamental group $\pi_1(X, x)$ is called covering, semicovering, and generalized covering subgroup if there is a covering, semicovering, and generalized covering map $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ such that $H = p_*\pi_1(\widetilde{X}, \tilde{x})$, respectively.

A collection Σ of subgroups of a group G is called a *neighborhood family* if for any $H, K \in \Sigma$, there is a subgroup $S \in \Sigma$ such that $S \subseteq H \cap K$. As a result of this property, the collection of all left cosets of elements of Σ forms a basis for a topology on G, which is called *the subgroup topology* determined by Σ, and we denote it by G^{Σ}. The subgroup topology on a group G specified by a neighborhood family was defined in [5, Section 2.5] and considered by some recent researchers such as [1]. Bogley and Sieradski [5] defined *infinitesimal* subgroup for the neighborhood family Σ, the intersection $\cap\{H \mid H \in \Sigma\}$ and denoted it by S_{Σ}.

Let H be a subgroup of a group G. Then we define $\Sigma^H = \{K \leqslant G \mid H \subseteq K\}$. It is easy to see that Σ^H is a neighborhood family. We consider the subgroup topology on G determined by Σ^H and denote it by G^H. Note that the infinitesimal subgroup for the neighborhood family Σ^H is H. Some famous neighborhood families of the fundamental group $\pi_1(X, x_0)$ are as follows:
$\Sigma^{Span} = \{\pi(\mathcal{U}, x_0) \mid \mathcal{U}$ is an open cover of $X\}$, where $\pi(\mathcal{U}, x_0)$ is the Spanier subgroup (see [1]).
$\Sigma^{pSpan} = \{\widetilde{\pi}(\mathcal{V}, x_0) \mid \mathcal{V}$ is a path open cover of $X\}$, where $\widetilde{\pi}(\mathcal{V}, x_0)$ is the path Spanier subgroup (see [1]).
Σ^{gcov} is the collection of all generalized covering subgroups of $\pi_1(X, x_0)$. Note that the intersection of all generalized covering subgroups is also a generalized covering subgroup and denoted by $\pi_1^{gc}(X, x_0)$ (see [4, Theorem 2.36]).
$\Sigma^s = \{H \leq \pi_1(X, x_0) \mid \pi_1^s(X, x_0) \subseteq H\}$, where $\pi_1^s(X, x_0)$ is a small subgroup, the collection of all small loops at x_0.
$\Sigma^{sg} = \{H \leq \pi_1(X, x_0) \mid \pi_1^{sg}(X, x_0) \subseteq H\}$, where $\pi_1^{sg}(X, x_0)$ is a small generated subgroup.

If (X, x_0) is a locally path connected space and $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, then there exists the following chain of famous subgroups of $\pi_1(X, x_0)$ (see [2, Theorem 2.6]):

$$\{e\} \leq \pi_1^s(X, x_0) \leq \pi_1^{sg}(X, x_0) \leq \pi_1^{gc}(X, x_0) \leq \overline{\pi_1^{sg}(X, x_0)}$$

$$\leq \widetilde{\pi}_1^{sp}(X, x_0) \leq \pi_1^{sp}(X, x_0) \leq p_*(\widetilde{X}, \widetilde{x}_0) \leq \pi_1(X, x_0).$$

In this talk, we define a Σ-covering map for a neighborhood family Σ of subgroups of $\pi_1(X, x_0)$. Also, we compare these generalized covering maps with each other according to their neighborhood families. In particular, we compare these generalized covering maps with the most famous notion, coverings, semicoverings, and generalized coverings.

2. Main results

Definition 2.1. Let Σ be a neighborhood family of subgroups of the fundamental group $\pi_1(X, x_0)$. Then we call a map $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ a Σ-*covering map* for X if and only if p is an onto continuous map and there exists $K \in \Sigma$ such that $K \subseteq \pi_1(p)(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

First, we investigate Σ-covering maps for some famous neighborhood families of subgroups of the fundamental group $\pi_1(X, x_0)$ and then compare these generalized covering maps with the most famous notion, coverings, semicoverings, and generalized coverings.

Theorem 2.2. *If X is a connected, locally path connected space and $\Sigma = \Sigma^{Span}$, then Σ^{Span}-covering maps are the same of usual covering maps.*

Theorem 2.3. *If X is a connected locally path connected space and $\Sigma = \Sigma^{pSpan}$, then Σ^{pSpan}-covering maps are equivalent to semicovering maps.*

Theorem 2.4. *If $\Sigma = \Sigma^{gcov}$, then every generalized covering map is a Σ^{gcov}-covering map.*

Remark 2.5. If $H \leq K \leq \pi_1(X, x_0)$, then it is clear $\Sigma^K \subseteq \Sigma^H$. Therefore, if $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a Σ^K-covering map, then p is a Σ^H-covering map.

Using the above remark, in the following chain, we compare some Σ-covering maps with each other according to their neighborhood families.

$$\mathrm{COV(X)} \overset{(1)}{=} \Sigma^{Span}\text{-}\mathrm{COV(X)} \overset{(2)}{\subseteq} \mathrm{SCOV(X)} \overset{(3)}{=} \Sigma^{pSpan}\text{-}\mathrm{COV(X)} \overset{(4)}{\subseteq} \mathrm{GCOV(X)}$$

$$\overset{(5)}{\subseteq} \Sigma^{gcov}\text{-}\mathrm{COV(X)} \overset{(6)}{\subseteq} \Sigma^{sg}\text{-}\mathrm{COV(X)} \overset{(7)}{\subseteq} \Sigma^{s}\text{-}\mathrm{COV(X)}$$

Now, according to the enumeration in the above chain, we give references and complementary notes for each item.

(1) See Theorem 2.2.

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(2) Brazas in [3, Proposition 3.7] showed that every covering map is a semicovering map but the converse is not true in general (see [3, Example 3.8]). For connected and locally path connected spaces, the equality holds if and only if X is semilocally small generated [2].
(3) See Theorem 2.3.
(4) It is proved in [2, Lemma 2.9] that every semicovering map has unique lifting property so every semicovering map is a generalized covering map. The converse is not true in general (see [6, Example 4.15]). For connected and locally path connected spaces the equality holds if and only if X is semilocally path $\pi_1^{gc}(X, x_0)$-connected [2, Corollary 4.4].
(5) See Theorem 2.4.
(6) By Remark 2.5, if X is a locally path connected and semilocally simply connected space, then the subgroups contained in $\pi_1^{sp}(X, x_0)$ are equal,. Hence if X is locally path connected and semilocally simply connected, then the equality holds. The strict inequality holds for the space RX [2, Example 2.5] and constant map $p : (RX, x_0) \to (RX, x_0)$. Because $\pi_1^{sg}(X, x_0) = 1$ and $\pi_1^{gc}(X, x_0) \neq 1$ thus p is a Σ^{sg}-covering map, but it is not a Σ^{gc}-covering map.
(7) Similar to (6), if X is locally path connected and semilocally simply connected, then the equality holds. The strict inequality holds for the space $(\mathbb{HA}, b)$, where $b \neq 0$ and constant map $p : (\mathbb{HA}, b) \to (\mathbb{HA}, b)$. Because $\pi_1^{s}(\mathbb{HA}, b) = 1$ and $\pi_1^{sg}(\mathbb{HA}, b) \neq 1$ so p is a Σ^{s}-covering map, but it is not a Σ^{sg}-covering map.

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