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To cite this article: Mohammad Sal Moslehian & Assadollah Niknam (2002): On Contractibility of Matrix Algebras, Quaestiones Mathematicae, 25:3, 327-332
To link to this article: http://dx.doi.org/10.2989/16073600209486020

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ON CONTRACTIBILITY OF MATRIX ALGEBRAS

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Abstract. We show first that for each C*-algebra $A$, contractibility of $A$ implies contractibility of $M_n(A)$.
We next prove that an incidence algebra $A$ of upper triangular matrices, defined by a partially ordered set $\Omega$ on \{1, 2, ..., n\} satisfying $(p, q) \in \Omega \Rightarrow p \leq q$, is a contractible Banach algebra if there is no discordant couple of D-transitive triples of elements of $\Omega$.

Mathematics Subject Classification (2000): Primary 46H05, 46H25; Secondary 15A99.
Key words: Contractible Banach algebra, incidence algebra.

1. Introduction. Contractible Banach algebras are one of the homologically best algebras. A Banach algebra $A$ is said to be contractible if it has the following (equivalent) properties [2]:

(i) $H^1(A, X) = 0$ for any Banach $A$-bimodule $X$, i.e. each derivation of $A$ with values in any Banach $A$-bimodule is inner.

(ii) Every (one-sided or two-sided) Banach $A$-bimodule is projective.

(iii) $A$ is unital and as a Banach $A$-bimodule is projective.

(iv) $A$ is unital and the canonical morphism $\pi : \hat{A} \otimes A \longrightarrow A$ defined by $\pi(a \otimes b) = ab$ is a retraction, i.e. has a right inverse, in the category $\mathbf{A - mod - A}$ of Banach $A$-bimodules.

The reader is referred to [3] for details, undefined terms and notations on homology of Banach algebras.
2. Contractibility of square matrices with entries in a C*-algebra.
Throughout this section, $A$ is a unital C*-algebra and $M_n(A)$ denotes the C*-algebra of $n \times n$ matrices with entries in $A$. Note that $M_n(A)$ is *-isomorphic to $A \otimes M_n(C)$. If $a \in A$ and $[\lambda_{ij}] \in M_n(C)$, then $a \otimes [\lambda_{ij}]$ is identified to the matrix $[\lambda_{ij} a]$ in $M_n(A)$; in particular if $\{E_{ij} : i, j = 1, \ldots, n\}$ is the standard system of matrix units for $M_n(C)$, we denote the corresponding element to $a \otimes E_{ij}$ by $a E_{ij}$.

We have
\[
\| a_{ij} \| \leq \| [a_{ij}] \| \leq \sum_{1 \leq i, j \leq n} \| a_{ij} \| \quad \text{for all } 1 \leq i, j \leq n \quad [4].
\]

**Theorem 2.1.** Let $A$ be contractible, then so is $M_n(A)$.

**Proof.** Suppose that $\rho : A \to A \hat{\otimes} A$ is a right inverse morphism to the canonical morphism $\pi : A \hat{\otimes} A \to A$. Put $\bar{\rho}(a E_{ij}) = \lim_{n \to \infty} \sum_{k=1}^m u_k^* E_{ij} \otimes v_k^* E_{ij}$, in which $\rho(a) = \lim_n \sum_{k=1}^m u_k^* \otimes v_k^*$; and extend it by linearity to a mapping $\bar{\rho}$ from $M_n(A)$ to $M_n(A) \hat{\otimes} M_n(A)$:

\[
\bar{\rho}([a_{ij}]) = \bar{\rho} \left( \sum_{1 \leq i, j \leq n} a_{ij} E_{ij} \right) = \sum_{1 \leq i, j \leq n} \bar{\rho}(a_{ij} E_{ij}).
\]

If $\sum_{k=1}^m u_k \otimes v_k = 0$, then by [4, proposition 11.1.8] there exist an $m \times m$ complex matrix $[c_{jk}]$ such that $\sum_{j=1}^m c_{jk} u_j = 0 (k = 1, \ldots, m)$ and $\sum_{k=1}^m c_{jk} v_k = v_j (j = 1, \ldots, m)$. Hence, $\sum_{k=1}^m u_k E_{ij} \otimes v_k E_{ij} = 0$. Therefore, the map

\[
\omega : \sum_{k=1}^m u_k \otimes v_k \mapsto \sum_{k=1}^m u_k E_{ij} \otimes v_k E_{ij}
\]

is well-defined and furthermore,

\[
\left\| \sum_{k=1}^m u_k E_{ij} \otimes v_k E_{ij} \right\| \leq \inf \left\{ \sum_{k=1}^m \| u_k' \| \left\| v_k' \right\| : \sum_{k=1}^m u_k \otimes v_k = \sum_{k=1}^m u_k' \otimes v_k' \right\} = \left\| \sum_{k=1}^m u_k \otimes v_k \right\|.
\]
Thus,
\[
\| \tilde{\rho}(a_{ij}) \| = \lim_{n \to \infty} \| \sum_{k=1}^{m_n} u_k^n E_{ij} \otimes v_k^n E_{ij} \|
\leq \lim_{n \to \infty} \| \sum_{k=1}^{m_n} u_k^n \otimes v_k^n \| = \lim_{n} \sum_{k=1}^{m_n} u_k^n \otimes v_k^n \leq \| \rho \| \| a \|.
\]

It follows that \( \tilde{\rho} \) is well-defined and \( \| \tilde{\rho} \| \leq n^2 \| \rho \| \).

\( \tilde{\rho} \) is a morphism in \( M_n(A) \) mod \( M_n(A) \). In fact for each \([a_{ij}], [b_{ij}] \in M_n(A)\), by virtue of \([c_{ij}] = \sum_{1 \leq i,j \leq n} c_{ij} E_{ij}\), we have

\[
\tilde{\rho}(b_{pq}[a_{ij}]) = \sum_{1 \leq i,j \leq n} \tilde{\rho}(b_{pq}) \rho(a_{ij} E_{ij}) = \sum_{1 \leq i,j \leq n} (\sum_{p=1}^{n} b_{pq} E_{pq}) \rho(a_{ij} E_{ij})
= \sum_{1 \leq i,j \leq n} [b_{pq}] \rho(a_{ij} E_{ij}) = [b_{pq}] \rho([a_{ij}]).
\]

Similarly, \( \tilde{\rho}(b_{pq} [a_{ij}]) = \tilde{\rho}(a_{ij} E_{ij}) = \rho(a_{ij} E_{ij}) \).

It is straightforward to show that \( \tilde{\rho} \) is a right inverse to the canonical morphism \( \bar{\pi} : M_n(A) \otimes M_n(A) \to M_n(A) \). It follows that \( M_n(A) \) is contractible.

\begin{corollary}
\( M_n(C) \) is contractible.
\end{corollary}
\begin{proof}
\( C \) is contractible, since the canonical morphism \( \pi : C \otimes C \to C \) is an isomorphism.
\end{proof}

\begin{corollary}
If \( A \) has discrete primitive spectrum and all its irreducible representations are finite dimensional, then the same is true for \( M_n(A) \).
\end{corollary}
\begin{proof}
A unital C*-algebra is contractible iff its primitive spectrum is discrete and each of its irreducible representations is finite dimensional [1, Assertion IV.5.15].
\end{proof}

The following is a good question in the argued context:

\begin{question}
Is the biprojectivity of \( M_n(A) \) implies the biprojectivity of \( A \)?
\end{question}

In this section we characterize a class of contractible Banach algebras among matrix algebras. We shall set up our notation.

Let $\Omega$ be a partially ordered set on $\{1, 2, \ldots, n\}$ such that $(p, q) \in \Omega$ implies $p \leq q$, and $A(\Omega)$ be the subalgebra of $M_n(\mathbb{C})$ of all matrices $(a_{ij})$ such that $a_{ij} = 0$ unless $(i, j) \in \Omega$. Note that the condition $(p, q) \in \Omega \Rightarrow p \leq q$ ensures that each element of $A(\Omega)$ is upper triangular and the transitivity guarantees that $A(\Omega)$ is closed under multiplication. $A(\Omega)$ is a Banach algebra with respect to the usual norm which inheriting from $M_n(\mathbb{C})$. Applying the terminology of [1], $A(\Omega)$ is called an incidence algebra of upper triangular matrices (the other names are digraph algebra and finite dimensional CSL-algebra). We shall write $A$ for $A(\Omega)$. $E_{pq}$ stands as already for the matrix having zero everywhere except the $(p, q)$th place and the $(p, q)$th place has the entry 1. The multiplication in $A$ is determined by the products $E_{ij} E_{kl} = \delta_{ik} E_{jl}$ where $\delta$ denotes the Kronecker symbol. Evidently $\{E_{pq} | (p, q) \in \Omega\}$ is a linear basis for $A$.

A transitive triple is an ordered triple of the form $((p, q), (q, r), (p, r))$ of elements in $\Omega$; if in addition $p \neq q$ and $q \neq r$, it said to be D-transitive. Two distinct D-transitive triple are called discordant if their third components are equal or there exists a pair $(i, j)$ occurring in different components of them.

**Lemma 3.1.** Let $\rho : A \rightarrow A \hat{\otimes} A$ be a right inverse morphism to the canonical morphism in $A \mod A$ and $((p, q), (q, r), (p, r))$ be a D-transitive triple. Then

$$
\rho(E_{pq}) = E_{pq} \otimes E_{qq} + \sum_{i,j,k,l} \alpha_{ij}^{kl} E_{ij} \otimes E_{kl} \tag{1}
$$

$$
\rho(E_{qr}) = E_{qq} \otimes E_{qr} + \sum_{i,j,k,l} \beta_{ij}^{kl} E_{ij} \otimes E_{kl} \tag{2}
$$

$$
\rho(E_{pr}) = E_{pq} \otimes E_{qr} + \sum_{i,j,k,l} \gamma_{ij}^{kl} E_{ij} \otimes E_{kl} \tag{3}
$$

in which the summations are taken over all $i, j, k, l$ such that $((i, j), (k, l), (i, l))$ is not a transitive triple.

**Proof.** Notice first that $\{E_{ij} \otimes E_{kl} | (i, j), (k, l) \in \Omega\}$ is a linear basis for $A \hat{\otimes} A$.

Since $\pi \circ \rho = 1_A$, we have

$$
\rho(E_{pq}) = a_0 E_{pq} \otimes E_{pq} + a_1 E_{p(p+1)} \otimes E_{(p+1)q} + \cdots

\cdots + a_{l-p} E_{pq} \otimes E_{qq} + \sum_{i,j,k,l} \alpha_{ij}^{kl} E_{ij} \otimes E_{kl} \tag{4}
$$

$$
\rho(E_{qr}) = b_0 E_{qq} \otimes E_{qr} + b_1 E_{q(q+1)} \otimes E_{(q+1)r} + \cdots

\cdots + b_{r-q} E_{qr} \otimes E_{rr} + \sum_{i,j,k,l} \beta_{ij}^{kl} E_{ij} \otimes E_{kl}, \tag{5}
$$
\[ \rho(E_{pr}) = c_0 E_{pp} \otimes E_{pr} + c_1 E_{p(p+1)} \otimes E_{(p+1)r} + \cdots 
\cdots + c_{q-p} E_{pq} \otimes E_{qr} + \cdots 
+ c_{r-p} E_{pr} \otimes E_{rr} + \sum_{i,j,k,l} \gamma_{ij}^{kl} E_{ij} \otimes E_{kl}. \tag{6} \]

The summations are taken over all \( i, j, k, l \) such that \( (i, j), (k, l), (i, l) \) is not transitive triple. Note also that
\[ q-p \sum_{u=0}^{r-q} b_u = q-p \sum_{u=0}^{r-p} c_u = 1. \]

Since \( \rho \) is an \( A \)-bimodule morphism, (4) and (5) imply:
\[ \rho(E_{pr}) = \rho(E_{pq}, E_{qr}) = a_0 E_{pp} \otimes E_{pr} + a_1 E_{p(p+1)} \otimes E_{(p+1)r} + \cdots 
\cdots + a_{q-p} E_{pq} \otimes E_{qr} + \cdots \tag{7} \]
\[ \rho(E_{pr}) = \rho(E_{pq}, E_{qr}) = b_0 E_{pq} \otimes E_{qr} + b_1 E_{p(q+1)} \otimes E_{(q+1)r} + \cdots 
\cdots + b_{r-q} E_{pr} \otimes E_{rr} + \cdots \tag{8} \]

Now comparing (6), (7) and (8) we can conclude the following:
\[ a_0 = c_0 = 0, \ a_1 = c_1 = 0, \ldots, a_{q-p-1} = c_{q-p-1} = 0, \]
and then
\[ b_1 = c_{q-p+1}, \ b_2 = c_{q-p+2}, \ldots, b_{r-q} = c_{r-p} = 0. \]

Thus, \( a_{q-p} = b_0 = c_{q-p} = 1 \) and
\[ \rho(E_{pq}) = E_{pq} \otimes E_{qq} + \sum_{i,j,k,l} \alpha_{ij}^{kl} E_{ij} \otimes E_{kl} \]
\[ \rho(E_{qr}) = E_{qq} \otimes E_{qr} + \sum_{i,j,k,l} \beta_{ij}^{kl} E_{ij} \otimes E_{kl} \]
\[ \rho(E_{pr}) = E_{pq} \otimes E_{qr} + \sum_{i,j,k,l} \gamma_{ij}^{kl} E_{ij} \otimes E_{kl}. \]

We are now ready to prove our main result:

**Theorem 3.2.** An incidence algebra \( A(\Omega) \) of upper triangular matrices is contractible if there is no discordant couple of \( D \)-transitive triples of elements of \( \Omega \).

**Proof.** Suppose that there are no discordant couples of \( D \)-transitive triples and let \((m, n) \in \Omega\). We may consider two cases:

(i) Suppose \((m, n)\) is a component of a \( D \)-transitive triple:

(ii) If the triple is of the form \((m, n), (n, r), (m, r)\), we define
\[ \rho(E_{mn}) = E_{mn} \otimes E_{nn} \text{ according to (1)}. \]
(i - 2) If the triple is of the form \((s, m), (m, n), (s, n)\), we define
\[ \rho(E_{mn}) = E_{mm} \otimes E_{mn} \]
according to (2).

(i - 3) If the triple is of the form \((m, t), (t, n), (m, n)\), we define
\[ \rho(E_{mn}) = E_{mt} \otimes E_{tn} \]
according to (3).

(ii) If \((m, n)\) is not a component of any D-transitive triple, we put
\[ \rho(E_{mn}) = E_{mm} \otimes E_{mn}. \]

Then \(\rho\), being defined on a basis of \(A\), is a well-defined \(A\)-bimodule morphism and furthermore \(\pi \circ \rho = 1_A\). Hence, \(A\) is contractible.

If there is a discordant couple of D-transitive triples, Lemma 1 clearly yields a contradiction; e.g., if there exist \((i, j), (j, k), (i, k)\) and \((h, i), (i, j), (h, j)\) it follows from the lemma that
\[ \rho(E_{ij}) = E_{ij} \otimes E_{jj} + \ldots = E_{ii} \otimes E_{ij} + \ldots \]
which is contradictory to the linear independence of the \(E_{pq}\)’s.

**Example 3.3.** The incidence algebra of all \(4 \times 4\) matrices of the form
\[
\begin{pmatrix}
* & 0 & * & * \\
0 & * & 0 & 0 \\
0 & 0 & * & * \\
0 & 0 & 0 & *
\end{pmatrix},
\]
where the stars indicate arbitrary complex numbers, is contractible; but the incidence algebra of all upper triangular \(4 \times 4\) matrices is not contractible.

**Question 3.4.** What can we say about the contractibility of incidence algebra \(A(\Omega)\) without the condition \((p, q) \in \Omega \Rightarrow p \leq q\) on \(\Omega\)?

**References**


Received 31 July, 2001