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LINEAR AND NON-LINEAR OPTIMIZATION MODELS FOR ALLOCATION OF A LIMITED WATER SUPPLY†

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ABSTRACT

One partial solution to the problem of ever-increasing demands on our water resources is optimal allocation of available water. A non-linear programming (NLP) optimization model with an integrated soil water balance was developed. This model is the advanced form of a previously developed one in which soil water balance was not included. The model also has the advantage of low computer run-time, as compared to commonly used dynamic programming (DP) models that suffer from dimensionality. The model can perform over different crop growth stages while taking into account an irrigation time interval in each stage. Therefore, the results are directly applicable to real-world conditions. However, the time trend of actual evapotranspiration (AET) for individual time intervals fluctuates more than that for growth-stage AETs. The proposed model was run for the Ardak area (45 km NW of the city of Mashhad, Iran) under a single cropping cultivation (corn) as well as a multiple cropping pattern (wheat, barley, corn, and sugar beet). The water balance equation was manipulated with net applied irrigation water to overcome the difficulty encountered with incorrect deep percolation. The outputs of the model, under the imposed seasonal irrigation water shortages, were compared with the results obtained from a simple NLP model. The differences between these two models (simple and integrated) became more significant as irrigation water shortage increased. Oversimplified assumptions in the previous simple model were the main causes of these differences. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: irrigation optimization; deficit irrigation; Iran

RÉSUMÉ

L'allocation optimale des ressources d'eau disponibles est une réponse partielle au problème de la demande sans cesse croissante de consommation d'eau. Un modèle d'optimisation à programmation non linéaire (NLP) qui intègre un bilan hydrique a été développé. Ce modèle est une version avancée d'un modéle précédent qui n'intégrait pas ce bilan hydrique. Il présente l'avantage de nécessiter moins de puissance informatique en comparaison des modèles à programmation dynamique (DP) généralement utilisés. Le modèle peut s'appliquer à différentes étapes de la croissance des cultures et prend en compte des fréquences d'irrigation variables. Ainsi, les résultats sont directement applicables aux conditions réelles. Le modèle proposé a été utilisé sur une seule culture (maïs) dans la région d'Ardak à 45 km nord-ouest de Mashad, Iran, et sur de multiples cultures (blé, orge, maïs, betterave sucrière). L'équation de bilan hydrique a été calibrée pour maîtriser les difficultés rencontrées avec des mesures d'infiltration incorrectes. Les résultats du modèle, dans le cadre de restrictions d'irrigation saisonnière imposées, ont été comparés avec ceux obtenus par un modèle simple NLP. Les différences entre ces deux modèles (simple et intégré) deviennent plus significatives à mesure que les restrictions d'irrigation augmentent. Les

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† Modèles d'optimisation linéaire et non linéaire d'allocation d'une quantité d'eau restreinte.

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hypothèses trop simplistes du modèle simple sont la cause de ces différences. Copyright © 2004 John Wiley & Sons, Ltd.

MOTS CLÉS: optimisation de l'irrigation; déficit d'irrigation; Iran

INTRODUCTION

Crop growth is hindered by low soil water content during the growing season. Inadequate irrigation in arid or semi-arid regions usually leads to low soil water content. In such a condition, the crop yield does not reach its potential. The reduced yield can be estimated through the so-called dated water production function, which is a mathematical relationship between crop ET and its ultimate yield (e.g. Nairizi and Rydzewski, 1977). The optimized allocation of a limited water supply aims at minimizing crop yield loss. The literature contains numerous references in this field. Nearly all of these references deal with dynamic programming (DP), a powerful means of mathematical programming. By using a structured language, all model constraints can be managed by DP (Rao *et al.*, 1988b). These constraints are derived from a soil water balance, the relationship between actual evapotranspiration (AET) and soil water content, and some upper and lower bounds for variables. The objective of such models is to maximize total crop yield. There are also some models (Pleban *et al.*, 1984) that intend to minimize irrigation cost (e.g. labour). Such models, however, do not include soil water variations or crop yield estimation. Dynamic programming in general suffers from a well-known inadequacy, i.e. dimensionality (Labadie, 1993). That is, as the number of states in DP increases the computer run-time and/or required memory also increases. Other types of mathematical programming, although with different robustness, do not pose with such difficulty. Techniques such as the Dantzig-Wolfe decomposition principle (Trava *et al.*, 1977) and the mixed-integer optimization model (Pleban *et al.*, 1983) were developed to optimize short term irrigation schedule. These techniques, similar to previous techniques, focus on minimization of irrigation costs and do not include yield optimization. Literature review indicates that there is no independent report that deals with an integrated optimization except DP. Ghahraman and Sepaskhah in a series of papers used a simplified form of a non-linear programming (NLP) for a single crop (Ghahraman and Sepaskhah, 1991, 1997a,b), and for multi-cropping pattern (Ghahraman and Sepaskhah, 1996).

The objective of this study is to enhance the previously established models and to introduce an accurate NLP as well as an approximate LP that are able to handle integrated constraints in optimization of irrigation water allocation for single- and multi-cropping patterns in a field.

THEORY

Objective function

The crop water use/yield relations which include the effects of both timing and quantities of irrigation water are called dated water production functions (e.g. Nairizi and Rydzewski, 1977). These relationships are complex, as they must include the effects of crop water-stress in different growth stages. In general, the following two water production functions are more popular in literature:

$$Y_a/Y_p = \prod_{i=1}^{n} (AET_i/PET_i)^{\lambda_i} \tag{1}$$

$$Y_a/Y_p = \prod_{i=1}^{n} [1 - Ky_i \cdot (1 - AET_i/PET_i)] \tag{2}$$

where n is total number of crop growth stages, Y_a and Y_p are actual and potential yields, respectively, AET and PET are actual and potential *ET* (evapotranspiration), respectively, and λ and Ky are the yield response factors. Equation (1) was given by Jensen (1968), while Rao *et al.* (1988a) proposed Equation (2). Equation (2) covers all growth stages simultaneously and is an extended form of one growth stage-water production function developed by

Doorenbos and Kassam (1979). Values of Ky were specified by Doorenbos and Kassam (1979), for approximately 33 crops but there are only scattered reports on λ values. The yield response factors (Ky) of Doorenbos and Kassam (1979) were validated and used by several investigators to predict crop-yield responses at several locations in the USA, China, Korea (Hayes *et al.*, 1982; Terjung *et al.*, 1984a,b). However, Ghahraman and Sepaskhah (1999) showed some signs of anomalies in using them.

With a suitable dated water production function, one can define an objective function for optimum allocation of a limited water supply. Determination of AET, however, requires further calculations through the soil water balance. For this purpose, Nairizi and Rydzewski (1977) and Meyer *et al.* (1993) developed an approximation in Equation (1) by substitution of ET_a/ET_p with the ratio of applied water to potential water need W_a/W_p:

$$Y_a/Y_p = \prod_{i=1}^{n} \left(W_a/W_p\right)^{\lambda_i} \tag{3}$$

Maximization of the Y_a/Y_p ratio in Equation (3) would be a sound objective function for one crop cultivation pattern. However, for a multi-cropping pattern, extended data on cropping acreage, unit area cultivation cost and gross benefit are required for each crop. Therefore, under such conditions, the total net benefit (TNB) is to be maximized (Equation 4):

$$\text{TNB} = \sum_{c} A_c \cdot \left[B_c \cdot \left(Y_a/Y_p\right)_c - C_c\right] \tag{4}$$

where A is area of cultivation, B and C are gross benefit (revenue) and cost of cultivation per hectare, respectively, and c is the crop index.

Constraints

A set of equations as constraints were developed based on the definition of relative yield. Relative yield, as defined earlier, is a function of either relative ET (Equations 1 or 2) or relative irrigation water (Equation 3). For Equation (3) which is described for one crop condition, the constraint equations are simple and can be defined as follows (Ghahraman and Sepaskhah, 1997b):

$$0 \le (W_a)_i \le (W_p)_i \tag{5}$$

$$\sum_i (W_a)_i = (1-x) \cdot \sum_i (W_p)_i \tag{6}$$

where x stands for the fraction of water shortage imposed on the crop. Equation (6) presents the available water ready for allocation that also has a physical meaning. In many parts of developing countries, irrigation water is only applied on fixed interval basis. Therefore, for a sustainable situation, the total seasonal water that can be delivered to a farm can be considered fixed since it must be in a volume balance with natural annual groundwater recharge. The model (Equations 3, 5, 6) detailed solution was reported elsewhere (Ghahraman and Sepaskhah, 1997b). Using the model, the optimum water depth at each growth stage can be determined. Then the total water depth should be reallocated in smaller time intervals corresponding to the irrigation intervals. The required input data for the model are water requirements at different growth stages. These data are obtained by multiplying crop coefficients, K/c (Allen *et al.*, 1998), and reference crop ET (e.g. from Penman–Monteith).

The simple model (Equation 3) provides only approximated results. The model outcomes cannot be reliably applied to real-world conditions. For a real condition (Equations 1 and 2), the constraint equations are more complex. However, the model outcomes are more realistic and can be applied to real-world conditions with more certainty.

The integrated NLP model has several constraints. These constraints may be defined as follows.

Soil water balance. Based on irrigation intervals which is derived from the local data, a series of soil water balance equations can be defined for any given crop c and time interval t as follows:

$$\text{SM}_{c,t+1} \cdot \text{Root}_{c,t+1} = \text{SM}_{c,t} \cdot \text{Root}_{c,t} + \text{Rain}_t + \text{IR}_{c,t}\text{AET}_{c,t} - \text{DP}_{c,t} + \text{SM}_c \cdot \left(\text{Root}_{c,t+1} - \text{Root}_{c,t}\right) \tag{7}$$

where SM is the available soil water per unit depth, Root is the average root depth, Rain is the rainfall amount, IR is the gross irrigation water allocated, and AET and DP are the actual evapotranspiration and deep percolation, respectively. Subscript c stands for specific crop and subscripts t and $t+1$ are due to the beginning and end of the irrigation time interval, respectively. In general, the assumptions made for this model are summarized as follows:

1. Irrigation application efficiency (E_a) of less than 100% causes some percolation of water to below the root zone. Therefore, a constraint in the form of $DP_{c,t} \geq \{IR_{c,t} \cdot (1 - E_a)\}$ needs to be included in the model structure.
2. It was assumed that application efficiency remains constant for both deficit and full irrigation conditions. The quantitative variation of application efficiency under deficit irrigation conditions has not been reported in the literature.
3. An irrigation uniformity of 100% is assumed for simplicity.
4. The dynamic aspect of root growth and its temporal variation was assumed using a sine function (Borg and Grimes, 1986).
5. The initial soil water content is considered constant through the deeper layers of the soil, so as the root deepens, a contribution of this extra soil water is added to the soil water balance equation (last term in Equation 7).
6. The occurrence of surface runoff was ignored under deficit irrigation.
7. The inputs to and outputs of the model are re-evaluated and updated at the beginning of each time interval instantaneously.
8. The exact variation of soil water content during irrigation intervals cannot be estimated. Therefore, for simplicity, it is suggested to be linear.
9. The available soil water content for any crop c and at any time t can only change within the following range:

$$PWP_c \leq SM_{c,t} \leq FC_c \tag{8}$$

where PWP is permanent wilting point and FC is field capacity.

Initial conditions for the soil water profile. In Iran, with arid or semi-arid climatic conditions, a calendar year is usually divided into two distinct seasons of unequal lengths, i.e. the dormant season (between November 22 and February 9) and non-dormant season (between February 9 and November 22 of the next year). Due to higher rainfall during the dormant season, it is assumed that soil water content at the beginning of the non-dormant season is at FC. This is supposed to be an initial condition for summer crops as well as for winter ones as they become active at the beginning of the non-dormant season. Winter crops, however, require an extra boundary condition. In Iran, these crops are cultivated at mid-autumn, prior to which there is nearly as long a period of no rainfall. Therefore, a PWP for soil water content is considered for such crops.

The length of the growing season for most of summer crops in Iran is generally less than the non-dormant season. Therefore, with an irrigation interval of 10 days considered in this study, it is not necessary to handle 28 separate soil water balance equations. As a consequence, the initial boundary condition for soil water content at the time of planting for summer crops becomes indefinite, while it was set to FC at the start of the non-dormant season. Bare soil evaporation is responsible for soil moisture reduction during this period. While total available soil water (FC – PWP) over the root depth has not been depleted by fraction p, it is assumed (Doorenbos and Kassam, 1979) that evapotranspiration (ET) is equal to potential evapotranspiration (PET). Beyond this limit, however, evapotranspiration will depend linearly on the remaining soil water and on PET. This concept can be generalized to a bare soil; ET is substituted by evaporation. The fraction p is not defined for a bare soil, as there are no roots in a bare soil. A hypothetical short-rooted crop can be considered and then fraction p is defined thereafter (Doorenbos and Kassam, 1979, Table 20). Based on the above assumptions and concepts, the average soil water content for consecutive time intervals applied in the model can be obtained through water balance calculations.

As was discussed earlier, it can reasonably be assumed that at the end of the dormant season the root zone soil water profile is at FC. From this time to the planting date (e.g. third week of April for corn), soil water content decreases due to the combined effects of surface evaporation and internal drainage. Therefore, at planting time, the average soil root zone water content will most likely be less than FC. For the time intervals after planting, as the roots deepen, the soil water content in the root depth increment can contribute to the soil water balance. At the same time, the soil water content below the current root profile (lower profile) increases as the irrigation application efficiency is less than 100%. So the soil water profile becomes FC as a plateau, i.e. balances the effect of surface evaporation during the past fallow intervals.

Crop evapotranspiration. There are many methods for estimation of reference evapotranspiration (ETo) from meteorological data (Burman and Pochop, 1994 among others). However, the Penman–Monteith method has been accepted as a standard (Allen *et al.*, 1998). PET for any crop can be calculated by multiplying ETo and the appropriate crop coefficient (*Kc*, Allen *et al.*, 1998).

For a given crop, AET is determined by the evaporative demand of the air when available soil water does not restrict evapotranspiration (i.e. AET = PET). Beyond the depletion of the total available soil water by the fraction p, AET falls below PET and its value depends on the remaining soil water and PET. Therefore, the following constraint governs:

$$\mathrm{AET}_{c,t} \leq \frac{\{\mathrm{SM}_{c,t}\} - \mathrm{PWP}_c}{(1 - p_{c,t}) \cdot (\mathrm{FC}_c - \mathrm{PWP}_c)} \cdot \mathrm{PET}_{c,t} \tag{9}$$

where $\{\mathrm{SM}_{c,t}\}$ is average soil water content, $(\mathrm{FC}_c - \mathrm{PWP}_c)$ is total available soil water attributed to crop c, and $p_{c,t}$ is the critical soil water depletion fraction, which depends on specific crop and PET_t (Doorenbos and Kassam, 1979). The upper limit for AET is PET, as shown below:

$$\mathrm{AET}_{c,t} \leq \mathrm{PET}_{c,t} \tag{10}$$

Based on quantity of relative available water for evapotranspiration $(\{SM_{c,t}\} - \mathrm{PWP}_c)/([1 - p_c] \cdot [\mathrm{FC}_c - \mathrm{PWP}_c])$, one of the constraints in Equations (9) and (10) will be dominant. For a given growth stage of a specific crop, total AET can be calculated as follows:

$$\mathrm{AET}_{c,\mathrm{g}} = \sum_{i=1}^{n} \left(\mathrm{AET}_{c,t}\right)_i \tag{11}$$

Water sensitivity factors (*Ky*, Equation 2) are valid only under a mild stress condition of less than or equal to 0.5 (Doorenbos and Kassam, 1979). There is no reported response factor for severe water stress conditions. Due to high yield loss occurrence, stresses greater than 0.5 are not considered realistic in an economically designed irrigation project.

Allocation constraints. Total amount of irrigation water at consecutive time intervals of all crops can not exceed the seasonal available water ready for allocation (U):

$$10^5 \cdot \sum_c \sum_t \mathrm{IR}_{c,t} \cdot A_c = U \cdot E_\mathrm{c} \tag{12}$$

where IR and U have the units of millimetres and million cubic metres, respectively, and E_c is conveyance efficiency. For a deficit irrigation plan, U will be a fraction (x) of total full seasonal irrigation water needs of a project. The value of U is not constant and varies with changes in rainfall and PET of crops under study.

STUDY LOCATION, AND CROPS DATA

The proposed integrated NLP model can be applied for any location with adequate data. In this case, the Ardak area (45 km NW of the city of Mashhad, Khorasan province, Iran) was considered as the location of study. The specifications of this area have been reported elsewhere (Ghahraman and Sepaskhah, 1999).

The adapted cropping pattern in the Ardak area with a medium-textured soil (FC = 0.3, PWP = 0.15 $\mathrm{cm}^3/\mathrm{cm}^3$) is composed of corn (12.6%), sugar beet (17.8%), winter wheat (40.7%), and winter barley (28.9%). Due to temperature limitations in the Ardak area, the crop calendar is nearly fixed. Sugar beet and corn are planted mainly in the third and last week of April, respectively, and harvested in the first and third week of October, respectively.

Table I. Some characteristics of crops under study in the Ardak area (near Mashhad, SE Iran), Corn (planting at April 27 and harvesting at October 2) and sugar beet (planting at April 18 and harvesting at October 22) are summer crops. Wheat (planting at October 2 and harvesting at June 20 with a dormant season between November 22 and and February 9) and barley (planting at September 23 and harvesting at June 10 with a dormant season similar for wheat) are winter crops. Potential evapotranspirations (PET) were calculated from ETo times *Kc*, where the Penman–Monteith equation was adopted for reference evapotranspiration, ETo (Allen *et al.*, 1998), and *Kc* is crop coefficient. Yield response factors (*Ky*) were extracted from Doorenbos and Kassam (1979). Growth stage durations and maximum root depth for different crops were chosen based on local observations. Gross benefits and production costs are typical for the study area. Accepted relative cultivated area of crops is common for the Ardak area

		Crop			
Growth stage	Parameter	Corn	Sugar beet	Wheat	Barley
Establishment	PET (mm)	71.4	67.4	29.8	33.1
	Ky	0.01	0.12[b]	0.01	0.01
	d[a]	30	30	20	20
Vegetative	PET (mm)	248.14	300.2	34.3[c] 132.8[d]	52.7 99.1
	Ky	0.4	2.0	0.2 0.2	0.2 0.2
	d	40	50	30 60	40 50
Flowering	PET (mm)	178.7		80.6	72.2
	Ky	1.5		0.6	0.6
	d	20		20	20
Yield formation	PET (mm)	314.0	417.2	183.7	165.3
	Ky	0.5	0.36	0.5	0.5
	d	50	50	40	40
Ripening	PET (mm)	23.4	190.3	36.4	27.9
	Ky	0.2	0.12	0.01	0.01
	d	20	50	20	10
Maximum root depth (cm)		120	100	110	110
Gross benefit (Rls 10^3)		1762.5[e]	3015	1400	1184.4
Production cost (Rls 10^3)		543.1	1196.2	362.44	304.2
Relative cultivated area (%)		12.6	17.8	40.7	28.9

[a] Growth stage, day.
[b] After Hill *et al.* (1983).
[c] Early vegetation.
[d] Late vegetation.
[e] 5000 Rls = 1 US dollar.

Planting dates for wheat and barley are the first week of October and last week of September, while their harvest dates are the third and second week of June, respectively. So it is concluded that there is mutual competition for water between summer and winter crops during their active growth periods. There is also an overall competition for water between all four crops during their planting and harvesting periods. Table I shows some critical data for the indicated crops.

In the Ardak area, the amount of rainfall varies during different years. For this study, however, the rainfall pattern of an average year (182.5 mm per non-dormant season) was selected and used. A constant irrigation interval of 10 days, as practised in the area, was adopted for the optimization model. A fixed irrigation interval is almost a rule in many arid and semi-arid regions of the world (Rao *et al.*, 1988b).

RESULTS AND DISCUSSION

Single crop

We assume corn is cultivated in only 1 ha. One may assume that corn potential evapotranspiration (PET, Table I), and the amount of rainfall at consecutive 10-day intervals during the growing season, are known by any appropriate probabilistic (or stochastic) model. Yet seasonal irrigation requirements may not be assumed to be the simple difference between PET and rainfall, because deep percolation is not known. Through a trial and error procedure, we assumed different values for U and ran our optimization model. The minimum seasonal irrigation water at the farm level with relative corn yield of unity was $U = 0.012832$ MCM (million cubic metres). In this routine water application efficiency was set at 0.6 which is a common value for the study area. On the other hand we neglect the conveyance efficiency for reasons of simplicity.

The corn-growing season extends from the 9th to the 24th irrigation interval. There is a little rainfall amount during this period, however, which is common in arid or semi-arid regions.

Initial conditions for the soil water profile of corn. Variations of soil water content in the active root zone for corn (from the soil surface up to the maximum root depth) for three different historical rainfall conditions (low, medium and high occurrence) are illustrated in Figure 1. The soil water content decreased during fallow intervals, with different rates, which depends on rainfall characteristics. Such a decrease shows that in these regions summer fallow cannot allow for the storage of soil water for the next season. (More field data supports are also available for a similar area in a semi-arid region of Iran, Khalili *et al.*, 2001.) Figure 1 clearly showed that as rainfall changes in amount, its time distribution could change drastically. Such variation is a rule, rather than a exception, for arid regions. During the plant growth period, however, the soil water content of the active root zone (profile) increased, while approaching FC as a plateau. Yet with this rainfall variability, the number of required time intervals for the soil water content to reach FC was nearly independent of rainfall regime and is not high (about six intervals, Figure 1). So a rough value of 0.85 for normalized water content $(\theta - \text{PWP})/(\text{FC} - \text{PWP})$ seems acceptable for the whole profile at corn planting time. The same reasoning holds true for the other summer crop, sugar beet. By so doing, a large number of constraints in the optimization model were removed. These constraints are due to the modifying

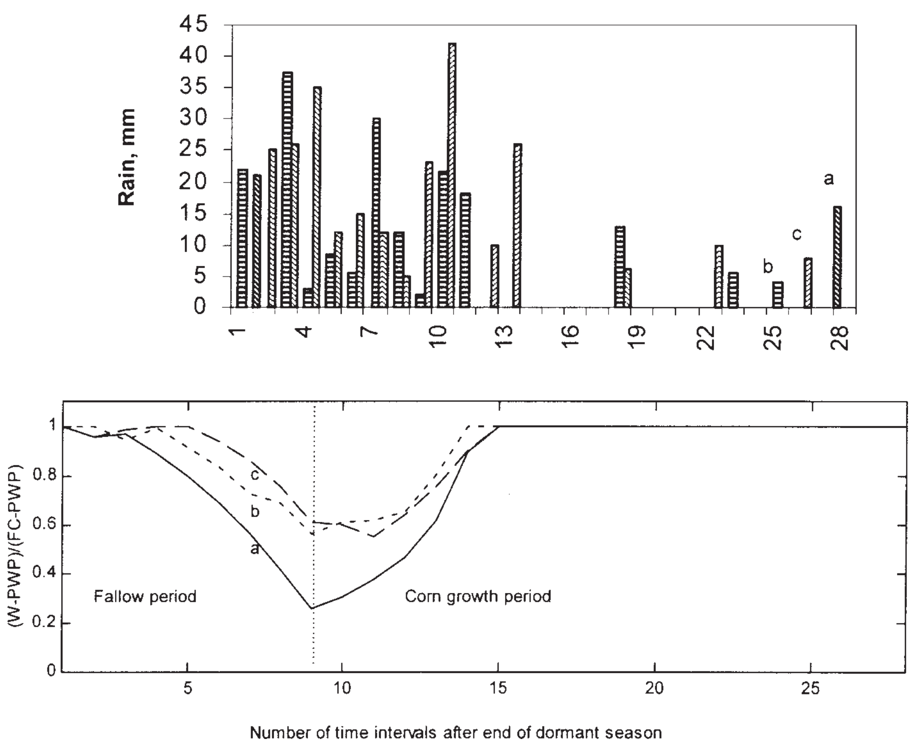

Figure 1. Lower profile soil water under corn cultivation and for different rainfall regimes, (a) low occurrence rainfall, 37 mm, (b) mean occurrence rainfall, 182.5 mm, (c) high occurrence rainfall, 255 mm. Corn is a summer crop for which there is negligible rainfall during its growth period in an arid/semi-arid region. Time distribution of rainfall changes drastically as its amount varies. There is a decreasing trend for soil water content during the fallow period, due to bare soil evaporation. Lower profile (below active root depth) soil water content increases in time when crop is irrigated, because of unwanted deep percolation, while approaching FC as a plateau

soil water content within the active root zone, as well as in the lower layers (below the active root depth) as a result of the fact that water application efficiency never reaches 100%. Therefore, the deep percolation can increase the water content of the lower soil water profile.

Results of irrigation optimization model (single crop). The results of applying the model showed that there was a considerable amount of deep percolation, which was in complete harmony with the irrigation amounts. However, the soil water contents in the active root zone at the end of about 10 time intervals were less than FC. Therefore, a positive deep percolation value is, logically, incorrect. Penalizing the objective function, to minimize deep percolation amounts as a secondary objective function, could not remove the difficulty (Ghahraman, 2000). Therefore, we decided to use another approach. The term $IR_{c,t}$ is the gross applied water, out of which $DP_{c,t}$ is a loss. Excluding surface runoff, we put $\underline{IR}_{c,t} = IR_{c,t} - DP_{c,t}$, net applied water. One may include this concept in Equation (12) which becomes

$$10^5 \cdot \sum_c \sum_t \underline{IR}_{c,t} \cdot A_c = U \cdot E_c \cdot E_a \quad (13)$$

In this new model, the constraint of $DP_{c,t} \geq IR_{c,t}(1 - E_a)$ should be discarded. Running this optimization model with redefined constraints (water balance and allocation constraints) showed that the primary parameter, AET, was unchanged and equal to PET (data not shown). Moreover, there was no deep percolation except for a very small amount (0.095 mm) for the 13th time interval. This is due to the fact that all deep percolation amounts were subtracted before entering the model. This may be an explanation to express the malfunctioning of the penalty term (Ghahraman, 2000).

The preference of the optimization model would be more pronounced with conditions of deliberately reduced seasonal full irrigation water needs (deficit irrigation). Under these conditions at different water reduction (x) of seasonal maximum irrigation demands, however, ΣAET fell below ΣPET due to insufficient water being applied to meet PET for producing a full yield. Temporal distributions of AET, as compared with PET, are presented for different x values in Figure 2. As water reduction (x) increased, AET also decreased and fell below the PET values. But these declines were not the same at different time intervals, due to unequal yield response factors (*Ky*, Table I). Although time trends of AET_g (AET for different growth stages, Equation 11) were more smooth and nearly follow the time trend of PET_g, this was not the case for AET as compared with PET (Figure 2). Such undulations may have resulted from Equations (2), (4), and (11). In the model algorithm, the primary weight is on a combination of AET_g, and individual AETs do not play a direct role in the objective function. The model just searches toward an

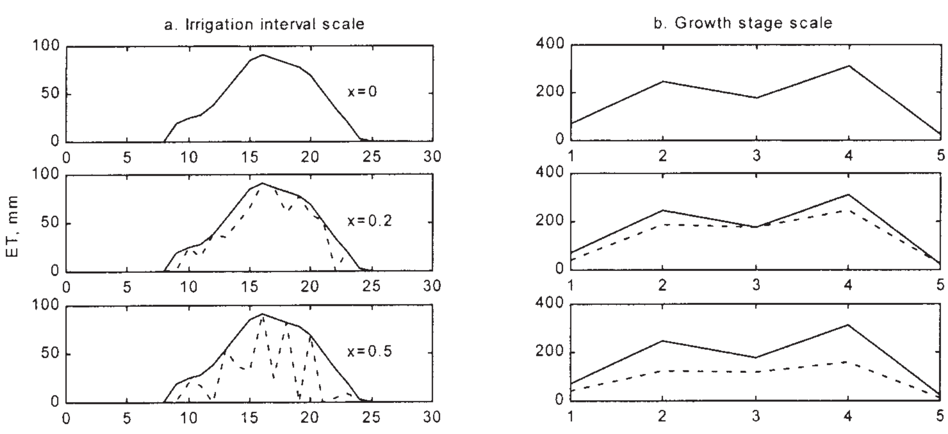

Figure 2. Time trend of AET at different x values under corn, as a single crop cultivation, (a) irrigation interval scale, (b) growth stage scale. Corn has five growth stages and cultivated at about 9th time interval in non-dormant season (April 27). With increasing water shortage level (i.e. x), crop AET falls more and more below crop PET and hence yield reduces accordingly. The structure of the optimization model is based on growth stages and, therefore, AET in a specific growth stage, and not in an irrigation interval, is a criterion for relative yield computations. It means that values of AET at irrigation interval scale do not play a direct role

optimum solution through some logical combinations of AET_g. Once such a solution is found, it does not say anything about individual AET for a known AET_g, except that all AET must be summed up to form an individual value of AET_g (Equation 11). However, the optimized solution is guaranteed by suitable combination of AET_g. Therefore, any 0 for AET_t does not mean that plant has died, while any 0 for AET_g does. The maximum relative corn yields at different water reductions of 10, 20, 30, 40, and 50% (x values of 0.1–0.5) are 0.888, 0.778, 0.675, 0.579, and 0.252. Imposing more water stresses was impossible, as the model could not find any feasible solution.

Multiple cropping pattern

Expanding the model to cover more than one crop was an easy task. With relative cultivated areas in Table I, a hypothetical 1 ha field size was assumed. A trial and error procedure resulted in the minimum total seasonal water needs for the potential yield of production for every crop. The true figure reads as 0.007795 MCM. Both crops of wheat and barley exhibited some occurrences of deep percolation at time intervals 1 and 3 (high rainy intervals). These deep percolation amounts were a total of 11.81 and 7.56 mm for wheat and barley, respectively, and are due to excess rainfall. Time distribution of AET and AET_g are identical to PET and PET_g, respectively, for a non-stress condition, as they must be so (see Figure 3 as a sample for corn and sugar beet; the other two winter crops are not reported here due to space limitation).

Reducing total seasonal irrigation water ready for allocation would impose water stress on crop(s). In this case, there is not only competition for water through the time intervals, but also there is competition among the active crops in any time interval. Therefore, the objective function (Equation 4) dictated the true water allocation to the model. It is wise to accept that as seasonal available water shortage (i.e. x) increases in amount, the more soil water content is reduced, and there is a lower chance of its replenishment. The diminishing trend of soil water content is transferred to AET and AET_g trends (Figure 3; as for Figure 2 any 0 value for AET_t does not mean plant death, while a 0 for AET_g does). AET are more fluctuating than AET_g and are due to the nature of model structure (Equations 2, 4, and 11), as was asserted previously for case of corn as a single crop. The full behaviour of other parameters are reported by Ghahraman (2000).

Table II presents the relative yields of each crop corresponding to different seasonal water shortage levels. A complex mixture of crop sensitivity to water reduction, area of cultivation, and crop gross benefit and cost caused a non-uniform effect of water shortage on crop yield, as is governed by such an integrated optimization model.

Comparison with a previous simple model

Ghahraman and Sepaskhah (1997b) have presented a simple optimization model, based on the relative yield of Equation (3). Simplifying substitution of $W_a/W_p = ET_a/ET_p$, as was used for Equations (1) and (3) for a Jensen (1968) type water production function, can be easily extended to the other water production functions (Equation 2), which becomes

$$Y_a/Y_p = \prod_i \left[1 - Ky_i\left(1 - W_a/W_p\right)_i\right] \tag{14}$$

With PET_g at different growth stages of crops in the cropping pattern (Table I), the procedure of Ghahraman and Sepaskhah (1997b) can be utilized. In addition to not considering soil water content variations, they neither regarded any rainfall in their analysis nor was water application efficiency included in the model. However, the water shortage fraction (x) can be applied directly to total seasonal water requirements (ΣPET_g). On the other hand, this study showed that rainfall can play a significant role throughout the growing season and on the average can maintain 7.5 and 35.3% of total water requirements for summer and winter crops, respectively, in the study region.

To apply rainfall to the previous simple model of Ghahraman and Sepaskhah (1997b), two different approaches may be used:

(a) One may add rainfall to the numerator (Equations 3 or 14), while keeping the denominator unchanged. This is based on the premise that any rainfall in a given growth stage can be treated as a help to applied water, so the

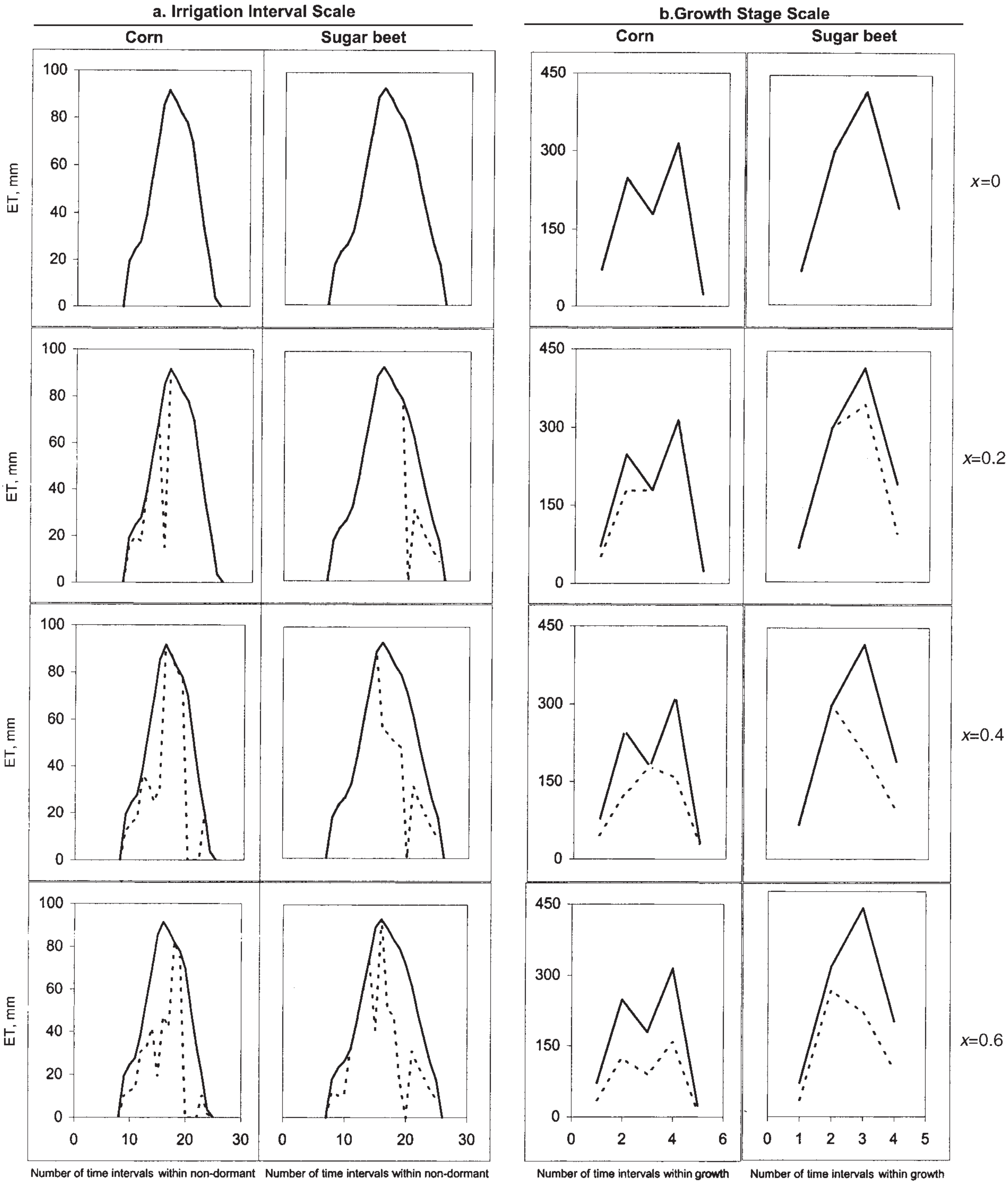

Figure 3. Time trend of AET at different x values under a mixed crop cultivation at (a) irrigation interval scale, and (b) growth stage scale. Dashed and continuous lines represent AET and PET, respectively

Table II. Comparison of relative yields of crops (multi-cropping pattern) at different water shortage fractions. A simple model is based on a dated water production function in which crop evapotranspiration is substituted by amount of water (for incorporation of rainfall, see the text). An integrated model is a comprehensive one based on a dated water production function in which evapotranspiration is traced through a water balance model, incorporating soil moisture content variations

Water shortage fraction (x)	Model type	Crop			
		Corn	Sugar beet	Wheat	Barley
0	abc	1	1	1	1
0.1	a	0.7003	1	0.9900	0.9900
	b	0.7412	1	0.9900	0.9900
	c	1	0.9400	0.9580	0.9900
0.2	a	0.5700	0.8782	0.9900	0.9900
	b	0.5700	0.9240	0.9900	0.9900
	c	0.8636	0.8819	0.8910	0.9568
0.3	a	0.5700	0.7246	0.9002	0.8910
	b	0.3540	0.7708	0.9900	0.9900
	c	0.7146	0.8400	0.8910	0.7450
0.4	a	0.1282	0.7246	0.7464	0.6014
	b	0.3540	0.7708	0.7717	0.9900
	c	0.5785	0.7708	0.8648	0.6140
0.5	a	0.1298	≈ 0	0.4210	0.4210
	b	0.3540	0.7708	0.7717	0.9900
	c	0.5700	0.7455	0.6014	0.5609
0.6	a	*	*	*	*
	b	*	*	*	*
	c	0.1282	0.4923	0.4210	0.4215

[a] Simple model, without rainfall.
[b] Simple model, with rainfall.
[c] Integrated model.
* Not feasible.

irrigation depth at the corresponding growth stage can in fact be reduced. On the other hand, the constancy of the denominator will be due to the concept of unchanged maximum crop water requirement.

(b) One may subtract rainfall from the denominator in Equations (3) or (14), while the rainfall is less than PET_g at a corresponding growth stage, which is the case for the area of the study (data not shown). In this case the numerator is left unchanged. This procedure is based on the premise that the adjusted denominator will be treated as maximum irrigation water to be applied, while the constancy of the numerator is due to actual applied water.

Therefore, the water shortage fraction (x) can be applied to the total seasonal water requirement (PET of method (a) or $\Sigma[PET_g - Rain]$ of method (b)).

Single crop (corn). The results for the two different methods of (a) and (b) were not completely the same. The differences between two approaches were less than 1% for a seasonal water fraction (x) of less than 0.5 (probably due to low rainfall amounts in the growing season). The first method (a) failed to reach an optimized solution for $x = 0.5$, yet the second method (b) and a case with no rainfall found a solution. However, as with the definition of x (i.e. fraction of irrigation water requirement, which is more meaningful for the denominator of method (b)), it seems that method (b) is a more defendable one.

Figure 4 presents the maximum relative yield of the corn crop, with two alternatives of including or excluding rainfall, at different levels of seasonal water shortages. There is also included in this figure the maximum relative yield under the integrated optimization model. It is concluded that for x values of less than 0.3, the two curves of the simplified model (methods (a) and (b)), are relatively the same, and yet have no remarkable differences from the outcomes of the integrated optimization model. The two model (simple and integrated) outcomes diverged more as x increased.

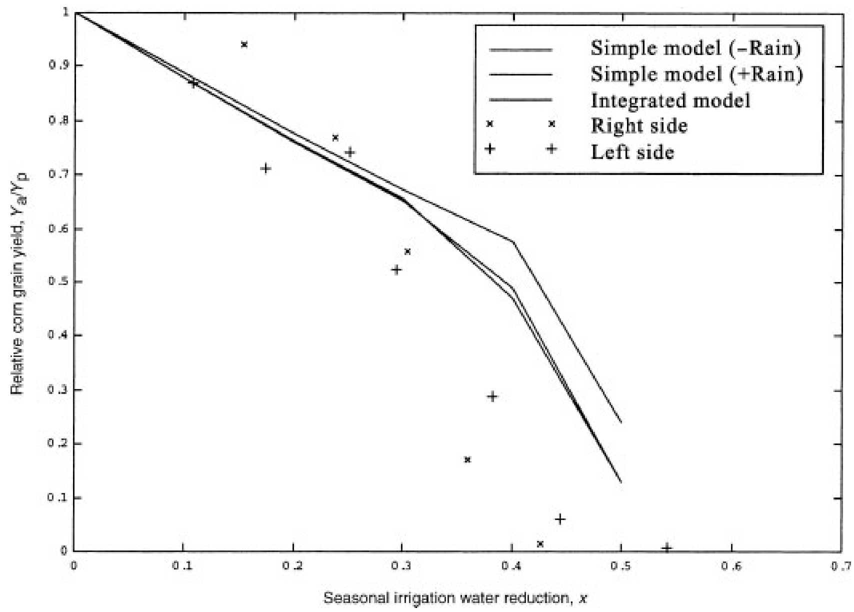

Figure 4. Maximum relative corn grain yield under different irrigation optimization models. Minor differences exist between models (see notes under Table II or refer to text for definitions of the models) up to a water reduction of 0.3. The measured relative corn grain yields in a similar arid region, extracted from a line source sprinkler irrigation experiment, are presented in the figure. The right and left sides are measured from the line source. The line source was not scheduled to maximize Y_a/Y_p, therefore the data points deviate more at higher water reductions

The field-measured yield of corn has not been reported at different water reductions in the study area. However, Sepaskhah *et al.* (1993) have measured the values of Y_a/Y_p at the various water reduction fractions in a line source sprinkler irrigation experiment for corn in the Bajgah area (16 km north of Shiraz, Fars province of Iran). In spite of the Ardak and Bajgah areas having different climatic conditions, their ETOs differ from each other by no more than 5% (Seyed Azizi, 1999). These measured data points, along with the results of the two optimization models, are also included in Figure 4. One can easily observe that the field-measured values almost follow the computed Y_a/Y_p values for both models, especially at water reduction values of less than 25%. As the line source was not scheduled to maximize Y_a/Y_p, the data points deviate more at higher water reductions. Wind distorting phenomena may explain the more scattered field data around the maximized Y_a/Y_p curve. Despite all the above assumptions, however, close agreement between theory and practice was observed.

Multiple cropping pattern. Maximum relative yields of a cropping pattern at different levels of seasonal water shortages and under two models, simple and integrated, are included in Table II. Almost the same scenario for the single crop, though more magnified, was also repeated here. There is also a sharp difference between including and excluding rainfall in the simple model. It seems that simple model is more violated under this condition.

Comparison with an integrated linear model

In addition to the multiplicative dated water production function of Equation (2), Rao *et al.* (1988a) have also addressed an additive one as follows:

$$Y_a/Y_p = 1 - \sum_i [Ky_i(1 - \mathrm{AET}_i/\mathrm{PET}_i)] \tag{15}$$

They supported that both Equations (2) and (15) bring similar results. In contrast to this strict similarity (Rao *et al.*, 1988a), however, we have magnified the performances of these two models for corn (Figure 5). Although at lower deficit values (relative yield >0.7) the two models converge, their divergence is marked at higher (yet valid–deficit less than 0.5, Doorenbos and Kassam, 1977) deficit limits (relative yield <0.7). The plotted points were not

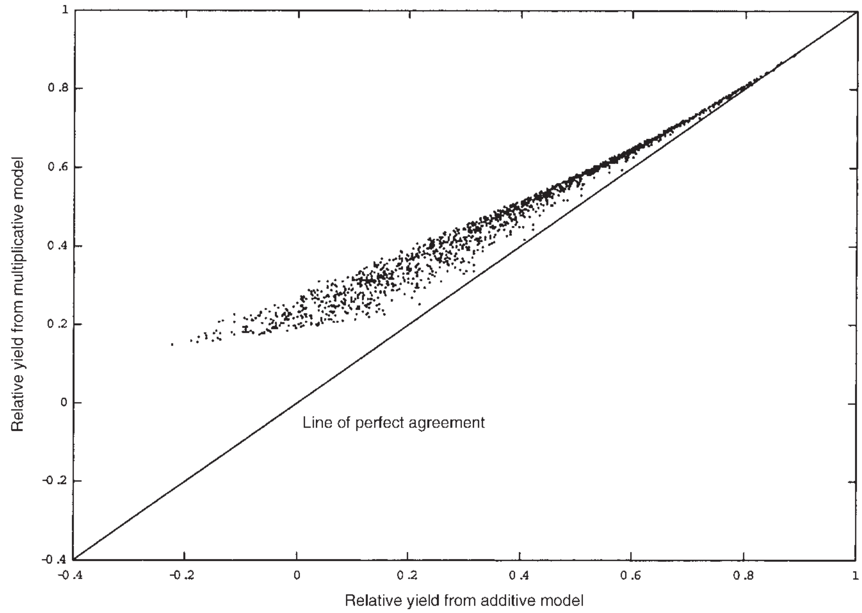

Figure 5. Comparing additive and multiplicative dated water production functions for corn plants. For a definition of additive and multiplicative relative yield models, see the comments on Table II or refer to the text

scattered uniformly around a 1:1 line of perfect agreement, which violates the assumption of Rao *et al.* (1988a). The other crops showed a similar trend (data not shown).

To overcome the above difficulty of not being similar to the multiplicative and additive dated water production functions, a regression-type approach is presented here. (The other approaches may be found in Ghahraman, 2000.) One may adopt a simple regression (which of course deviates from a 1:1 line of perfect agreement) between the outcomes (Y_a/Y_p) of the two functions. Once such a regression line was established, an estimate of the multiplicative function outcome can be made from a corresponding function, i.e.

$$Y_a/Y_p = \prod_i \lfloor 1 - Ky_i(1 - \text{AET/PET})_i \rfloor = a + b \sum_i \lfloor Ky_i(1 - \text{AET/PET})_i \rfloor \tag{16}$$

where a and b are parameters of regression line. To quantify these parameters, however, for each growth stage during a growing season of a specific crop a valid deficit level ($1 - \text{AET}_i/\text{PET}_i$) was chosen at random. So the relative yield for the whole growth period (Y_a/Y_p) can be obtained via both multiplicative and additive models of Equations (2) and (15), respectively. This procedure was repeated 1500 times independent of each other. Due to the random nature of deficit level selections, the results will differ for every run of the procedure. By equating Equations (2) and (15) (Rao *et al.*, 1988a) and the proposed method the relative yields were established. Then their differences from the results of the multiplicative water production function (Equation 2) were obtained as errors. A typical error plot for these two approaches was compared in Figure 6 for corn plants. This figure shows a drastic decrease in absolute error by shifting from Rao *et al.*'s approach to the simple regression method. An average error of about 6% (with variation of about 2% after 10 separate runs, data not shown) for the proposed method is quite satisfactory. So the proposed method was utilized to predict the assumed true multiplicative relative crop yield (Equation 2) from Equation (16) to linearize the optimization model.

A hypothetical 1 ha of field was simulated. The required seasonal water supply was determined on the basis of unit relative crop yields under a non-linear optimization model, however, the linear model did not maintain unit relative yields with the full seasonal water supply ($x = 0$). Yet the differences were not great, where the average of differences over the whole cropping pattern was less than 5% (Table III). These differences in relative yield under full irrigation conditions resulted in about 5.9% reduction in total net benefit under the linear model compared with the non-linear optimization model.

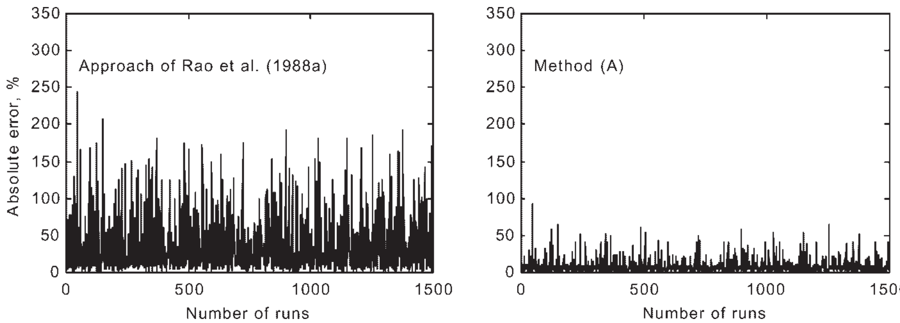

Figure 6. A typical error plot for two methods of estimating multiplicative model from and additive one (corn plant). Rao *et al*. (1988a) assumed that the outcomes of multiplicative and additive models are the same. In method (A), adopted in this paper, a relationship was fitted to these outputs (see Table III for more information).

Table III. Comparison of two models

Parameter	Crop				
	Model[a]	Corn	Sugar beet	Wheat	Barley
$x = 0.0$					
Y_a/Y_p (–)	NL	1	1	1	1
	L	0.9229	0.9747	0.9595	0.9595
Σ AET (mm)	NL	835.6	975.0	497.7	450.4
	L	835.6	975.0	497.7	450.4
Σ IR (mm)	NL	1284.2	1467.1	555.4	467.1
	L	1284.2	1467.1	555.4	467.1
$x = 0.2$					
Y_a/Y_p (–)	NL	0.8636	0.8819	0.8910	0.9569
	L	0.7658	0.9243	0.8751	0.8895
Σ AET (mm)	NL	746.7	808.2	398.2	403.3
	L	699.8	879.9	398.2	379.6
Σ IR (mm)	NL	1136.8	1193.3	391.5	388.6
	L	1057.9	1312.7	391.5	349.1
$x = 0.4$					
Y_a/Y_p (–)	NL	0.5785	0.7708	0.8648	0.6140
	L	0.5934	0.7729	0.8115	0.6976
Σ AET (mm)	NL	529.0	671.3	387.4	270.6
	L	535.7	671.3	367.8	296.9
Σ IR (mm)	NL	779.5	965.0	373.5	167.4
	L	784.5	965.0	340.8	211.4
$x = 0.6$					
Y_a/Y_p (–)	NL	0.1282	0.4923	0.4210	0.4215
	L	0	0.4462	0.3763	0.3770
Σ AET (mm)	NL	417.8	589.5	248.9	225.7
	L	419.7	588.3	248.9	225.7
Σ IR (mm)	NL	608.2	841.1	142.6	101.0
	L	611.0	839.1	142.6	101.0

[a] NL = Non-linear model, L = linear model.

Under an intentional water shortage, AET in general would fall below the PET. AET is determined from the availability of soil water content which in turn is influenced by applied irrigation water. Soil water balance (Equation 7) links these parameters to each other. A comparison of the seasonal results of the two optimization models has been made in Table III for different levels of seasonal water shortages. In general, the results showed that there were no marked differences between the model outputs, however, except that the computer run-time for the linear model was many times smaller than the non-linear model. There remains another important point, in that a linear model guarantees a global optimum solution, while a non-linear one only maintains a local optimum solution.tpb

CONCLUSION

An integrated soil water balance algorithm was coupled to a non-linear optimization model. Under two scenarios of single and multiple cropping patterns, by the model outputs, the optimal irrigation management decision can be made for consecutive time intervals coinciding with the irrigation intervals. This is a contribution over the previous simple NLP model with no inclusion of soil water balance, and also over commonly used DP models to overcome the curse of dimensionality. One difficulty encountered in the model application was attributed to the incorrect deep percolation amounts for all crops. These were due to fact that some fraction of irrigation water for each irrigation interval was considered to be wasted due to irrigation application efficiency of less than 100%. This problem persisted under a penalized objective function. Replacing gross applied water with a net amount did not change the results, except it removed the incorrect deep percolation. Present model formulation also produced a fluctuating AET trend, since the objective function acts over AET_gs and not on individual 10-day interval AETs. Although theoretically it seems possible to extend the model to cover more crops, few gains may be obtained in the real world for more than four crops in a cropping pattern. One must consider that including more crops needs more constraints in the model structure which may cause computational difficulties and/or more computer run-time and memory needs, although this increase is much less than that for a DP model with everything else the same.

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